Lecture notes

The intrinsic approach to differential growth and geometric frustration

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ICTS program on "Geometry, Mechanics and the Physics of Growth"

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Chapter 1

Introduction

The field of mechanics relates the forces acting on a physical body to the resulting motion and distortion it displays. In what follows we will restrict ourselves to classical systems (where both quantum and relatevistic effects are irrelevant), and focus primarily on solids (with a few exceptions). There is a vast body of literature on elasticity, both from a practical/engineering perspective and a theoretical axiomatic approach. In most accounts the basic elements of the theory are the stress tensor, measuring forces per unit area, and the strain tensor which measures the distortion from a stress-free rest-state. In what follows we will pursue a slightly different approach from the classical treatment by adopting a purely intrinsic geometric approach to mechanics, employing only quantities available to an observer residing within the material.

Biologic growth, while often regulated at the organism scale, is a local process redefining distances within the material rather than determine a speciic configuration. It is thus most natural to describe such systems using the intrinsic approach which is fundamentally local and is not configuration based. This is particularly true for systems exhibiting residual stresses. Residual stresses are internal forces present in a body in the absence of external forces or constraints. These are indicative of geometric frustration; mutually contradicting geometric tendencies in the body that cannot be simultaneously satisfied. One could think, for example of a planar triangle made from three planar head angles that together sum up to more than 180 degrees. As we will soon see, there are more subtle and in particular continuous ways of generating geometric frustration. The intrinsic approach makes extensive use of Riemannian geometry and is particularly suited for describing residually stressed and geometrically frustrated solids [1]. To a large extent, many of the aspects of the theory of elasticity are

[1] In 1954 Kazuo Kondo, a Japanese mathematical engineer wrote: *"Although the remarkable success of the general relativity theory impressed the importance of tensor calculus and Riemannian geometry on public opinion, it was unfortunate that it gave a metaphorical appearance to Riemannian expression, banishing it for a time from the attention of engineers. But the reader, after having studied the following analysis, will agree with us in the opinion that it is strange that the first practical field of application was not the theory of elasticity, especially of residual strains."* **Geometry of Elastic Deformation and Incomapatibility.**

results in Riemannian geometry. We will learn the necessary notions of differential geometry and in particular Riemannian geometry as they arise in their physical context. No prior knowledge in mechanics or geometry will be assumed except for some advanced calculus. For additional sources see [2].

Some motivation

Before we make the notion of geometric frustration more mathematical it is worth noting that frustrated structures are actually not that hard to find, and in many cases are very useful. Both examples are easiest to exemplify in terms of manmade structures, as here we know exactly the process through which the structure was formed. There are growth related analogs, though these are less didactitc. The most common frustrated structure is that of tempered or toughened glass. Almost all modern smart phones are covered with "Gorilla glass" an exceptionally thin toughened glass sheet. This greatly strengthens the glass against scratching and breakage. The side windows in our car are made of tempered glass. In the case of car windows one not only wants to strengthen the glass against breakage but also to make sure that when the glass does break it leaves behind no large shards.

In order to understand how tempering serves both these purposes we need to understand what is the mechanical state of tempered glass. The process of glass tempering usually follows in two steps. In the first step the glass in its final form is heated to allow some flow. It is then cooled abruptly using forced air drafts. This solidifies the outer surfaces of the glass while the bulk of the glass remains molten and hot. During the gradual cooling the bulk shrinks. However, as the external surfaces have already solidified the bulk is held tensed. This tension is balanced by a corresponding compression on boundary. As brittle failure advances cracks under tension, compressing the exposed boundary of a body implies that in order to propagate a crack through the material one needs to first overcome the surface compression. However, this compression is balanced by very strong tension in the interior of the material. If for some reason this tensed region would be exposed, and thus allow a crack to nucleate the tensed interior would propagate the crack without any external forces. This often results in many cracks. The Gorilla glass on you phone screen is typically only about half a millimeter thick. Thermal conduction is too efficient at these scales to allow the production of a significant temperature gradient across the thickness of the glass. To produce the compressive stress on the outer boundary ion exchange is employed. The sodium containing glass is dipped in a molten

K. Kondo, Memoirs of the unifying study of the basic problems in engineering sciences by means of geometry. Vol I p.361–373. (1955)

[2]Additional resources for differential geometry: Two of my favorite and most accessible sources for differential geometry of the kind we use here are:

1. **Lectures on Classical Differential Geometry.**
A textbook by D.J. Stuik. Dover Publications; Second edition (1988)

2. **The geometry of soft materials: a primer.**
A review paper by R. Kamien. Rev. Mod. Phys. **74**, 953 (2002)

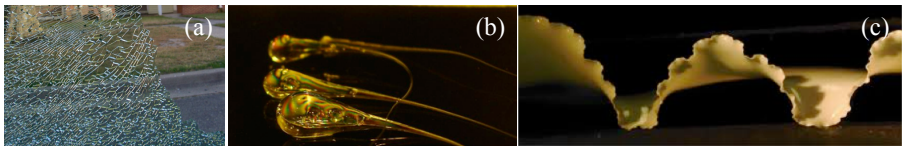

Figure 1.1: (a) A broken tempered glass. (b) Prince Rupert's drops, an extreme example of tempered glass. (c) A torn plastic sheet.

bath of Potassium. As the Potassium ions are larger, when they exchange the sodium ions the external layers of the glass differentially elongate (according to the amount of diffused Potassium), resulting in the desired stress.

The second example of a residually stressed system exemplifies a different feature of these systems, namely the emergence of complex ground states. A compatible system can fully comply with its all intrinsic local tendencies. Consequently, simple inputs lead to simple outputs. In contrast, incompatible (or frustrated) systems must exhibit some compromise between the mutually contradicting tendencies. The modes of compromise can be highly cooperative in nature and lead to complex behavior.

