Fusion Products and Chari-Venkatesh Modules for Current Lie Algebras of type A_2

(Jt work with Shushma Rani)

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The Setup:

The Setup

- $\mathfrak{g}$:= complex finite-dimensional simple Lie algebra ;
  $\mathfrak{h}$:= Cartan subalgebra of $\mathfrak{g}$.
- $\{\alpha_i : 1 \leq i \leq n\}$:= simple roots of $\mathfrak{g}$,
  R^+:= set of positive roots, R:= set of roots of $\mathfrak{g}$,
- $\{e_\alpha, f_\alpha, h_i : \alpha \in R^+,\ 1 \leq i \leq n\}$:= Chevalley basis of $\mathfrak{g}$,
  For $\alpha \in R^+$, let $h_\alpha = [e_\alpha, f_\alpha]$.
- $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathbb{C}e_\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha \in R^+} \mathbb{C}f_\alpha$
- $\{\omega_1, \cdots, \omega_n\}$:= fundamental weights of $\mathfrak{g}$,
  $P^+ = \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z}_+, \forall\ \alpha \in R^+\}$ dominant integral weights of $\mathfrak{g}$
- P^+ paramatrizes the set of finite-dimensional irreducible $\mathfrak{g}$-modules.
  For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible $\mathfrak{g}$-module with highest weight λ.

Current Lie algebra

- The current Lie algebra associated to $\mathfrak{g}$ is the Lie algebra with underlying vector space
$$\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$$
and Lie bracket operation :
$$[x \otimes P, y \otimes Q] = [x, y]_{\mathfrak{g}} \otimes PQ, \qquad \forall\, x, y \in \mathfrak{g},\ P, Q \in \mathbb{C}[t].$$
- For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, let
  - $\mathfrak{a}[t] := \mathfrak{a} \otimes \mathbb{C}[t]$.
  - $U(\mathfrak{a}[t])$:= be the universal enveloping algebra of $\mathfrak{a}[t]$.
- The $\mathbb{Z}$-grading in $\mathbb{C}[t]$ induces a grading on $\mathfrak{a}[t]$ and $U(\mathfrak{a}[t])$.

  For $s \in \mathbb{N}$, let
$$U(\mathfrak{a}[t])[s] := \{(x_{i_1} \otimes t^{r_1}) \cdots (x_{i_k} \otimes t^{r_k}) : \sum_{j=1}^{k} r_j = s,\ x_{i_j} \in \mathfrak{a}\,\forall\, j\}$$

Representations of Current Lie algebra

- Given $\lambda \in P^+$, the **local Weyl module**, $W_{loc}(\lambda)$ is defined as the cyclic, $\mathfrak{g}[t]$-module generated by vector w_λ which satisfies following relations:

$$(e_\alpha \otimes \mathbb{C}[t])w_\lambda = 0, \qquad (f_\alpha \otimes 1)^{\lambda(h_\alpha)+1} w_\lambda = 0, \quad \forall\, \alpha \in R^+$$
$$(h \otimes t^r)w_\lambda = \lambda(h)\delta_{0,r} w_\lambda, \qquad \forall h \in \mathfrak{h},\ r \in \mathbb{Z}_{\geq 0}$$

  Any cyclic finite-dimensional highest weight $\mathfrak{g}[t]$-module with the highest weight λ is a quotient of $W_{loc}(\lambda)$.

- For $a \in \mathbb{C}$, using the map

$$\phi_a : \quad \mathfrak{g} \otimes \mathbb{C}[t] \to \mathfrak{g}$$
$$x \otimes t \mapsto ax,$$

  a $\mathfrak{g}$-module V can be considered as a $\mathfrak{g}[t]$-module. Such a module is called an **evaluation module** and is denoted by $ev_a(V)$.
  Clearly $ev_a(V(\lambda))$ is a graded $\mathfrak{g}[t]$-module when $a = 0$.

