Pedagogical Lectures on MBL

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January, 2024

The Plan

Lecture 1: *Analytical tools for a proof of Anderson localization*
Lie-Schwinger rotations provide a graphical framework for stepwise diagonalization of the Hamiltonian. Nonperturbative regions are controlled probabilistically with moment estimates and the Markov inequality.

Lecture 2: *Existence of an MBL phase*
I will describe competing effects on the density of nonperturbative regions. In the RG, isolated nonperturbative regions can be eliminated, while nearby ones have to be merged. Percolation estimates ensure that these regions are compact and rare, maintaining a minimum exponential decay rate and forestalling the avalanche mechanism.

Lecture 3: *The MBL transition*
In order to understand the nature of the transition between the MBL and ETH phases, I will use a series of approximations to develop RG flow equations based on elimination and merging of nonperturbative regions. These equations resemble the Kosterlitz-Thouless (KT) flow equations, but there are important differences that place the MBL transition in a new universality class.

Outline of Lecture 2[1]

1. Goals for a proof of MBL
2. Defining resonances
3. Perturbative analysis away from resonant regions
4. Effects of resonant regions on spins nearby
5. Preserving exponential decay
6. Conclusions

[1] Based on my paper "On many-body localization for quantum spin chains", JSP2016

Phenomenology of MBL

For a many-body quantum system with disorder, we may observe the following, which may be thought of as essential features of many-body localization (MBL):

1. Absence of transport
2. Anderson localization in configuration space (as in, e.g. IPR measures)
3. Area law entanglement
4. Violation of ETH (eigenstate thermalization hypothesis)
5. Absence of level repulsion
6. Logarithmic growth of entanglement for an initial product state

Typical example: disordered spin chain

Spin chain with random interactions and a weak transverse field on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^{K} h_i S_i^z + \sum_{i=-K}^{K} \gamma_i S_i^x + \sum_{i=-K-1}^{K} J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable.

Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

Ergodicity breaking and the emergence of an extensive set of local integrals of motion (LIOMs)

Loosely speaking, ergodicity should mean the spreading of wavepackets throughout the system. In an extreme case, there may be a complete set of of conserved quantities (quasilocal in nature) – a complete failure of ergodicity.

How do we know if a system has a complete set of quasilocal LIOMs? Can we construct them?

We seek a quasilocal unitary that diagonalizes H. That is, $D = U^\star HU$ is diagonal, and quasilocality means that the effect of U on a set of spins that span a distance L in the lattice should be (identity) + (exponentially small in L). There may be rare, nonpercolating regions where this property fails (resonant regions).

Then we may define LIOMs $\tau_i = US_i^z U^\star$.

It is clear that $[H, \tau_i] = [D, S_i^z] = 0$.

Likewise $[\tau_i, \tau_j] = 0$.

Properties 1-6 listed above for MBL should follow if one can find a complete set of LIOMs[2]

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[2]Huse, Nandkishore, Oganesyan, PRB '14; Serbyn, Papic, Abanin, PRL '13

One spin

For guidance, consider what happens for a single spin. Then

$$H = \begin{pmatrix} h & \gamma \\ \gamma & -h \end{pmatrix}$$

and for $\gamma \ll h$ the eigenfunctions are close to $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. The eigenfunctions resemble the basis vectors. This means the basis vectors can be used to label the eigenfunctions.

At the other extreme, if $\gamma \gg h$ the eigenfunctions are close to $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$. With complete hybridization, there is no meaningful way to associate eigenfunctions with basis vectors.

Perturbative and non-perturbative approaches

One may construct LIOMs perturbatively[3].

But rare regions where perturbation theory breaks down have the potential to spoil MBL. I will outline a nonperturbative construction (which, however, depends on a physically reasonable assumption on eigenvalue statistics – essentially that the level spacings in a system of n spins are no smaller than some exponential in n.)

It is especially important to have a nonperturbative proof of an MBL phase, as some are questioning the numerical evidence for MBL[4].

[3] Integrals of motion in the many-body localized phase, Ros, Müller, Scardicchio NP '15

[4] Quantum chaos challenges many-body localization, Šuntajs, Bonča, Prosen, Vidmar arXiv:1905

What about the level spacing condition?

Assumption LLA(ν, C). *Consider the Hamiltonian H in boxes of size n. Its eigenvalues satisfy*

$$P\left(\min_{\alpha\neq\beta}|E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n,$$

for all $\delta > 0$ *and all n.*

I have been developing tools for proving level-spacing conditions in simpler systems (noninteracting).