Consider a torn plastic sheet, and the edge of a curly lettuce or a beet leaf. The striking similarity between these two systems is not coincidental, they were both formed by a similar mechanism. Before we come to understand how these systems were shaped, let us first think of the shaping mechanism that would bend a otherwise straight plastically deformable ribbon (think of a ruler made of copper). When we bend it in one direction the plastic deformations elongate the outer portion of the surface and compress the inner portion of the surface. We thus break the up/down symmetry in the internal structure of the ribbon. If we would like to incorporate then the opposite bend to form a wavy shape we would need to switch between the sides that underwent elongation and contraction. Bending something into a ruffled shape will thus require a tedious and complex process. If however we examine the shaping mechanism that produced the ruffles on the edge of the torn plastic sheet we would observe that the process is not only simple and featureless along the tear path, but also preserves the up/down symmetry of the sheet. In particular the expected irreversible elongation is identical on the top and bottom layer, implying we are not plastically "bending" the material. Why then does the sheet ruffle? The tearing process differentially elongates the material in the direction of the tear path. The ruffles arise spontaneously to accommodate the excess length. In fact the observed sheet is the least bent configuration that accommodates the excess length. The ruffling of edges of leaves is similarly created by differential growth. The resulting thin sheet would like to not bend (not break the up/down symmetry), yet must bend to accommodate the excess length.

This brings us to the need to formulate an elastic description for frustrated structures. We will first briefly review standard elasticity, and its shortcoming in

describing frustrated systems. We will then digress into some differential geometry and learn the tools which will serve us to formulate the metric description of elasticity.

Chapter 2

Frustrated elastic solids in two and three dimensions

2.1 Classical Elasticity

The main paradigm of elasticity is that in elastic media the force a certain body exerts in its deformed state depends only on the distortion from its initial stress free state. To quantify these notions one needs a measure of distortion; this is commonly captured by the elastic strain. Assuming a body is in its stress free rest state Ω parametrized by the cartesian coordinates $\mathbf{x}$ undergoes a deformation, and that the deformation may be expressed by the mapping $\mathbf{r}(\mathbf{x})$ we may write the displacement vector

$$\mathbf{u} = \mathbf{r} - \mathbf{x}.$$

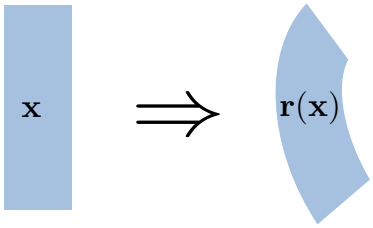

Figure 2.1: The rest reference state and the deformed configuration

In standard elasticity literature the stress is defined as the force per unit area, and the strain employed to measure the degree of deformation is expressed using

the displacement vector. The most commonly used strain is the linearized strain reading

$$\epsilon_{ij}^{Linear} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

This strain measure enjoys the advantage of being linear in the displacement field, but actually fails when large rotations are present. If the displacement is a rigid rotation, i.e. $\mathbf{r} = O\mathbf{x}$ where $O^T O = I$, then we expect no distortions, yet $\epsilon^{Linear} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u}^T + \boldsymbol{\nabla}\mathbf{u}) = \frac{1}{2}(O^T + O - 2I)$ which for a $\pi/2$ rotation reads $\epsilon^{Linear} = -I$ despite being associated with no distortions. The linearized strain is useful when analyzing waves, and infinitesimal distortion in solids but fails when dealing with large rotations as are commonly present with elastomeric materials and thin sheets.

We will thus use the nonlinear strain

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_j}) = \frac{1}{2}(\sum_l \frac{\partial r_l}{\partial x_i}\frac{\partial r_l}{\partial x_j} - \delta_{ij}).$$

The quantity $g_{ij} = \sum_l \frac{\partial r_l}{\partial x_i}\frac{\partial r_l}{\partial x_j} = \frac{d\mathbf{r}}{d\mathbf{x}_i} \cdot \frac{d\mathbf{r}}{dx_j}$ is called the metric of the mapping $\mathbf{r}$ with respect to the coordinates $\mathbf{x}$ and is used to relate distances in the deformed state to coordinate variations

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j \equiv g_{ij} dx^i dx^j,$$

where above we used the Einstein summation convention by which if an index appears twice in a multiplicative product (one lower index and one upper index) then we sum over the values it assumes.

As later on we'll see an elastic potential that is quadratic in the strain it may seem that we have exchanged a nonlinear problem (in the displacement field gradients) with a linear one (with respect to the metric). This is however not true. Not every 3 symmetric matrix may be the metric of a region in $\mathbb{R}^3$. The non-linear constraints imposed on admissible metrics preserve the complexity of the problem. It does, however, in many cases, render the elastic problem more transparent.

2.2 A short geometric digression

We have already encountered the metric tensor of a deformed three dimensional body. We will continue to review some results in Riemannian geometry from this context. The first notion to review is that of tangent vectors. They correspond to directions in the deformed configuration induced by coordinate variations in the undeformed state. The three independent coordinate variations correspond to three independent vectors

$$\boldsymbol{e}_i \equiv \partial_i \mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i}.$$

The second derivatives of these vectors also yield vectors in $\mathbb{R}^3$. Therefore they could be written as a sum of the tangent vectors

$$\partial_i \boldsymbol{e}_j = \partial_i \partial_j \mathbf{r} = \Gamma^k_{ij} \partial_k \mathbf{r} = \Gamma^k_{ij} \boldsymbol{e}_k.$$

Whenever an index appears in a product one up and once down we sum over its range of values; in the case above the index k is summed over the values $1, 2$ and 3. This defines the Christoffel symbol Γ.

Exercise 2.2.1. *As a home exercise participants could prove that using the definition of the metric the above definition gives*

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{2.1}$$

where g^{kl} that appears above is the inverse metric i.e.

$$g^{kl} = \left(g^{-1}\right)^{kl} \Rightarrow g^{ik} g_{kj} = \delta^i_j.$$

In differential geometry the location of the indices is meaningful. Index quantities (unless otherwise noted) form covariant or contravariant (or mixed) tensors. These terms refer to how these index quantities vary under a change of coordinate system. Any quantity that transforms similarly to tangent vectors is called covariant. Consider an alternative coordinate system $\tilde{x}^i$. The tangent vectors will transform according to

$$\tilde{\boldsymbol{e}}_i = \tilde{\partial}_i \mathbf{r} = \frac{\partial \mathbf{r}}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial_i \mathbf{r}}{\partial x^j} = \Lambda^j_i \boldsymbol{e}_j.$$

Any set of quantities that transforms as $\tilde{v}_i = \Lambda^j_i v_j$ with $\Lambda^j_i = \frac{\partial x^j}{\partial \tilde{x}^i}$ are said to constitute a covariant vector. Index quantities that transform with the inverse of Λ, i.e. $(\Lambda^{-1})^j_i = \frac{\partial \tilde{x}^j}{\partial x^i}$, are called contravariant vectors.

