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The different faces of Higgs bundles

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Ramanujan Lectures Bangalore February 2026

Oxford Mathematics


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MVLTA ET MIRANDA DE NVMERIS REPPERIT
SINGVLARE INGENIVM AGNOVIT GOTOFRIDVS HARDY
IN ANGLIAM VEXIT STVDIA PROMOVEBAT
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GODFREY HAROLD HARDY
MATHEMATICAE OXONII MOX CANTABRIGIAE PROFESSOR
DISCIPLINAE APVD VTROSQVE EMENDATOR PRAECLARVS
IN STVDIIS PROPRIIS PER ORBEM TERRARVM INTER PRIMOS
OMNIVM CONSENSV EMINENS
VIR OPINIONVM IMPATIENS ALIENARVM
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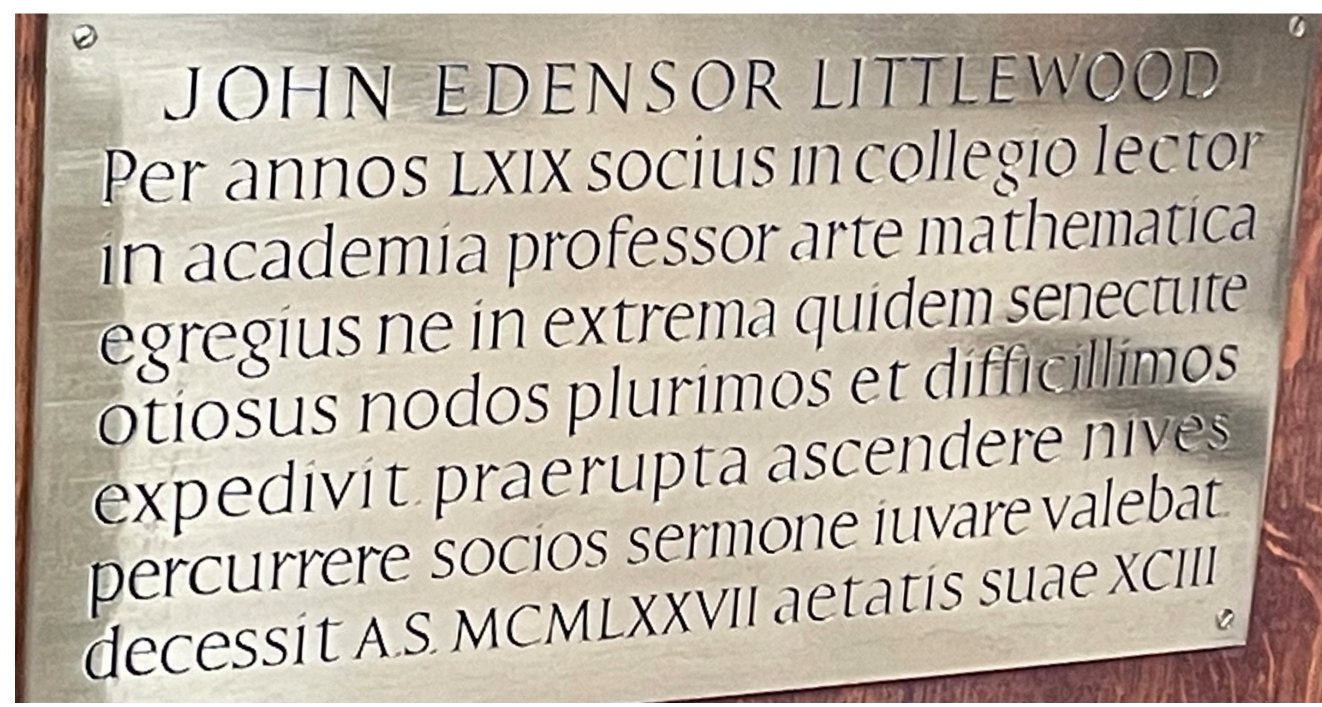
JOHN EDENSOR LITTLEWOOD
Per annos LXIX socius in collegio lector
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THE DIFFERENT FACES OF HIGGS BUNDLES