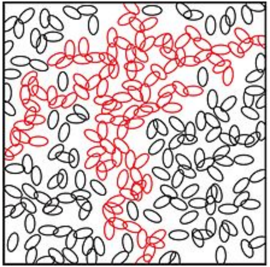
But in this talk I will focus on explaining the key mechanisms at work in the proof.

In Lecture 3, I will connect these to recent work on the nature of the transition out of the MBL phase.

Percolation picture validated for large disorder or weak interactions in 1d

Proof controls the probability of resonance for processes, and shows that the graph of resonances is non-percolating.

Then is possible to define quasilocal similarity transformations on H that diagonalize it, deforming the tensor product basis vectors into the exact eigenfunctions.



Results

Assume **LLA**(ν, c). Then MBL holds as follows:

(i) Existence of a labeling system for eigenstates by spin/metaspin configurations, with metaspins needed only on a dilute collection of resonant blocks. (Spin variables used to label basis vectors can also be used to label the exact eigenstates, but the correspondence becomes somewhat arbitrary in resonant regions, so we use the term "metaspin" instead.)

(ii) Faster-than-power-law decay of the probability of resonant blocks, which implies their diluteness. (This is critical to the whole concept of a labeling system – without it the labeling system would lose its meaning.)

(iii) Diagonalization of H via a sequence of local rotations defined via convergent graphical expansions with exponential bounds. (Locality means that graphs depend only on the random variables in their immediate vicinity.)

(iv) Bounds establishing closeness of expectations of local observables in any eigenstate to their naïve ($\gamma = 0$) values, when observables are not in resonant regions. (This makes precise the idea that eigenstates resemble the basis vectors.)

(v) Exponential decay of truncated expectations, except on a set of rapidly decaying probability. (This shows the exponential loss of entanglement with distance for the subsystems associated with the observables.)

(vi) Other good stuff...

Theorem

(c.f. (iv),(v) above) Assume **LLA(ν,c)** *for some fixed ν, c. Then there exists a $\kappa > 0$ such that for γ sufficiently small,*

$$\mathbb{E}\,\mathrm{Av}_\alpha\,|\langle S_0^z\rangle_\alpha| = 1 - O(\gamma^\kappa), \tag{1}$$

where $\langle\cdot\rangle_\alpha$ denotes the expectation in the eigenstate α, and Av_α denotes an average over α. Furthermore, for any i, j,

$$\max_\alpha |\langle \mathcal{O}_i;\mathcal{O}_j\rangle_\alpha| \le \gamma^{|i-j|/3} \text{ with probability } 1-(\gamma^\kappa)^{1+c_3(\log(|i-j|/8\vee 1))^2},$$

for some constant $c_3 > 0$. Here $\langle\mathcal{O}_i;\mathcal{O}_j\rangle_\alpha \equiv \langle\mathcal{O}_i\mathcal{O}_j\rangle_\alpha - \langle\mathcal{O}_i\rangle_\alpha\langle\mathcal{O}_j\rangle_\alpha$, with $\mathcal{O}_i$ any operator formed from products of $S_{i'}^x$ or $S_{i'}^z$, for i' near i. All bounds are uniform in Λ.

No thermalization

Consider infinite temperature, so Av_α is a uniform weighting of eigenstates. Then with thermalization (ETH), averages of eigenstate expectations of S_0^z should go to zero as $\Lambda \to \infty$. This would contradict (1) above.

Another consequence of (iv) is that essentially all of the eigenstates have a nonuniform spatial distribution of energy, which persists for all time. So in a basic sense, there is no transport in the system.

Resonances in the first step

Initially, the only off-diagonal term is $\gamma_i S_i^x$, which is local, so we may start by looking at single-flip resonances.

Let the spin configuration $\sigma^{(i)}$ be equal to σ with the spin at i flipped.
Let the associated change in energy be $\Delta E_i \equiv E_\sigma - E_{\sigma^{(i)}}$

We say that the site i is *resonant* if $|\Delta E_i| < \varepsilon \equiv \gamma^{1/20}$ for at least once choice of $\sigma_{i-1}, \sigma_{i+1}$. Then for nonresonant sites the ratio $\gamma_i / \Delta E_i$ is $\leq \gamma^{19/20}$.

A site is resonant with probability $\sim 4\varepsilon$. Hence resonant sites form a dilute set where perturbation theory breaks down.