Fusion Product Modules

Fusion Product of Evaluation Modules

For $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k) \in (P^+)^k$ and $\mathrm{a} = (a_1, \cdots, a_k)$ a k-tuple of pairwise distinct complex numbers, set

$$V(\boldsymbol{\lambda}, \mathrm{a}) := ev_{a_1} V(\lambda_1) \otimes \cdots \otimes ev_{a_k}(V(\lambda_k))$$

.

- The $\mathbb{N}$-grading in $\mathrm{U}(\mathfrak{g}[t])$ induces a $\mathfrak{g}$-equivariant grading on $\mathrm{V}(\boldsymbol{\lambda}, \mathrm{a})$ which is given by
$$\mathrm{V}(\boldsymbol{\lambda}, \mathrm{a})[r] = \bigoplus_{0 \leq s \leq r} \mathrm{U}(\mathfrak{g}[t])[s].v_1 \otimes \cdots \otimes v_k,$$
where v_i is the generator of $V(\lambda_i)$ for $1 \leq i \leq k$.

- The **fusion product** of the modules $\{ev_{a_i}(V(\lambda_i)) : 1 \leq i \leq k\}$ at $\mathrm{a} = (a_1, \cdots, a_k)$ is the associated graded $\mathfrak{g}[t]$-module
$$\bigoplus_{r \in \mathbb{N}} \mathrm{V}(\boldsymbol{\lambda}, \mathrm{a})[r] / \mathrm{V}(\boldsymbol{\lambda}, \mathrm{a})[r-1].$$
It is denoted by $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$.

Conjecture [Feigin, Loktev, 1999]

Let $\mathfrak{g}$ *be a simple Lie algebra and* $V_1, V_2, \ldots, V_k$ *be cyclic* $\mathfrak{g}[t]$*-modules. Then for arbitrary* k*-tuples of distinct complex numbers,* $(z_1, \ldots, z_k)$, $(a_1, \ldots, a_k) \in \mathbb{C}^k$,

$$V_1^{z_1} * \ldots * V_k^{z_k} \cong V_1^{a_1} * \ldots * V_k^{a_k}$$

as $\mathfrak{g}[t]$*-modules.*

[0] Feigin, B. and Loktev, S., On generalized Kostka polynomials and the quantum Verlinde rule; Differential topology, infinite-dimensional Lie algebras and applications, *Amer. Math. Soc. Transl. Ser. 2*, 194, 61–79, 1999

History

This conjecture has been proved in several cases:

- For $\mathfrak{g} = \mathfrak{sl}_2$, independepent proofs of the conjecture have been given by Feigin and Loktev in 1999, Chari and Venkatesh, 2013.
- When the V_is are fundamental representations of $\mathfrak{g}$,
  - Chari and Loktev proved the conjecture for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ in 2006,
  - Fourier and Littelmann proved it for $\mathfrak{g}$ simply-laced, in 2007,
- In 2015, for $\mathfrak{g} = \mathfrak{sl}_{n+1}$, Naoi proved the conjecture in the case when highest weight of the fusion product module is a multiple of a fundamental weight.
- For fusion product of two graded irreducible $\mathfrak{g}[t]$-modules, the conjecture was proved by
  Fourier in 2015, for some special cases when $\mathfrak{g} = \mathfrak{sl}_n$
  Kus and Barth in 2020, in the case when $\mathfrak{g}$ is of rank 2.

Properties of $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$

- $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$ is a finite dimensional, highest weight $\mathfrak{g}[t]$-module with highest weight $\sum_{i=1}^{k} \lambda_i$.

  Hence $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$ is a quotient of $W_{loc}(\sum_{i=1}^{k} \lambda_i)$.

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- $\dim V(\lambda_1)^{a_1} * \ldots * V(\lambda_k)^{a_k} = \prod_{i=1}^{k} \dim V(\lambda_i)$.