Exercise 2.2.2. *Show directly from the definition of contravatiant and covariant vectors that the product of any contravatiant and covariant yields quantity that is independent of the coordinates used.*

$$\tilde{\phi} = \tilde{v}^k \tilde{v}_k = v^k v_k = \phi$$

We call such quantities scalars.

Consider a vector field $\boldsymbol{V}(\mathbf{r})$, we may parametrize $\mathbf{r}$ through various coordinate systems, yet the local value of $\boldsymbol{V}$ will not be affected. Assuming we employ a coordinate system x^i we may construct the local tangent vectors $\boldsymbol{e}_i = \partial_i \mathbf{r}$ and express the vector field as

$$\boldsymbol{V} = v^i \boldsymbol{e}_i.$$

The quantities v^i constitute the contravariant components of the vector. We may alternatively use the duals to the

$$\boldsymbol{e}^j \cdot \boldsymbol{e}_i = \delta^j_i,$$

and express the same vector as

$$\boldsymbol{V} = v_i \boldsymbol{e}^i.$$

The quantities v_i constitute the covariant components of the vector $\boldsymbol{V}$. One of the corollaries of this definition of the covariant and contravariant components of a vector is that raising and lowering of indices is not trivial but performed with respect to the metric and its inverse

$$v_i = g_{ij} v^j, \qquad v^i = g^{ij} v_j.$$

This implies that the scalar product between vectors is prescribed by the metric

$$\boldsymbol{V} \cdot \boldsymbol{U} = u^i \boldsymbol{e}_i \cdot \boldsymbol{e}^j v_j = u^i v_i = u_i \boldsymbol{e}^i \cdot \boldsymbol{e}_j v^j = g_{ij} u^i v^j = g^{ij} u_i v_j$$

Differentiation of vectors is also not trivial. If we would like to examine how a vector whose components are given by v^j changes in space we need to also consider the variation of the tangent vectors in space. We recall that the components v^j correspond to the vector $\boldsymbol{v} = v^j \partial_j \mathbf{r}$, and therefore

$$\partial_i \boldsymbol{V} = \partial_i (v^j \partial_j \mathbf{r}) = \partial_i v^j \partial_j \mathbf{r} + v^j \partial_i \partial_j \mathbf{r} = \partial_i v^j \partial_j \mathbf{r} + v^j \Gamma^k_{ij} \partial_k \mathbf{r} = (\partial_i v^j + v^k \Gamma^j_{ik}) \partial_j \mathbf{r}.$$

The covariant derivative of the components v^j are thus defined to read

$$\nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k.$$

Similarly for covariant vectors (lower indices) we have

$$\nabla_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k.$$

Because we are only considering presently embedded objects we can also analyze them as functions. In particular we may require that the mixed derivatives of the configuration $\mathbf{r}$ with respect to coordinates commute: $\partial_i \partial_j \mathbf{r} = \partial_j \partial_i \mathbf{r}$. Requiring this equality actually implies that the Christoffel symbol is symmetric (which is true by construction from the metric). One may ask then if any metric corresponds to a parametrization of a region in $\mathbb{R}^3$. The answer to such questions is most conveniently and transparently obtained by an explicit construction. We will attempt to construct the configuration from knowledge of the metric. To do so we first identify that to obtain the configuration as a function of the coordinates $\mathbf{r}(x)$ we only need to integrate the tangent vectors, treating their definition as a partial differential equation (PDE):

$$\partial_i \mathbf{r} = \boldsymbol{e}_i.$$

This determines $\mathbf{r}$ up to an additive constant, which we may interpret as translation. Recall that for a PDE to be solvable it needs to satisfy a compatibility condition, namely that integrating it along different paths gives the same answer. In the above case this corresponds to $\partial_j \boldsymbol{e}_i = \partial_i \boldsymbol{e}_j$, which is of course

assured by the symmetry of the Christoffel symbols. However, the above require knowing the tangent vectors as a function of the coordinates x^i. Given the metric we can construct the Christoffel symbol directly, and then use the definition through tangent vectors as a PDE

$$\partial_i \boldsymbol{e}_j = \Gamma^k_{ij} \boldsymbol{e}_k.$$

Here too the tangent vectors could be integrated up to a constant, which is now interpreted as prescribing the initial directions of the tangent vectors, and corresponds to rigid rotations. Requiring compatibility yields

$$0 = \partial_j \partial_k \partial_i \mathbf{r} - \partial_k \partial_j \partial_i \mathbf{r} = (\partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^l_{ik}\Gamma^m_{lj} - \Gamma^l_{ij}\Gamma^m_{lk})\partial_m \mathbf{r} = R^m_{ijk}\partial_m \mathbf{r}.$$

The quantity R^k_{ijk} is called the Riemann curvature tensor. For a metric g_{ij} to describe an object in $\mathbb{R}^3$ each of the $3^4 = 81$ components of the tensor must vanish identically. There are, however, only 6 independent components in R^k_{ijk}. Their vanishing is also a sufficient condition for the existence of an a deformed state whose metric is given by g_{ij}. In general, an n-dimensional Riemann curvature tensor has $n^2(n^2-1)/12$ independent entries. For $n = 3$ there are only six independent entries. It is thus more convenient to us the Ricci curvature tensor (which for $n = 3$ captures all the information), given by the partial trace $Ricci_{ij} = R^k_{ikj}$. For $n = 2$ there is only one independent entry, and thus it suffices to consider the Gaussian curvature given by half the trace of the Ricci tensor $K = \frac{1}{2} g^{ij} Ricci_{ij}$.

At this point it is important to mention that the Christoffel symbol is NOT a mixed tensor. It is an essential component of our algebra but as we will soon understand its role in correcting the simple differentiation of vectors, it could not be a tensor.