1. THE INTEGRABLE SYSTEM

2. A UNIVERSAL MODULI SPACE

3. THE ODD INTEGRABLE SYSTEM

THE INTEGRABLE SYSTEM

- complex symplectic manifold M^{2n}

- n independent Poisson-commuting functions f_i

- $F(f_1, \ldots, f_n)$ defines a Hamiltonian vector field

  f_i are constants of the motion

- e.g. geodesics on the ellipsoid

A LITTLE HISTORY

- Stony Brook 1983-84

  NJH, A. Karlhede, U. Lindström & M. Roček *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–589.

- hyperkähler manifold:

  complex structures I, J, K with $I^2 = J^2 = K^2 = IJK = -1$

  Riemannian metric, Kähler forms $\omega_1, \omega_2, \omega_3$

- quotient construction:

  G acting on M hyperkähler, preserving ω_i $i = 1, 2, 3$

  moment map for $\omega_1, \omega_2, \omega_3$, $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$

$$\Rightarrow \mu^{-1}(0)/G \text{ is hyperkähler}$$

- Oxford 1984-85

  M.F.Atiyah & NJH, *Low energy scattering of nonabelian monopoles*, Phys. Lett. A **107** (1985), 21–25.

- M infinite-dimensional hyperkähler manifold $= \{A, \phi\}$

  A connection on a G-bundle over $\mathbf{R}^3$, $\phi : \mathbf{R}^3 \to \mathfrak{g}$ (Higgs field)

  quaternionic affine space: $\phi + A_1 i + A_2 j + A_3 k$

- $\mathcal{G}$ gauge transformations

  moment map: Bogomolny equations $F_A = * d_A \phi$

- $\Rightarrow$ hyperkähler metric on monopole moduli space

- on $\mathbf{R}^4$, $A_0 + A_1 i + A_2 j + A_3 k \Rightarrow$

  self-dual Yang-Mills equations $F_A = *F_A$

- on $\mathbf{R}^3$, $\phi + A_1 i + A_2 j + A_3 k \Rightarrow$

  Bogomolny equations $F_A = *d_A \phi$

- on $\mathbf{R}^2$, $\phi_1 + \phi_2 i + A_1 j + A_2 k \Rightarrow$

  Higgs bundle equations $F_A + [\Phi, \Phi^*] = 0, \bar{\partial}_A \Phi = 0$

M.A.Lohe, *Two- and three-dimensional instantons*, Phys.Lett. **70B** (1977) 325–328

term have acquired special values. One can obtain first order equations which imply the field equations, and these are deduced from eqs. (6):

$$F_{ij}^{a} = \pm e\, \epsilon_{ij} \epsilon^{abc} \psi^{b} \phi^{c} ,$$

$$D_i \psi^{a} \pm \epsilon_{ij} D_j \phi^{a} = 0 . \qquad (11)$$

Unfortunately, in this case the model is not interesting because the action is always zero. We obtain

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Unfortunately, in this case the model is not interesting because the action is always zero. We obtain

- $F_A - [\Phi, \Phi^*] = 0 \sim$ harmonic maps from a surface to $U(n)$

$\Rightarrow$ define equations on a compact Riemann surface C

NJH, *The self-duality equations on a Riemann surface,* Proc. London Math. Soc. **55** (1987), 59–126.

influenced by:

M.F.Atiyah & R.Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. Lond. A, **308** (1983) 523–615.