Rotate away interaction terms $J(i) \equiv \gamma_i S_i^x$ for nonresonant sites i by defining

$$A \equiv \sum_{\text{nonresonant } i} A(i) \text{ with } A(i)_{\sigma\sigma^{(i)}} = \frac{J(i)_{\sigma\sigma^{(i)}}}{E_\sigma - E_{\sigma^{(i)}}}$$

and a renormalized Hamiltonian:

$$H^{(1)} = e^A H e^{-A} = H + [A, H] + \frac{[A,[A,H]]}{2!} + \ldots = H_0 + J^{\mathrm{res}} + J^{(1)}.$$

Properties of the new Hamiltonian:

The new interaction $J^{(1)}$ is quadratic and higher order in γ – the leading-order term has been eliminated.

Note that $A(i)$ commutes with $A(j)$ or $J(j)$ if $|i - j| > 1$.

Thus we preserve quasi-locality of $J^{(1)}$; it can be written as $\sum_g J^{(1)}(g)$, where g is a sum of connected graphs involving spin flips $J(i)$ and associated energy denominators.

Define resonant blocks by taking connected components of the set of sites belonging to resonant graphs.

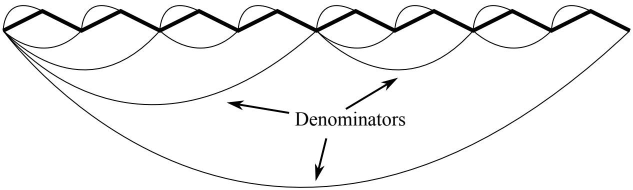
As in the last lecture on the Anderson model, we follow the procedures of quasidegenerate perturbation theory and perform exact rotations O in small, isolated resonant blocks to diagonalize the Hamiltonian there.

Graph-based notion of resonance. Moment bounds control probability.

Use a sequence of length scales $L_k = (15/8)^k$, and continue rotating away interactions of lower order than γ^{L_k}.

$J^{(k)}$ is a sum of connected graphs $J^{(k)}_{\sigma\tilde{\sigma}}(g)$; quasilocality is preserved.

(Each graph g is a walk in spin-configuration space, whose trace in physical space is connected)



A graph of order L_k is resonant if $A^{(k)}_{\sigma\tilde{\sigma}}(g) \equiv \frac{J^{(k)}_{\sigma\tilde{\sigma}}(g)}{E^{(k)}_{\sigma} - E^{(k)}_{\tilde{\sigma}}} > (\gamma/\varepsilon)^{L_k}$.

Fractional moment bounds on graphs and the Markov inequality imply that the probability that g is resonant is $< \varepsilon^{L_k}$; then it is OK to sum over $\exp(O(L_k))$ graphs in the associated percolation problem.

Fractional moment bounds and Markov inequality

Use fractional moment bounds to control the probability of a resonant graph, i.e.

$$\mathbb{E}\left|A^{(k)}_{\sigma\tilde{\sigma}}(g)\right|^s \leq \gamma^{|g|}\prod_i \int \frac{dh_i}{\left|E^{(k)}_\sigma - E^{(k)}_{\tilde{\sigma}}\right|^s} \leq |C\gamma|^{|g|}.$$

Here s is the fractional moment, it must be less than 1 for finiteness of the integral.
Note: $E^{(k)}_\sigma - E^{(k)}_{\tilde{\sigma}}$ is essentially the sum of the h_i's for sites flipped between σ and $\tilde{\sigma}$.

Then the Markov inequality implies that

$$\left|A^{(k)}_{\sigma\tilde{\sigma}}(g)\right|^s \leq |C\gamma/\varepsilon|^{|g|} \text{ with probability } 1-\varepsilon^{|g|}$$

However, this assumes each site is different (forward approximation).

Backtracking

The moment method breaks down for walks that return to previously visited sites in physical space.

As in yesterday's discussion of the Anderson model, there are complications if the graph g involves a significant number of repeated spin flips. With repeated spin flips, energy denominators can appear to a high power, or there can be a large number of relations between them. If this is the case, then the fractional moment will no longer be finite, because of the lack of integrability of $|h|^{-sp}$ for $p \geq 1/s$.

But backtracking sections can be handled with L^∞ bounds as they have a greater decay rate, which arises from the greater degree of connectivity of such graphs.

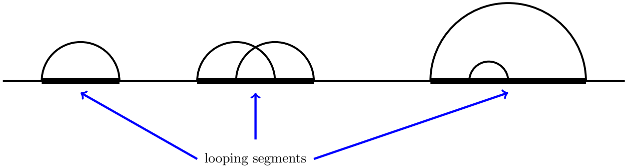
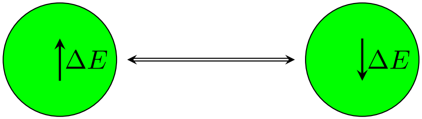


Figure 6: Timeline of the walk. Arches connect pairs of times where the walk is at the same site/block.