2.2.1 Generation of incompatibility in non-uniform isotropic expansion

We now come to exemplify how difficult it is to construct a flat metric through a specific example. Consider a strain-free body, parameterized by Cartesian coordinates, i.e. $g = I$. Allow every point in the body to expand isotropically but non-homogeneously by a factor $\lambda(\mathbf{x})$, thus giving rise to a reference metric $\bar{g} = \lambda^2 I$. Such expansion may result for example from thermal expansion, or in growth induced by turgor pressure in plants' cells. We now ask a simple question: what isotropic growth profiles will result in a compatible reference metric, i.e. will be realizable by an Euclidean metric, g, and will therefore not induce residual stress?

To answer this question we write down the components of the Riemannian curvature tensor of the metric $\bar{g}$ in terms of the expansion factor λ and its derivatives. Taking independent linear combination of the covariant components of the Riemannian curvature tensor (essentially the components of the Ricci

tensor $Ricci_{ij} = R^k_{ikj}$) yields the following compatibility conditions:

$$\begin{aligned} &2(\partial_1\lambda)^2 - \lambda\partial_1\partial_1\lambda - \lambda\triangle\lambda = 0, \qquad 2\partial_1\lambda\partial_2\lambda - \lambda\partial_1\partial_2\lambda = 0, \\ &2(\partial_2\lambda)^2 - \lambda\partial_2\partial_2\lambda - \lambda\triangle\lambda = 0, \qquad 2\partial_1\lambda\partial_3\lambda - \lambda\partial_1\partial_3\lambda = 0, \\ &2(\partial_3\lambda)^2 - \lambda\partial_3\partial_3\lambda - \lambda\triangle\lambda = 0, \qquad 2\partial_2\lambda\partial_3\lambda - \lambda\partial_2\partial_3\lambda = 0, \end{aligned} \tag{2.2}$$

where $\triangle = \nabla \cdot \nabla = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the standard Laplacian operator. It takes straightforward algebra and integration to find that the only non-constant solution of the above equations is

$$\lambda = \frac{C^2}{|\mathbf{x} - \mathbf{x_0}|^2},$$

for some constants C and $\mathbf{x_0}$.

Exercise 2.2.3. *Show that the six independent components of the Riemann Curvature tensor can be reorganized into the form of Eq. 2.2.*
Integrate Eq. 2.2 to obtain $\lambda = \frac{C^2}{|\mathbf{x}-\mathbf{x_0}|^2}$.

Every other isotropic expansion profile of an initially Euclidean 3D body will give rise to a non-Euclidean metric and inevitably result in a residually stressed body. This result, may be surprising when considering growth profiles. However it is a consequence of a well-known geometric result whereby all conformal mappings in $\mathbb{R}^3$ are inversions of a sphere. It implies that any growth that does not result in residual stress requires delicate global control, or some mechanical feedback.

2.3 Frustrated solid bodies

To better understand the notion of frustrated bodies we first rewrite the non-linear strain with respect to general coordinates. Recall that tensor (multi indexed quantities) should be considered as the external product of vectors, and that the simplest vector is a derivative with respect to coordinates predicts how should one change a tensor under a change of coordinates.

$$\tilde{v}_i = \frac{\partial\phi}{\partial\tilde{x}^i} = \frac{\partial x^j}{\partial\tilde{x}^i}\frac{\partial_i\phi}{\partial x^j} = \Lambda^j_i v_j.$$

When we implement this rule on the strain tensor

$$\tilde{\varepsilon}_{ij} = \Lambda^k_i\Lambda^l_j\varepsilon_{kl} = \frac{1}{2}(\Lambda^k_i\Lambda^l_j g_{kl} - \Lambda^k_i\Lambda^l_j\delta_{kl}) = \frac{1}{2}(\tilde{g}_{ij} - \bar{g}_{ij}).$$

The quantity $\Lambda^k_i\Lambda^l_j\delta_{kl} = \bar{g}_{ij}$ is nothing but the metric of the body prior to the deformation with respect to the new coordinates, while $\tilde{g}$ is the metric of the deformed state with respect to the same new coordinates. This gives a more general for to the strain as the difference between two metrics

$$\varepsilon_{ij} = \frac{1}{2}(g_{ij} - \bar{g}_{ij}).$$

It is now easy to see how can a body become frustrated. If a body is locally growing then it varies $\bar{g}$. If this growth rule is local, then as we saw in the isotropic growth example it may easily become incompatible i.e. have a non zero Riemannian curvature. This implies that g_{ij} must deviate from $\bar{g}_{ij}$, as g must have a vanishing Riemann curvature.

2.3.1 Back to Elasticity

To formulate the elastic description of residually stressed bodies we may implement coordinate transformations to express the elastic response tensor relating the stress and the strain in general coordinates. However, it may be more insightful to follow a different path which generalizes the notion of hyper-elasticity: *The elastic energy stored within a deformed elastic body can be written as a volume integral of a local elastic energy density that depends only on (i) the local value of the metric tensor and (ii) local metrial properties that are independent of the configuration.*

Under the assumptions that both space and the elastic body are homogeneous and isotropic and that strains are small we obtain

$$E = \int_{\Omega} \mathcal{W}(g, \bar{g}) \sqrt{|\bar{g}|} dx^1 dx^2 dx^3,$$

where the energy density is given by

$$\mathcal{W} = A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \mathcal{O}(\varepsilon^3),$$

and the elasticity tensor may be expressed entirely in terms of the reference metric and the two elastic response coefficients: the Young's modulus, Y and the Poisson ratio ν:

$$A^{ijkl} = \tfrac{Y}{1+\nu} \left(\frac{1}{2} \left(\bar{g}^{ik} \bar{g}^{jl} + \bar{g}^{il} \bar{g}^{jk} \right) + \tfrac{\nu}{1-2\nu} \bar{g}^{ij} \bar{g}^{kl} \right).$$

The stress may be obtained from the elastic energy density

$$S^{ij} = \frac{\partial \mathcal{W}}{\partial \varepsilon_{ij}} = A^{ijkl} \varepsilon_{kl} = \tfrac{Y}{1+\nu} \left(\varepsilon^{ij} + \tfrac{\nu}{1-2\nu} \bar{g}^{ij} \varepsilon^k_k \right).$$

This description recovers the familiar elastic description of regular bodies when substituting $\bar{g} = I$. However, its main advantages is in cases where $\bar{g}$ is non-Euclidean, where it gives rise to residual stress.