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- Given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k)$ and $\mathrm{a} = (a_1, \cdots, a_k)$ -k-tuple of distinct complex numbers, if $v^*_{\lambda,a}$ is the image of the generator of $\bigotimes\limits_{i=1}^{k} ev_{a_i}(V(\lambda_i))$ in $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$, then

$$\mathrm{x} \otimes \mathrm{t}^{\mathbf{s}}.\mathrm{v}^*_{\lambda,\mathbf{a}} = \mathrm{x} \otimes (\mathrm{t} - \mathrm{a}_1) \cdots (\mathrm{t} - \mathrm{a}_{\mathbf{s}}).\mathrm{v}^*_{\lambda,\mathbf{a}}, \quad \forall\, \mathrm{x} \in \mathfrak{g}, \text{ and } \mathrm{a}_1, \cdots, \mathrm{a}_{\mathbf{s}} \in \mathbb{C}.$$

Given a dominant integral weight $\lambda \in P^+$, a $|R^+|$-tuple of partitions $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ is said to be λ-compatible if

$$\xi^\alpha = (\xi_1^\alpha, \cdots, \xi_r^\alpha), \qquad \sum_{i=1}^{r} \xi_i^\alpha = \lambda(h_\alpha)$$

Definition

Given $\lambda \in P^+$ and a $|R^+|$-tuple of λ-compatible partitions, the Chari-Venkatesh module $V(\boldsymbol{\xi})$ is defined to be the graded quotient of $W_{loc}(\lambda)$ by the submodule generated by the graded elements

$$(x_\alpha^+ \otimes t)^s (x_\alpha^- \otimes 1)^{r+s} w_\lambda,$$

with $s, r \in \mathbb{N}$ such that $s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha$, for some $k \in \mathbb{N}$, $\alpha \in R^+$.

Chari-Venkatesh Modules...

Given a fusion product module $V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$, for each $\alpha \in R^+$ let

$$\xi^\alpha := \text{ majorization of } (\lambda_1(h_\alpha), \cdots, \lambda_k(h_\alpha)).$$

Clearly, this gives a $|R^+|$-tuple of partitions, namely $\xi(\boldsymbol{\lambda})$, which is λ-compatible.

Let $V(\xi(\boldsymbol{\lambda}))$ denote the corresponding Chari-Venkatesh modules.

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Lemma : Given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k) \in (P^+)^k$, there exists a $\mathfrak{g}[t]$-module surjective map

$$V(\xi(\boldsymbol{\lambda})) \twoheadrightarrow V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k};$$

for all k-tuple of distinct complex numbers $(a_1, \cdots, a_k)$.

Consequently, $\dim V(\xi(\boldsymbol{\lambda})) \geq \prod_{i=1}^k \dim V(\lambda_i)$.

Fusion Product and Chari-Venkatesh Modules for $\mathfrak{sl}_2[t]$

Chari and Venkatesh proved in [CV-15][1],

- When $\mathfrak{g} = \mathfrak{sl}_2$,

$$V(\xi(\boldsymbol{\lambda})) \cong_{\mathfrak{sl}_2[t]} V(\lambda_1)^{a_1} * \cdots * V(\lambda_k)^{a_k}$$

- For the proof, they determined a basis of $V(\xi(\boldsymbol{\lambda}))$ using a series of short exact sequences of CV-modules and showed that

$$\dim V(\xi(\boldsymbol{\lambda})) \leq \prod_{i=1}^{k} \dim V(\lambda_i) = \dim V(\lambda_1)^{a_1} * \ldots * V(\lambda_k)^{a_k}.$$

- Using dimension arguments and the fact that $V(\lambda_1)^{a_1} * \ldots * V(\lambda_k)^{a_k}$ is a quotient of $V(\xi(\boldsymbol{\lambda}))$ it followed that $V(\xi(\boldsymbol{\lambda})) \cong_{\mathfrak{sl}_2[t]} V(\lambda_1)^{a_1} * \ldots * V(\lambda_k)^{a_k}$.