S.K.Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom. **18** (1983) 269–277

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- C^∞ hermitian vector bundle E, complex structure $\bar\partial_A \in \mathcal{A}$,

  $\Phi \in \Omega^{1,0}(\mathrm{End}_0\, E)$ with $\bar\partial_A \Phi = 0$

- consider a minimizing sequence for $\|F_A + [\Phi, \Phi^*]\|^2$

  on a complex gauge orbit

- **Aim:** Find an L^2 bound on F_A

  and apply Uhlenbeck weak compactness

- $\bar{\partial}_A \Phi = 0 \Rightarrow$ Weitzenböck formula

  $\Rightarrow (F_A, [\Phi, \Phi^*]) + c\|\Phi\|^2 \geq 0$

- $n \times n$ matrix B, bound on spectrum + bound on $\|[B, B^*]\|$

  $\Rightarrow$ bound on $\|B\|$

- spectrum of Φ invariant by gauge transformations

  $\Rightarrow$ bound on $\|F_A + [\Phi, \Phi^*]\|$ + Weitzenböck + spectrum

  $\Rightarrow$ bound on F_A and Φ.

- moduli space $\mathcal{M}$

  $\det(x - \Phi) = x^n + a_1 x^{n-1} \cdots + a_n, \ a_m \in H^0(C, K^m)$

  $p(A, \Phi) = (a_1, \ldots, a_n)$

  $p : \mathcal{M} \to H^0(C, K) \oplus \cdots \oplus H^0(C, K^n) = \mathbf{C}^{\dim \mathcal{M}/2}$

  proper map

- generic fibre = Jac(S)

  S = spectral curve $x^n + a_1 x^{n-1} \cdots + a_n = 0$

- completely integrable Hamiltonian system

Christieville, St Sauveur, Québec

AN EXAMPLE

- C genus 2, Higgs bundle (E, Φ)

  E rank 2, $\Lambda^2 E$ fixed, odd degree, $\operatorname{tr} \Phi = 0$

  $C : y^2 = (z - \mu_1) \dots (z - \mu_6)$

- moduli space $\mathcal{N}$ of stable bundles E

  $\mathcal{N}$ = intersection of quadrics $Q \cap Q_\mu$ (P.Newstead 1968)

  $q = x_1^2 + \cdots + x_6^2 = 0, \quad q_\mu = \mu_1 x_1^2 + \cdots + \mu_6 x_6^2 = 0$

- cotangent bundle $T^*\mathcal{N} \subset \mathcal{M}$ open dense

- T^*Q restricted to $Q_\mu \cap Q$, symplectic form degenerate

  $T^*(Q_\mu \cap Q) =$ quotient by degeneracy foliation

- $x_1 y_1 + \cdots + x_6 y_6 = 0, \quad x_1^2 + \cdots + x_6^2 = 0$

  $x \wedge y \in \mathfrak{so}(6) \cong \mathfrak{so}(6)^*$ coadjoint orbit $\subset T^*Q$

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- $$f_i = \sum_{j \neq i} \frac{(x_iy_j - x_jy_i)^2}{\mu_j - \mu_i} = \sum_{j \neq i} \frac{X_{ij}^2}{\mu_j - \mu_i} \qquad X_{ij} \in \mathfrak{so}(6)$$

  f_i constant along the foliation

A.Beauville, A.Höring, J.Liu & C.Voisin, *Symmetric tensors on the intersection of two quadrics and Lagrangian fibration*, Moduli (2024);1:e4.

- genus 2 curve $C \to \mathsf{P}^1$

- Higgs field descends:

$$\Phi(e_1) = -\sum_{i=1}^{N} \frac{x_i y_i}{z-\mu_i} e_1 - \sum_{i=1}^{N} \frac{y_i^2}{z-\mu_i} e_2$$

$$\Phi(e_2) = \sum_{i=1}^{N} \frac{x_i^2}{z-\mu_i} e_1 + \sum_{i=1}^{N} \frac{x_i y_i}{z-\mu_i} e_2$$

(Garnier system)

- $\det(x - \Phi) = 0$: hyperelliptic curve S of genus 3

  generic fibre of $p : \mathcal{M} \to \mathbf{C}^3$ = finite covering of $\mathrm{Jac}(S)$

- What has this to do with the hyperkähler metric?