2.3.2 Obtaining the equilibrium equations

We are now faced with the well defined task of finding the configuration of a solid (as captured by the metric g) that minimizes the elastic energy with respect to some reference metric $\bar{g}$. We might be tempted to write the Euler-Lagrange associated with variation in the strain ε, or equivalently in the metric

g. However, this would give us the wrong answer. The metric is constrained to be flat, i.e. have a vanishing Riemannian curvature. We could constrain the metric to be Riemannianly flat using Lagrange multipliers, yet we will require six such multipliers, each of which could vary in space. It is far simpler to circumvent the problem. To do so we recall that while we were striving to work with a local description, we are perturbing a realized configuration $\mathbf{r}$, and assume that the perturbed configuration is also realizable, meaning it could be written as $\mathbf{r}+\delta\mathbf{r}$. We may thus write

$$g_{ij}+\delta g_{ij}=(\partial_i\mathbf{r}+\partial_i\delta\mathbf{r})\cdot(\partial_j\mathbf{r}+\partial_j\delta\mathbf{r})=\partial_i\mathbf{r}\cdot\partial_j\mathbf{r}+\partial_i\delta\mathbf{r}\cdot\partial_j\mathbf{r}+\partial_i\mathbf{r}\cdot\partial_j\delta\mathbf{r}+\mathcal{O}(\delta^2)$$

$$\delta E=2\int_\Omega A^{ijkl}\varepsilon_{jk}\delta\varepsilon_{ij}\sqrt{|\bar{g}|}dx^1dx^2dx^3=2\int_\Omega A^{ijkl}\varepsilon_{jk}\partial_i\mathbf{r}\cdot\partial_j\delta\mathbf{r}\sqrt{|\bar{g}|}dx^1dx^2dx^3,$$

We now identify the stress and can differentiate in parts

$$\delta E=B.T.-2\int_\Omega\partial_j\left(S^{ij}\sqrt{|\bar{g}|}\partial_i\mathbf{r}\right)\cdot\delta\mathbf{r}dx^1dx^2dx^3$$

Where $B.T.$ stands for boundary terms. We may thus identify the equilibrium equations as

$$\partial_jS^{ij}\sqrt{|\bar{g}|}\partial_i\mathbf{r}+S^{ij}\partial_j\sqrt{|\bar{g}|}\partial_i\mathbf{r}+S^{ij}\sqrt{|\bar{g}|}\partial_j\partial_i\mathbf{r}\Rightarrow\partial_jS^{ij}+S^{ij}\bar{\Gamma}^k_{jk}+S^{kj}\Gamma^i_{kj}$$

where we have made use of $\partial_j\sqrt{|\bar{g}|}=\bar{\Gamma}^k_{jk}$. The above result doesn't look tensorial yet. We can however massage it a bit. We may add and subtract a Christoffel symbol to obtain

$$\partial_jS^{ij}+S^{ij}\bar{\Gamma}^k_{jk}+S^{kj}\bar{\Gamma}^i_{kj}+S^{kj}\Gamma^i_{kj}-S^{kj}\bar{\Gamma}^i_{kj}=\bar{\nabla}_jS^{ij}+\left(\Gamma^i_{kj}-\bar{\Gamma}^i_{kj}\right)S^{kj}=0$$

2.3.3 The self-averaging nature of residual stress

Consider a free standing body (not subjected to any external forces), yet frustrated due to metric incompatibility. We know that non-trivial spatially varying residual stresses would arise in the body. However, the lack of external forces implies that the body is the one exerting the forces on itself. Consequently, the forces on the body must be self balancing. This statement is somewhat opaque when the stress and strains are expressed in terms of arbitrary variables x^i. However, there is a compact for to express this self averaging. Consider a scalar field χ with a non vanishing gradient $v_i=\nabla_i\chi$, which satisfies

$$\nabla_iv_j=\nabla_i\nabla_j\chi=0,$$

where the covariant derivative above is taken with respect to the metric g. For all such test functions χ the quadratic form $S^{ij}v_iv_j$ must average to zero when integrated over the entire body, i.e.,

$$\int_\Omega v_iv_jS^{ij}\sqrt{|\bar{g}|}dx^1dx^2dx^3=0.$$

The proof follows immediately from integration by parts and explicit substitution of the divergence equation for the stress. In particular, as the integrand is a quadratic form in the gradient, v_i, every non trivial residual stress field must contain both tension and compression.

2.3.4 Incompatibility and residual stress: The tempered glass case

We now look into the tempered glass system. Away from the edges of the glass sheet we have that the attempted area of the solid along the yz plan grows with x, and are symmetric for positive and negative x-values. The simplest reference metric that obeys this is

$$\bar{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha x^2 & 0 \\ 0 & 0 & 1+\alpha x^2 \end{pmatrix}$$

We will seek a solution of the form

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(x)^2 & 0 \\ 0 & 0 & f(x)^2 \end{pmatrix}$$

Writing the Ricci tensor for the above ansatz yields $f''(x) = f'(x) = 0$, so $f(x) = const$. This elucidates how restricted Euclidean states really are. The constant f will lay somewhere between 1 and $1 + \alpha x_{max}^2$. This will mean that the strain (and consequently the stress) will be compressive near the boundary, and tensile in the center. Let's

2.3.5 Incompatibility and residual stress: A uniformly frustrated system

In this section we demonstrate the main difference between residual stress and residual strain. While the latter is a local property, the former depends also on global properties such as the domain size and shape. Let us consider a two dimensional problem of embedding a piece of a sphere in the plane. The metric of a spherical cap or radius R reads

$$\bar{g} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \sin(r/R)^2 \end{pmatrix}$$

We will consider a small section of the spherical cap $0 \leq r \leq r_{max} \ll R$.

A general embedding respecting the symmetry would read

$$g = \begin{pmatrix} \phi^2(r) & 0 \\ 0 & \psi^2(r) \end{pmatrix}$$

Restricting the metric to be Riemannianly flat amounts to

$$\frac{\phi'(r)}{\phi(r)} = \frac{\psi''(r)}{\psi'(r)} \Rightarrow \phi(r) = C\psi'(r)$$

The constant C captures both the units of the azimuthal coordinate θ as well as captures the behavior at the origin and will be now set to 1. We thus have

$$g = \begin{pmatrix} \psi'^2(r) & 0 \\ 0 & \psi^2(r) \end{pmatrix}$$

The resulting Euler Lagrange equations are second order in ψ and non-linear. Solving them numerically yields the stresses plotted below.