[3] Vyjayanthi Chari, R. Venkatesh. *Demazure modules, fusion Products,and Q-systems.* Commun.Math.Phys.333,(2015),no.2, 566–593.

Our Results

Chari-Venkatesh Modules for $\mathfrak{sl}_3[t]$

Let $\mathfrak{g} = \mathfrak{sl}_3$, $\theta :=$ highest root of $\mathfrak{g}$.

Given $\lambda, \mu \in P^+$, the Chari-Venkatesh module $V(\xi(\lambda,\mu))$ is defined as the cyclic module generated by a non-vector vector $v_{(\lambda,\mu)}$ which satisfies the following relations:

$$\mathfrak{n}^+[t].v_{(\lambda,\mu)} = 0, \qquad x \otimes t^k.v_{(\lambda,\mu)} = 0, \forall\ k \geq 2,$$

$$(f_\alpha \otimes 1)^{\lambda+\mu(h_\alpha)+1} v_{(\lambda,\mu)} = 0, \quad (f_\alpha \otimes t)^{\min\{\lambda(h_\alpha),\mu(h_\alpha)\}+1} v_{(\lambda,\mu)} = 0, \qquad \forall\, \alpha \in R^+$$

$$(h \otimes t^r) v_{(\lambda,\mu)} = (\lambda+\mu)(h)\delta_{0,r} v_{(\lambda,\mu)}, \qquad \forall h \in \mathfrak{h},\ r \in \mathbb{Z}_{\geq 0}.$$

Theorem (K, S. Rani)

Let λ, μ be dominant integral weights of $\mathfrak{sl}_3$. For any distinct pair of complex numbers (a,b), $V^a(\lambda) * V^b(\mu)$ is isomorphic to $V(\xi(\lambda,\mu))$ as a $\mathfrak{g}[t]$-module.

The set $\mathcal{P}(\Lambda, 2)$

- Given a dominant integral weight Λ of $\mathfrak{sl}_3$, let

$$\begin{aligned} P(\Lambda, 2) &= \{(\lambda, \mu) \in P^+ \times P^+ : \lambda + \mu = \Lambda\}; \\ \mathcal{P}(\Lambda, 2) &= \{\xi(\lambda, \mu) : (\lambda, \mu) \in P(\Lambda, 2)\} \end{aligned}$$

- Given $(\lambda, \mu), (\lambda', \mu') \in P(\Lambda, 2)$, we say $\xi(\lambda, \mu) \succeq \xi(\lambda', \mu')$ if for each $\alpha \in R^+$,

$$\xi^\alpha(\lambda, \mu) \trianglelefteq \xi^\alpha(\lambda', \mu')$$

with respect to the dominance ordering $\trianglelefteq$ on partitions of $\Lambda(h_\alpha)$.

- **Lemma**: Let $\Lambda \in P^+$. Given $(\lambda, \mu), (\lambda', \mu') \in P(\Lambda, 2)$ there exists a $\mathfrak{g}[t]$-module surjective map

$$V(\xi(\lambda, \mu)) \twoheadrightarrow V(\xi(\lambda', \mu')),$$

whenever $\xi(\lambda, \mu) \succeq \xi(\lambda', \mu')$.

Steps towards the proof of the Theorem:

Step 1:

Proposition [S Rani, -K]

Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, $\mu = \mu_1\omega_1 + \mu_2\omega_2 \in P^+$ with $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ and $\lambda_1 \geq \mu_1$. Set

$$\xi(\lambda,\mu)^+ = \begin{cases} \xi(\lambda+\omega_1, \mu-\omega_1) & \text{if } \mu_2 = 0, \text{ and } \mu_1 > 0, \\ \xi(\lambda+\omega_2, \mu-\omega_2) & \text{if } \lambda_2 \geq \mu_2 > 0, \\ \xi(\lambda+(\mu_2-\lambda_2)\omega_2, \mu_1\omega_1+\lambda_2\omega_2) & \text{if } \lambda_2 < \mu_2 \end{cases}$$

Then there exists a short exact sequence

$$0 \to \ker(\lambda,\mu) \to V(\xi(\lambda,\mu)) \xrightarrow{\phi(\lambda,\mu)} V(\xi(\lambda,\mu)^+) \to 0$$

where the kernel, $\ker(\lambda,\mu)$ admits a filtration whose successive quotients are the direct sum of finitely many CV- modules.