Resonances with blocks and the LLA

In later steps, graphs may connect resonant blocks with nearby sites or with other blocks. For graphs connecting different blocks, the fractional-moment bound depends on having some control over the probability that an energy difference in a block is close to that of a given nearby transition. One can obtain the necessary bounds using the LLA level-spacing assumption.

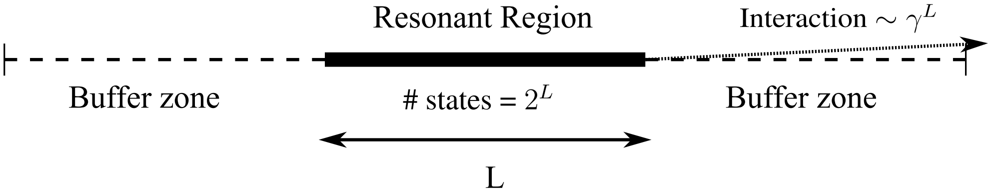


A block-block resonance.

Resonant regions (= Griffiths regions) need buffer zones

These are regions where we have failure of the bounds needed to control the rotations.

Buffer zones are needed so that the smallness $\sim \gamma^L$ of a graph crossing the buffer is much smaller than the typical $\Delta E = 2^{-L}$ in the resonant region.



The buffer zone is expected to be thermalized by the resonant region.

In 1-d the buffer zone has volume comparable to that of the resonant block, so we can diagonalize H in the combined region, eliminating internal interactions while keeping the level-spacing larger than the interactions with spins outside.

Preserving exponential decay

Resonant blocks interrupt or short-circuit the exponential decay along graphs, which leads to a reduction in the overall decay rate ("Rule of Halted Decay"[5]). Following FS83[6] this effect is kept under control by gathering blocks into loosely connected groups separated by large gaps with uninterrupted decay. The loss of decay rate in each step forms a convergent series, ensuring that $|\cdot|^{(j)}$ remains comparable with $|\cdot|$. Thus if the initial decay rate is high enough, exponential decay is preserved uniformly in the RG. Furthermore, resonant blocks are rare and their percolation connectivity function has faster-than-power-law decay.

[5] Thierry, Huveneers, Müller, de Roeck, Many-Body Delocalization as a Quantum Avalanche, 2018
[6] Fröhlich, Spencer: Absence of diffusion in the Anderson tight binding model, CMP 1983

Renormalization group picture

In RG terms, the rotations removing terms in the Hamiltonian up to order γ^L is analogous to "integrating out" short distance degrees of freedom in traditional RG.

At the same time, resonant regions up to some size R are "eliminated" once L is large enough so that the remaining interaction terms are smaller than the level spacing in the region (with its buffer zone, total size $R + 2L$). At that point, the region hosts a "metaspin" which takes 2^{R+2L} values, but the interactions are so small that there is little hybridization with spins elsewhere.

Deep in the localized region, this RG has the property that the density of remaining resonant regions (including their buffer zones with width given by the running RG length L) goes to zero with L.

Note two effects are in play:
(1) Elimination of smaller resonant regions reduces the density.
(2) Fattening of the buffer zones on the remaining regions increases the density.
The proof shows that (1) dominates (2) deep in the weak coupling/strong disorder region, and the density goes to zero as $L \to \infty$.

Conclusion of proof; LIOMs

We have established that the resonant regions are rare and that the rotation generators exhibit exponential decay away from resonant regions, This implies that with high probability, local observables take values close to what they would be in the original tensor product basis.

LIOMs are defined by applying the rotations to the original spin variables. Large rotations are required only on a dilute set of sites; elsewhere the rotations are small with exponential tails.

How small is small?

The MBL theorem states only that an infinite volume MBL phase exists for γ sufficiently small. It does not say how small. However, it should be clear from the constructions that the resonant regions create short-circuits that depress the decay rate. Consequently, the initial decay rate must be quite large, so that the RG flow does not push it below the threshold for avalanches. Recent studies have confirmed this quantitatively.[7,8]

The mutual interaction between the decay rate and the density of resonant regions has implications for the nature of the MBL transition. This will be probed in greater detail in lecture 3.

[7] Morningstar, Colmenarez, Khemani, Luitz, Huse, PRB2022
[8] Sels, PRB2022