THE ASYMPTOTIC METRIC

- $\Phi \mapsto \lambda\Phi$, $\lambda \in \mathbf{C}^*$, action on moduli space $\mathcal{M}$

  Φ not nilpotent $\Rightarrow$ $(E, \lambda\Phi)$ unbounded as $\lambda \to \infty$

- $\det(x - \lambda\Phi) = x^n + \lambda a_1 x^{n-1} \cdots + \lambda^n a_n$

  $x = \lambda y$ isomorphic spectral curves: same Jacobian

  $\Rightarrow$ integrable system visible at infinity in $\mathcal{M}$

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R.Mazzeo, J.Swoboda, H.Weiss & F.Witt, *Ends of the moduli space of Higgs bundles*, Duke Math. J. **165** (2016), 2227–2271.

L.Fredrickson, *Exponential decay for the asymptotic geometry of the Hitchin metric*, Comm. Math. Phys. **375** (2020), 1393–1426.

. . ..

T.Mochizuki, *Semi-flat metrics of the moduli spaces of Higgs bundles in the non-zero degree case*, arXiv:2501.12741

SEMI-FLAT METRICS

- $$\omega_3 + i\omega_1 = -\frac{1}{2}\sum \omega_{ij} d(x_j + i\xi_j) \wedge d(x_k + i\xi_k)$$

$$\omega_2 = \sum \frac{\partial^2 f}{\partial x_j \partial x_k} dx_j \wedge d\xi_k$$

(ω_{ij} constant symplectic form)

- nonlinear equation for f:

$$0 = \sum \omega^{ij} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial^2 f}{\partial x_j \partial x_\ell} dx_k \wedge dx_\ell$$

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- $p : \mathcal{M}^{2N} \to \mathbf{C}^N$ $\longleftarrow$ space of spectral curves

  hypersurface $D \subset \mathbf{C}^N$ of singular curves

- torus fibration over $\mathbf{C}^N \backslash D$

  flat symplectic connection (Gauss-Manin) on cohomology

  $H^1(\mathcal{M}_s, \mathbf{R})$, $s \in \mathbf{C}^N \backslash D$

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- spectral curve $S \subset T^*C \quad x^n + a_1 x^{n-1} \cdots + a_n = 0$

  $\pi : T^*C \to C$, $x \in H^0(T^*C, \pi^* K)$ tautological section

- $\pi^* K \subset T^*(T^*C)$, $x \sim$ canonical 1-form θ

- $$f = -\frac{i}{4} \int_S \theta \wedge \bar{\theta}$$

D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, Adv. Theor. Math. Phys. **23** (2019), 1981–2024.

- $f = -\dfrac{i}{4}\displaystyle\int_S \theta \wedge \bar{\theta}$ well-defined and C^1 for all spectral curves

- nodal spectral curves S: subintegrable system

NJH, *Integrable systems and Special Kähler metrics*, EMS Surveys in Mathematical Sciences, **8** (2021) 163–178.

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NJH, *Integrable systems and Special Kähler metrics*, EMS Surveys in Mathematical Sciences, **8** (2021) 163–178.

- flat connection logarithmic on smooth part of $D \subset \mathbf{C}^N$

Z.Huang, S.Huang & B.Xu, *Local models for special Kähler metric singularities along the discriminant locus of the $SL(2,\mathbf{C})$ Hitchin base*, **arXiv** 2601.0376

THE $P = W$ CONJECTURE

- hyperkähler metric: complex structures I, J, K

  I: Higgs bundles (E, Φ), J: character variety

  $= \mathrm{Hom}(\pi_1(C), GL(n, \mathbf{C}))$ modulo conjugation

- $\mathbf{C}^*$ action $x \in S^2 \backslash \{\pm 1, 0, 0\}$,

  $x_1 I + x_2 J + x_3 K \cong$ character variety

- different as algebraic varieties:

  $I = \mathcal{M}_{Dol}$ has compact subvarieties, $J = \mathcal{M}_B$ is affine

$\mathbf{W}$ = weight filtration on $\mathcal{M}_B$

- $0 \subset W_0 \subset \cdots \subset W_{2i} = H^i(\mathcal{M}, \mathbf{Q})$

mixed Hodge structure: quotients W_j/W_{j-1} have the usual (p,q) decomposition for a compact Kähler manifold.