Proof of main theorem continued...

In particular, setting τ_r is the grade shift operator for $r \in \mathbb{Z}$, we show:

i. When $\mu_2 = 0$ and $\mu_1 > 0$, $\ker \phi(\lambda, \mu)$ admits a filtration by modules that are quotients of :

$$\tau^*_{\mu_1} V(\lambda + w_0\mu + j\alpha_2), \qquad \max\{0, \mu_1 - \lambda_2\} \leq j \leq \mu_1.$$

Proof of main theorem continued...

In particular, setting τ_r is the grade shift operator for $r \in \mathbb{Z}$, we show:

i. When $\mu_2 = 0$ and $\mu_1 > 0$, $\ker \phi(\lambda, \mu)$ admits a filtration by modules that are quotients of :

$$\tau^*_{\mu_1} V(\lambda + w_0\mu + j\alpha_2), \qquad \max\{0, \mu_1 - \lambda_2\} \leq j \leq \mu_1.$$

ii. When $\mu_2 > 0$ and $\mu_1 > 0$, $\ker \phi(\lambda, \mu)$ admits a filtration by CV modules that are quotients of :

$$\bigoplus_{a \in S_j^{\lambda,\mu}} \tau^*_{|\mu|}(V(\lambda + w_0\mu + a\alpha_2 + (j-a)\alpha_1), \qquad 0 \leq j < \mu_1 + \mu_2,$$

$$\tau^*_{|\mu|} \mathcal{F}_{\lambda + \mu_2(\omega_1 - \omega_2), \mu_1\omega_1},$$

with $S_j^{\lambda,\mu} = \{a \in \mathbb{Z} : \begin{array}{c} 0 \leq a \leq \mu_1,\ 0 \leq j - a < \mu_2, \\ \mu_2 - \lambda_1 \leq j - 2a \leq \lambda_2 - \mu_1 \end{array} \}$

Continued..

iii. When $\mu_2 > \lambda_2$, $\ker \phi(\lambda, \mu)$ admits a filtration by modules that are quotients of :

$$\bigoplus_{a \in S_{\lambda,\mu}^{(\ell,k)}} \tau^*_{\mu_1+\lambda_2+\ell} V(\lambda + w_0\mu + (\mu_2 - \lambda_2 - \ell)\theta + a\alpha_2 + (k-a)\alpha_1),$$

for $1 \leq \ell \leq \lambda_2 - \mu_2, \quad \leq k \leq \mu_1 + \lambda_2 \;$,

where, $S_{\lambda,\mu}^{(\ell,k)} = \{a \in \mathbb{Z} : \begin{array}{l} 0 \leq a \leq \mu_1,\, 0 \leq k - a \leq \lambda_2, \\ \mu_1 - \mu_2 + \ell \leq 2a - k \leq \lambda_1 - \lambda_2 - \ell \end{array} \}.$

Step 2:

Using the short exact sequences and by applying induction on $\min\{\lambda(h_\theta), \mu(h_\theta)\}$, we get a set of recurrence relations on the dimensions of the associated CV-modules.

From these we deduce $\dim V(\xi(\lambda, \mu) \leq \dim V(\lambda). \dim V(\mu)$ and thus prove the theorem.