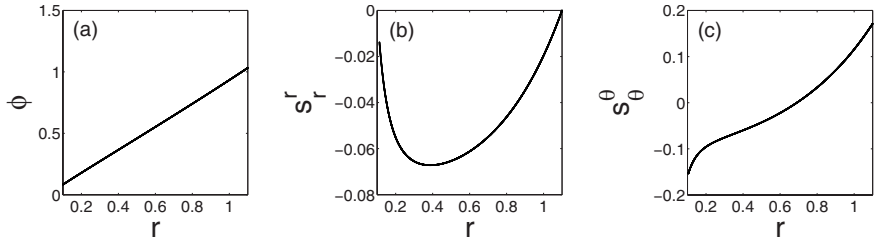

Figure 2.2: The solution to the plane stress problem of a flattened spherical cap. Adapted from PRE **80**, 016602 (2009)

One notable property of the above stress distribution is that for small enough spherical caps the elastic energy grows as

$$E(A) \propto A^3.$$

How can we understanding this area to the third power growth rate? Well it suffices to understand that our compatibility conditions (in the present case vanishing of the Gaussian curvature of the two dimensional metric constitute a second order differential equation. We may express it in terms of the components of the strain tensor.

$$\varepsilon = \frac{1}{2}(g - \bar{g}) \Rightarrow g = \bar{g} + 2\varepsilon$$

We may now expand the compatibility condition and ε in orders of r and seek to eliminate as many orders as we can. One can show that in the present case we could eliminate the zeroth order of the strain to zero, as well as the linear order of the strain and still satisfy the compatibility conditions up to second order (by the appropriate choice of the higher order components of the strain). We thus expect the strain to grow as $\varepsilon \propto r^2$. The energy would thus grow as

$$E \propto \epsilon^2 dA \propto r^4 r dr \propto r^6 \propto A^3.$$

2.3.6 Superextensive energy in long-range systems: Electrostatics

Thermodynamics teaches us that the total energy of a system must grow extensively for the thermodynamic limit to make sense. Our analysis is limited to finite size structures, and thus one should think of the phenomena described as finite size effects. Moreover, we know well other examples that yield superextensive energy terms in systems with long range interactions. One such system is electrostatics. Consider a finite ball of radius R uniformly charged with charge density ρ. The Electrostatic energy of such a system is given by

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\frac{4}{3}\pi r^3}{r}\rho 4\pi r^2 dr \propto \int r^4 dr \propto R^5 \propto V^{5/3},$$

where V is the volume of the uniformly charged ball. This implies that superextensive energy is not exceptional, however, in the system above it arises in a system with short ranged interactions. Thus geometric frustration is a vehicle for making a system with short range interactions behave as a system. with long range interactions.

2.4 Frustrated thin elastic sheets

Frustrated thin elastic sheets deserve dedicated and special attention. In these systems the intrinsic geometric frustration may manifest very strongly.

2.4.1 Theory of surfaces

Up until now we considered only two dimensional manifolds embedded in a flat two dimensional space. Such embeddings are called tight embeddings (or of co-dimension 0). The compatibility conditions in these cases were that the Riemannian curvature tensor (which in 2D has only one independent component) mush vanish. We now come to consider a two dimensional manifold in a three dimensional Euclidean space.

Such embeddings are characterized by the mapping

$$\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3.$$

We use greek indices to express two dimensional indexed quantities such as the tangent vectors $\partial_\alpha \mathbf{r}$ with which we define the metric tensor (which for surfaces is sometimes called the first fundamental form)

$$a_{\alpha\beta} = \partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r}.$$

We note however that to express every vector in a three dimensional space the two tangent vectors do not suffice and we need an additional vector. We thus define the normal

$$\hat{\mathbf{N}} = \frac{\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}}{|\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}|} = \frac{\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}}{|a|}.$$

We may now express every vector locally in terms of $\partial_\alpha \mathbf{r}$ and $\hat{\mathbf{N}}$. We thus write

$$\partial_\alpha \partial_\beta \mathbf{r} = \Gamma^\gamma_{\alpha\beta} \partial_\gamma \mathbf{r} + b_{\alpha\beta} \hat{\mathbf{N}}.$$

Γ above is the familiar Christoffel symbol of the second kind, defined through first derivatives of the metric. $b_{\alpha\beta} = \partial_\alpha \partial_\beta \mathbf{r} \cdot \hat{\mathbf{N}}$ is called the curvature tensor or the second fundamental form. Note, that by its definition it measures curvatures not as orientation change per unit length but rather per unit of x^1 and x^2; these might as well be measured in nautical miles, or oranges (or any other arbitrary dimension). We note that the above definition implies

$$b_{\alpha\beta} = \partial_\alpha \partial_\beta \mathbf{r} \cdot \hat{\mathbf{N}}$$

which also gives

$$0 = \partial_\alpha (\partial_b \mathbf{r} \cdot \hat{\mathbf{N}}) = b_{\alpha\beta} + \partial_\alpha \hat{\mathbf{N}} \cdot \partial_\beta \mathbf{r} \quad \Rightarrow -\partial_\alpha \hat{\mathbf{N}} \cdot \partial_\beta \mathbf{r} = b_{\alpha\beta}.$$

From the above we may deduce also $\partial_\alpha \hat{\mathbf{N}} = -a^{\gamma\delta} b_{\delta\alpha} \partial_\gamma \mathbf{r}$. While in general b has strange coordinates, when considering a single point p we may use a special set of coordinates in which $a = I$. With respect to these coordinates b is symmetric and has dimensions of curvature (reciprocal length) and may be diagonalized to read

$$b = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

κ_1 and κ_2 are called the principal normal curvatures of the surface. At every point and at every direction we may ask what is the normal curvature of the surface at that point in that direction. The above argument shows that as a function of the direction the normal curvature has one minimum and maximum and these are obtained in perpendicular directions.