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mixed Hodge structure: quotients W_j/W_{j-1} have the usual (p, q) decomposition for a compact Kähler manifold.

$\mathbf{P}$ = perverse Leray filtration on $\mathcal{M}_{Dol}$
for $p : \mathcal{M}^{2N} \to \mathbf{C}^N$

- $0 \subset P_0 \subset \cdots \subset P_{2i} = H^i(\mathcal{M}, \mathbf{Q})$

Conjecture: These coincide (de Cataldo, Hausel, Migliorini 2012) perverse filtration for $p : \mathcal{M}^{2N} \to \mathbf{C}^N$ more concretely:

- $P^i H^j(\mathcal{M}, \mathbf{Q}) = \mathsf{ker}(H^i(X, \mathbf{Q}) \to H^i(p^{-1}(A^{j-i-1}), \mathbf{Q}))$

A^{j-i-1} is a general affine subspace of $\mathbf{C}^N$ of dimension $j-i-1$

$=$ a shadow of the integrable system in the algebraic

geometry of the character variety

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V. Hoskins, *Two proofs of the $P = W$ conjecture [after Maulik-Shen and Hausel-Mellit-Minet-Schiffmann]*, Séminaire Bourbaki, **1213** (2023–2024).

DEFORMATIONS OF THE METRIC

- hyperkähler metric:

  complex structures I, J, K, symplectic forms $\omega_1, \omega_2, \omega_3$

- first order deformation $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$

  $\dot{\omega}_i$ closed, type $(1,1)$ with respect to all I, J, K

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- $T^* \otimes \mathbf{C} \cong E \otimes H$, H = quaternions $\mathbf{C}^2$

  E = complex symplectic vector bundle, $S^2 E \otimes \Lambda^2 H \subset \Lambda^2 T^*$

- $S^2 E \otimes S^2 H \subset S^2 T^*$, $S^2 H = \mathfrak{su}(2) \otimes \mathbf{C}$

  variation of metric: $\dot{g} = \dot{\omega}_1 \otimes i + \dot{\omega}_2 \otimes j + \dot{\omega}_3 \otimes k$

- moduli space of Higgs bundles $\mathcal{M}$ on C

vary complex structure on $C \Rightarrow$ deformation of hyperkähler metric

- $\nabla_A + \Phi + \Phi^*$ flat connection

  $\Rightarrow \mathcal{M}_B =$ complex structure $J =$ character variety

  <u>only depends on $\pi_1(C)$</u>

- Goldman symplectic form $\omega_3 + i\omega_1$ fixed

$$\Rightarrow \dot{\omega}_3 = 0 = \dot{\omega}_1$$

- isometric circle action $\Phi \mapsto e^{i\theta}\Phi$

- holomorphic on $(\mathcal{M}, I)$, preserves Kähler form ω_1

  vector field X, moment map $f = -\|\Phi\|^2/2$

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- holomorphic on $(\mathcal{M}, I)$, preserves Kähler form ω_1

  vector field X, moment map $f = -\|\Phi\|^2/2$

- on $(\mathcal{M}, J)$ $\omega_2 =$ Kähler form $\omega_2 = -dJdf$

  $\Rightarrow$ $f : (\mathcal{M}, J) \to \mathbf{R}$ determines the metric.

and $\underline{\dot{\omega}_2 = -dJd\dot{f}}$

- flat connection $\nabla_A + \phi \qquad f = -\frac{1}{2}\int_C \mathrm{tr}\, \phi \wedge *\phi$