Graded Character

- Let $V = \oplus_{r\in\mathbb{N}} V[r]$ be a finite-dimensional graded $\mathfrak{g}[t]$-module. Then for each $r \in \mathbb{N}$, $V[r]$ is completely reducible as $\mathfrak{g}$-module.
- For a dominant integral weight ν, let

$$[V : \tau_p(V(\nu))] := \text{multiplicity of } V(\nu) \text{ in } V[p]$$

and let

$$[V : V(\nu)]_q = \sum_{p\geq 0} [V : \tau_p(V(\nu))]\, q^p.$$

- By definition the **graded character** of a $\mathfrak{g}[t]$-module V is :

$$ch_{gr} V = \sum_{\nu\in P^+} [V : \tau_p(V(\nu))]_q ch_{\mathfrak{g}} V(\nu),$$

where $ch_{\mathfrak{g}} V(\nu)$ is the $\mathfrak{g}$-character of $V(\nu)$.

- $\lim_{q\to 1}[V : V(\nu)]_q :=$ numerical multiplicity of $V(\nu)$ in V.
$\lim_{q\to 1}[V(\xi(\lambda,\mu)) : V(\nu)]_q :=$ Littlewood Richarson coefficient $c^{\nu}_{\lambda,\mu}$, for $\lambda, \mu, \nu \in P^+$,

Graded Character and LR coefficients

Theorem (K, S Rani)
Let $(\lambda, \mu) \in P^+(\lambda + \mu, 2)$ *such that* $\lambda(h_\alpha) \geq \mu(h_\alpha)$ *for* $\alpha \in \{\alpha_1, \theta\}$. *If* $\mu(h_{\alpha_2}) = 0$, *Then*

$$[V(\xi(\lambda, \mu)) : V(\nu)]_q = q^{\mu_1 - j}$$

for $\nu = \lambda + w_0\mu + j\theta + a_j\alpha_2$, *and 0 otherwise.*

This shows that the $\mathfrak{sl}_3$-modules $V(\lambda) \otimes V(m\omega_1)$ is multiplicity free.

Graded Character and LR coefficients

Theorem (K, S.Rani)

Let $(\lambda, \mu) \in P^+(\lambda+\mu, 2)$ be such that $\lambda(h_\alpha) \geq \mu(h_\alpha) > 0$ for $\alpha \in R^+$. Then,

$$[\, V(\xi(\lambda,\mu)) : V(\nu)]_q = \begin{cases} \sum_{s=0}^{a} q^{|\mu|-\ell-s-j}, & \text{if } \nu = \lambda + w_0\mu + (l+j)\theta + (\mu_2 - j)\alpha_1 + a\alpha_2, \text{ with } (j,l,a) \in A \\ \sum_{s=0}^{\min\{a,b\}} q^{|\mu|-s-j}, & \text{if } \nu = \lambda + w_0\mu + j\theta + b\alpha_1 + a\alpha_2, \text{ with } (j,a,b) \in B \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\begin{aligned} A &= \{(j,\ell,a) \in \mathbb{Z}^3 : j = 0,\ 0 \leq \ell \leq \mu_1,\ \max\{0, |\mu| - \lambda_2 - \ell\} \leq a \leq \mu_1 - \ell\} \\ &\sqcup \{(j,\ell,a) \in \mathbb{Z}^3 : 1 \leq j \leq \mu_2,\ 0 \leq \ell \leq \mu_1,\ a = \mu_1 - \ell\} \\ &\sqcup \{(j,\ell,a) \in \mathbb{Z}^3 : 1 \leq j \leq \mu_2,\ 0 \leq \ell \leq \mu_1,\ \ell + 2j \leq |\mu| - \lambda_2,\ a = |\mu| - \lambda_2 - (\ell + 2j)\} \end{aligned}$$

$$\begin{aligned} B &= \{(j,a,b) \in \mathbb{Z}^3 : j = 0, 0 \leq a \leq \mu_1,\ 0 \leq b \leq \mu_2 - 1,\ \mu_2 - \lambda_1 \leq b - a \leq \lambda_2 - \mu_1\} \\ &\sqcup \{(j,a,b) \in \mathbb{Z}^3 : 1 \leq j \leq \mu_2 - 1,\ 0 \leq a \leq \mu_1, 0 \leq b < \mu_2 - j, b - a = \lambda_2 - \mu_1 + j\} \\ &\sqcup \{(j,a,b) \in \mathbb{Z}^3 : 1 \leq j \leq \mu_2 - 1,\ 0 \leq a \leq \mu_1, 0 \leq b < \mu_2 - j, b - a = \mu_2 - \lambda_1 - j\} \end{aligned}$$