To obtain the curvatures in a more manageable form (in particular with the correct dimensions of orientation variation per unit length) we resort to the shape operator defined as

$$s^\alpha_\beta = a^{\alpha\gamma} b_{\gamma\beta}.$$

The shape operator is very informative, but is not symmetric and has mixed indices. Consequently, most analysis of surfaces is carried out using a and b. We note, however, that the two most important characteristics of a surface, the mean curvature and Gaussian curvature, are given by

$$H = \frac{1}{2} Tr(s) = \frac{1}{2} s^\alpha_\alpha = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \frac{|b|}{|a|} = |s| = \kappa_1 \kappa_2.$$

We may now ask what are the conditions under which we could interpret the equation above as a PDE prescribing the tangent vectors given the first and second fundamental forms, a and b. As before the integrability of the equations for the configuration $\mathbf{r}$ from the prescribed tangent vectors require $\partial_\alpha \partial_\beta \mathbf{r} = \partial_\beta \partial_\alpha \mathbf{r}$ which is satisfied trivially by the symmetry of Γ and b. The integrability condition for the equations of the tanget vectors read

$$\partial_\gamma \partial_\alpha \partial_\beta \mathbf{r} - \partial_\alpha \partial_\gamma \partial_\beta \mathbf{r} = 0.$$

Let us expand these equations

$$\begin{aligned}
0 =&\partial_\gamma(\Gamma^\delta_{\alpha\beta}\partial_\delta\mathbf{r} + b_{\alpha\beta}\hat{\mathbf{N}}) - \partial_\alpha(\Gamma^\delta_{\gamma\beta}\partial_\delta\mathbf{r} + b_{\gamma\beta}\hat{\mathbf{N}})\\
=&\partial_\gamma\Gamma^\delta_{\alpha\beta}\partial_\delta\mathbf{r} + \Gamma^\delta_{\alpha\beta}\partial_\gamma\partial_\delta\mathbf{r} + \partial_\gamma b_{\alpha\beta}\hat{\mathbf{N}} + b_{\alpha\beta}\partial_\gamma\hat{\mathbf{N}} - \partial_\alpha\Gamma^\delta_{\gamma\beta}\partial_\delta\mathbf{r} - \Gamma^\delta_{\gamma\beta}\partial_\alpha\partial_\delta\mathbf{r}\\
&- \partial_\alpha b_{\gamma\beta}\hat{\mathbf{N}} - b_{\gamma\beta}\partial_\alpha\hat{\mathbf{N}}\\
=&\partial_\gamma\Gamma^\delta_{\alpha\beta}\partial_\delta\mathbf{r} + \Gamma^\delta_{\alpha\beta}\Gamma^\eta_{\gamma\delta}\partial_\eta\mathbf{r} + \Gamma^\delta_{\alpha\beta}b_{\gamma\delta}\hat{\mathbf{N}} + \partial_\gamma b_{\alpha\beta}\hat{\mathbf{N}} - b_{\alpha\beta}a^{\eta\nu}b_{\gamma\eta}\partial_\nu\mathbf{r}\\
&- \partial_\alpha\Gamma^\delta_{\gamma\beta}\partial_\delta\mathbf{r} - \Gamma^\delta_{\gamma\beta}\Gamma^\eta_{\alpha\delta}\partial_\eta\mathbf{r} - \Gamma^\delta_{\gamma\beta}b_{\alpha\delta}\hat{\mathbf{N}} - \partial_\alpha b_{\gamma\beta}\hat{\mathbf{N}} + b_{\gamma\beta}a^{\nu\eta}b_{\nu\alpha}\partial_\eta\mathbf{r}\\
=&\partial_\delta\mathbf{r}\big(\partial_\gamma\Gamma^\delta_{\alpha\beta} - \partial_\alpha\Gamma^\delta_{\gamma\beta} + \Gamma^\nu_{\alpha\beta}\Gamma^\delta_{\gamma\nu} - \Gamma^\nu_{\gamma\beta}\Gamma^\delta_{\alpha\nu} + b_{\gamma\beta}a^{\nu\delta}b_{\nu\alpha} - b_{\alpha\beta}a^{\eta\delta}b_{\gamma\eta}\big)\\
&+ \hat{\mathbf{N}}\big(\partial_\gamma b_{\alpha\beta} - \partial_\alpha b_{\gamma\beta} + \Gamma^\delta_{\alpha\beta}b_{\gamma\delta} - \Gamma^\delta_{\gamma\beta}b_{\alpha\delta}\big)
\end{aligned}$$

We thus deduce Gauss's equation

$$\partial_\gamma\Gamma^\delta_{\alpha\beta} - \partial_\alpha\Gamma^\delta_{\gamma\beta} + \Gamma^\nu_{\alpha\beta}\Gamma^\delta_{\gamma\nu} - \Gamma^\nu_{\gamma\beta}\Gamma^\delta_{\alpha\nu} = a^{\eta\delta}b_{\alpha\beta}b_{\gamma\eta} - a^{\eta\delta}b_{\gamma\beta}b_{\eta\alpha},$$

as well as the pair of equations called the Peterson, Mainardi Codazzi equations (depending on your location on the globe), which can be elegantly rephrased using the covariant derivative with respect to a as

$$\nabla_\alpha b_{\beta\gamma} = \nabla_\beta b_{\alpha\gamma}.$$

Gauss' equation yields just one equation (despite the four indices), while the latter provide two distinct equations. Satisfying these implies that a and b could be integrated to give a surface. Failure to satisfy them means that there exists no surface for which a and b and the first and second fundamental forms, respectively. In Gauss' equation we may identify on the left hand side the Riemann curvature tensor (which for surfaces has only one independent component). Some index gymnastics may be used to show that it may be rephrased as

$$\tfrac{1}{2}a^{\alpha\beta}R^\delta_{\alpha\delta\beta} = K = \frac{|b|}{|a|},$$

where K is Gaussian curvature of the surface.

2.4.2 Dimensionally reduced theory for frustrated thin sheets

We studying slender three dimensional bodies having one small dimension we would like to describe their state and response through surface properties. This is the motivation behind dimensionally reduced 2D elasticity models such as the Foppl Von Karman theory or the Koiter theory for thin plates. For frustrated this structures we can follow similar lines and obtain a dimensionally reduced functional for frustrated thin elastic structures.