- fix gauge equivalence class of the flat connection

  ... and vary the complex structure on C

- infinitesimal complex gauge transformation:

  $\nabla_A(\psi_1 + \psi_2) + [\phi, \psi_1 + \psi_2]$

  ψ_1 is skew-Hermitian and ψ_2 Hermitian

- $$\dot{f} = -\frac{1}{2}\int_C \mathrm{tr}(\dot{\phi} \wedge *\phi + \phi \wedge \dot{*}\phi + \phi \wedge *\dot{\phi}).$$

$$\dot{\phi} = \nabla_A \psi_2 + [\phi, \psi_1]$$

- Stokes + $d_A * \phi = 0$ and $d_A \phi = 0 \Rightarrow$

$$\dot{f} = -\frac{1}{2}\int_C \mathrm{tr}(\phi \wedge \dot{*}\phi).$$

- $\dot f = -\frac{1}{2}\int_C \mathrm{tr}(\dot\phi \wedge *\phi + \phi\wedge \dot{*}\phi + \phi\wedge *\dot\phi).$

$$\dot\phi = \nabla_A\psi_2 + [\phi,\psi_1]$$

- Stokes $+\ d_A * \phi = 0$ and $d_A\phi = 0 \Rightarrow$

$$\dot f = -\frac{1}{2}\int_C \mathrm{tr}(\phi\wedge \dot{*}\phi).$$

- $*^2 = -1$ so $\dot{*}* + *\dot{*} = 0$ and $\dot{*} = \mu \in \Omega^{0,1}(K^*)$

$$\dot f = \text{real part of } \int_C \mu\, \mathrm{tr}(\phi^{1,0})^2$$

- $\mathrm{tr}\,\Phi^2 \in H^0(C,K^2)$, $[\mu]\in H^1(C,K^*)$ Kodaira-Spencer class

- $\dot{f} + i\dot{g} = \int_C \mu \operatorname{tr} \Phi^2$

  $=$ a Hamiltonian function of the integrable system

  $\Rightarrow d(\dot{f} + i\dot{g}) = i_Z(\omega_2 + i\omega_3)$

  Z holomorphic Hamiltonian vector field

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  Z holomorphic Hamiltonian vector field

- $i_Z\omega_2 = i(i_Z\omega_3) \Rightarrow d(\dot{f} + i\dot{g}) = 2i(i_Z\omega_3)$

  $\Rightarrow Jd(\dot{f} + i\dot{g}) = 2iJ(i_Z\omega_3) = 2i(i_Z\omega_1) \in \Omega_I^{0,1}$

  $\Rightarrow \dot{\omega}_2 = d(Jd\dot{f}) = 2\, \mathrm{Re}\, id(i_Z\omega_1)$

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  $\Rightarrow \dot{\omega}_2 = d(Jd\dot{f}) = 2\,\mathrm{Re}\, id(i_Z\omega_1)$

- $[i_Z\omega_1] \in H^1(\mathcal{M}, \mathcal{O})$

  Kodaira-Spencer class $H^1(\mathcal{M}, \mathcal{O}) \stackrel{\partial}{\to} H^1(\mathcal{M}, T^*) \stackrel{\sigma}{\to} H^1(\mathcal{M}, T)$

  $\sigma = (\omega_2 + i\omega_3)^{-1}$ Poisson tensor

Remark:

- ρ closed type $(1,1)$ wrt I, J, K

  holomorphic Poisson tensor $\sigma : T^* \to T$

- $\rho \in C^\infty(T^* \otimes \bar{T}^*), \sigma(\rho) \in C^\infty(T \otimes \bar{T}^*)$

  $\sim$ Kodaira-Spencer class in $H^1(\mathcal{M}, T)$

- $[\sigma(\rho), \sigma(\rho)] \in H^2(\mathcal{M}, T)$ obstruction

  $[\sigma(\rho), \sigma(\rho)] = \sigma\partial\langle\sigma, \rho \wedge \rho\rangle = 0$

  since $\langle\sigma, \rho \wedge \rho\rangle = 0$ $(\sigma = (\omega_2 + i\omega_3)^{-1})$