Graded Character and LR coefficients

Theorem (K, S.Rani)

Let $(\lambda, \mu) \in P^+(\lambda+\mu, 2)$ be such that $\lambda(h_\alpha) \geq \mu(h_\alpha) > 0$ for $\alpha \in \{\alpha_1, \theta\} R^+$ and $\lambda(h_{\alpha_2}) \leq \mu(h_{\alpha_2})$. Then

$$[\,V(\xi(\lambda,\mu)) : V(\nu)]_q = \begin{cases} \sum_{s=0}^{\min\{a,b\}} q^{|\rho_\lambda^\mu|-\ell-s}, & \text{if } \nu = \rho_\mu^\lambda + w_0\rho_\lambda^\mu - \ell\theta + a\alpha_2 + b\alpha_1, (\ell,a,b) \in C \\ \sum_{s=0}^{\min\{b_1,b_2\}} q^{|\rho_\lambda^\mu|-s-j}, & \text{if } \nu = \rho_\mu^\lambda + w_0\rho_\lambda^\mu + j\theta + b_1\alpha_2 + b_2\alpha_1, (j,b_1,b_2) \in B_{inv} \\ \sum_{s=0}^{a} q^{|\rho_\lambda^\mu|-\ell-s-j}, & \text{if } \nu = \rho_\mu^\lambda + w_0\rho_\lambda^\mu + (l+j)\theta + (\lambda_2 - j)\alpha_1 + a\alpha_2, (j,\ell,a) \in A_{inv} \\ 0 & \text{otherwise} \end{cases}$$

where A_{inv} and B_{inv} are obtained from A and B respectively by interchanging λ_2 with μ_2 and vice versa and

$$\begin{aligned} C = \; & \{(\ell,a,b) : \ell = 0, 0 \leq a \leq \mu_1, 0 \leq b \leq \lambda_2, \mu_2 - \lambda_1 \leq a_2 - a_1 \leq \lambda_2 - \mu_1\} \\ & \sqcup \{(\ell,a,b) : 1 \leq \ell \leq \mu_2 - \lambda_2 - 1, 0 \leq a \leq \mu_1, 0 \leq b \leq \lambda_2, b - a = \mu_2 - \lambda_1 - \ell\} \\ & \sqcup \{(\ell,a,b) : 1 \leq \ell \leq \mu_2 - \lambda_2 - 1, 0 \leq a \leq \mu_1, 0 \leq b \leq \lambda_2, b - a = \lambda_2 - \mu_1 + \ell\}. \end{aligned}$$

Computing the Littlewood Richardson coefficients from the graded character of $V(\xi(\lambda,\mu))$ we observe the following:

Theorem (K, S. Rani)

For a dominant integral weight $\lambda = m_1\omega_1 + m_2\omega_2$ *of* $\mathfrak{sl}_3$*, let*

$$|\mathrm{p}_\lambda| = m_1 + 2m_2.$$

Given a triple (λ,μ,ν) *of dominant integral weights of* $\mathfrak{sl}_3(\mathbb{C})$ *such that* $|\mathrm{p}_\lambda| + |\mathrm{p}_\mu| \equiv |\mathrm{p}_\nu| \mod 3$*, for all* $N \in \mathbb{Z}_+$*, we have*

$$c^{N\nu}_{N\lambda,N\mu} = N(c^{\nu}_{\lambda,\mu} - 1) + 1.$$

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Thank you !