MIRROR SYMMETRY

- SYZ mirror symmetry:

  Calabi-Yau manifold M + special Lagrangian torus fibration

  mirror $M^{\vee}$ = fibration by dual tori

- Higgs bundle moduli space $\mathcal{M}$ for simple group G

  I - holomorphic Lagrangians = special

  $\mathcal{M}^{\vee}$ = moduli space for Langlands dual group

- for smooth fibres:

  R.Donagi & T.Pantev, *Langlands duality for Hitchin systems*, Invent. Math. **189** (2012), 653–735.

- **BAA**-brane

  $=$ I-holomorphic Lagrangian submanifold $+$ flat line bundle

  BBB-brane

  $=$ hyperholomorphic bundle on a hyperkähler submanifold

  $=$ connection with curvature of type $(1,1)$ wrt I, J, K

- mirror symmetry is supposed to interchange these two

Issues:

- hyperkähler submanifolds:

  . subgroups $H \subset G$ (but can be singular locus)

  . fixed points of automorphisms of C

- hyperholomorphic bundles:

  . hypercohomology $\mathbf{H}^1$ of $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

  for some representation of G

  (e.g. adjoint representation = tangent bundle)

  . universal bundle over $p \in C$

- simplest example:

  one fibre of $p : \mathcal{M} \to \mathbf{C}^N$ + line bundle

  $\sim$ point in $\mathcal{M}^\vee$ = hyperkähler submanifold

- Lagrangian = character variety for a real form

  split real form = section $\mathcal{M}_0 \subset \mathcal{M}$ of $p : \mathcal{M} \to \mathbf{C}^N$

  $\Rightarrow$ mirror = trivial line bundle over $\mathcal{M}^\vee$

- real form $SU(n,n)$

  $\Rightarrow$ mirror supported on $Sp(n,\mathbf{C})$ moduli space

  hyperholomorphic bundle

  $=$ exterior powers of $\mathbf{H}^1$ for the representation on $\mathbf{C}^{2n}$

  NJH, *Higgs bundles and characteristic classes*, in "Arbeitstagung Bonn 2013" Birkhäuser Progress in Mathematics **319** (2016) 373–382.

Justification:

- data on smooth fibres corresponds,

  hyperholomorphic bundle well-defined everywhere

- Lagrangian = upward flow from a fixed point of $\mathbf{C}^*$-action

- Hecke transforms of section $\mathcal{M}_0 \Rightarrow$ hyperholomorphic bundle

  = tensor products of exterior powers of universal bundle

NJH & T.Hausel, *Very stable Higgs bundles, equivariant multiplicity and mirror symmetry*, Invent. math. **228** (2022) 893–989.

D.Fang, *On Fourier-Mukai transforms of upward flows for Hitchin systems*, arXiv:2504.04309

ANOTHER EXAMPLE

- deformation $\dot{\omega}_2 = dJd\dot{f}$

  $=$ exact 2-form of type $(1,1)$ wrt I, J, K

  $\Rightarrow$ hyperholomorphic line bundle L, $c_1(L) = 0$

$$\boxed{\text{What Lagrangian is its mirror?}}$$

- $(Jd\dot{f})^{0,1} = i(i_Z\omega_1)$

  Z tangential to fibres, ω_1 Kähler form

- $\bar{\partial} + i_Z\omega_1$ holomorphic structure on trivial line bundle

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- integrate the vector field Z to time $t = 1$

  $\Rightarrow$ holomorphic Hamiltonian diffeomorphism $h : \mathcal{M}^\vee \to \mathcal{M}^\vee$

- Lagrangian submanifold $= h(\mathcal{M}_0^\vee)$

  $=$ translate of the Hitchin section