One of the main advantages of using a coordinate invariant formulation is that we are allowed to exploit convenient sets of coordinates, and not worry about transforming the equations. One of the most useful sets of coordinates

in differential geometry are called semi-geodesic coordinates. These are sets of coordinates where one of the parametric curves families are composed of geodesic lines. In $3D$ we identify a surface $\mathcal{S}$ and extend from it perpendicular geodesics parametrized by arc length. In this case the metric reads

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & 1 \end{pmatrix}, \qquad g_{ij} = g_{ij}(x^1, x^2, x^3).$$

The geodesic equation for an affine parametrization reads

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = 0.$$

The curve $\dot{x^3} = 1$ and $\dot{x}^1 = \dot{x}^2 = 0$ is a geodesic. We may thus deduce that $\Gamma^i_{33} = 0$ for all i values.

$$\Gamma^i_{33} = g^{ik}\partial_3 g_{3k} = 0, \quad \Rightarrow \partial_3 g_{3k} = 0.$$

As $g_{3k} = 0$ at $x^3 = 0$ we may deduce that $g_{3k} = 0$ for all x^3 values.

$$\bar{g} = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{12} & 0 \\ \bar{g}_{21} & \bar{g}_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $x^3 \in [-t/2, t/2]$, where t is the local thickness. We define the reduced two dimensional reference fundamental forms

$$\bar{a}_{\alpha\beta} = \bar{g}_{\alpha\beta}\big|_{x^3=0}, \qquad \bar{b}_{\alpha\beta} = -\tfrac{1}{2}\partial_3\bar{a}_{\alpha\beta}\big|_{x^3=0}.$$

As long as the surfaces of the thin body are not loaded we may show that the realized metric g has the same form as $\bar{g}$, i.e. semi geodesic with respect to x^3. Using the reference fundamental forms we may express the three dimensional elastic energy in terms of a two dimensional energy density

$$\begin{aligned} E &= \int\int\int \mathcal{W}(g, \bar{g})\sqrt{|\bar{g}|}dx^1dx^2dx^3 \\ &\approx \int\int \mathcal{W}^{2D}(a, \bar{a}, b, \bar{b})\sqrt{|\bar{a}|}dx^1dx^2. \end{aligned}$$

Carrying out a formal expansion of the elastic energy density in powers of the thickness, we obtain a reduced energy density,

$$\mathcal{W}^{2D}(x^1, x^2) = te_S(x^1, x^2) + t^3 e_B(x^1, x^2),$$

where

$$\begin{aligned} e_S(x^1, x^2) &= \frac{1}{8}\mathcal{A}^{\alpha\beta\gamma\delta}(a_{\alpha\beta} - \bar{a}_{\alpha\beta})(a_{\gamma\delta} - \bar{a}_{\gamma\delta}) \\ e_B(x^1, x^2) &= \frac{1}{24}\mathcal{A}^{\alpha\beta\gamma\delta}(b_{\alpha\beta} - \bar{b}_{\alpha\beta})(b_{\gamma\delta} - \bar{b}_{\gamma\delta}), \end{aligned}$$

and the reduced two dimensional elastic tensor is

$$\mathcal{A}^{\alpha\beta\gamma\delta} = \frac{Y}{1+\nu}\left[\frac{1}{2}(\bar{a}^{\alpha\gamma}\bar{a}^{\beta\delta} + \bar{a}^{\alpha\delta}\bar{a}^{\beta\gamma}) + \frac{\nu}{1-\nu}\bar{a}^{\alpha\beta}\bar{a}^{\gamma\delta}\right].$$

The elastic problem is defined as follows: given two reference fundamental forms $\bar{a}$ and $\bar{b}$ find the realized fundamental form by minimizing the elastic energy among all first and second fundamental forms that satisfy the compatibility conditions:

$$\begin{aligned}\partial_2 b_{11} - \partial_1 b_{12} &= b_{11}\Gamma^1_{12} + b_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - b_{22}\Gamma^2_{11}\\ \partial_2 b_{12} - \partial_1 b_{22} &= b_{11}\Gamma^1_{22} + b_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - b_{22}\Gamma^2_{12}\\ b_{11}b_{22} - b^2_{12} &= K(a_{11}a_{22} - a^2_{12}),\end{aligned}$$

2.4.3 Non-Euclidean plates and the Willmore functional

Non Euclidean plates are a special kind of thin frustrated bodies in which $\bar{b} = 0$ yet $\bar{a}$ is not Riemannianly flat.

$$\begin{aligned}w_B =& \frac{Y}{24(1+\nu)}\left(\frac{\nu}{1-\nu}\bar{a}^{\alpha\beta}\bar{a}^{\gamma\delta} + \bar{a}^{\alpha\gamma}\bar{a}^{\beta\delta}\right) b_{\alpha\beta}b_{\gamma\delta}\\ =& \frac{Y}{24(1+\nu)}\left(\frac{\nu}{1-\nu}a^{\alpha\beta}a^{\gamma\delta} + a^{\alpha\gamma}a^{\beta\delta}\right) b_{\alpha\beta}b_{\gamma\delta}\\ =& \frac{Y}{24(1+\nu)}\left(\frac{\nu}{1-\nu}s^\alpha_\alpha s^\gamma_\gamma + s^\alpha_\delta s^\delta_\alpha\right)\\ =& \frac{Y}{24(1+\nu)}\left(\frac{\nu}{1-\nu}Tr(s)^2 + Tr(s)^2 - 2\det(s)\right)\\ =& \frac{Y}{24(1+\nu)}\left(\frac{1}{1-\nu}4H^2 - 2K\right)\end{aligned}$$

where we used the Cayley Hamilton theorem states that every matrix satisfies its own characteristic equation. For a 2×2 matrix this implies

$$A^2 - tr(A)A + \det(A)I = 0 \Rightarrow tr(A^2) = tr(A)^2 - 2\det(A).$$

The functional

$$W = \int\left(\frac{1}{1-\nu}4H^2 - 2K\right)dA$$

is often called the Willmore functional.

2.5 Frustration in liquid 2D crystals

A two dimensional nematic liquid crystal is characterized by a unit vector field $\hat{\mathbf{n}}$ named the director which is indicative of a local preferred orientation of the constituents in the liquid. The constituents in a nematic liquid crystal, named *nematogens*, have a broken symmetry and are typically elongated rod-like structures. In its ground state a nematic liquid crystal attempts to align the nematogens, leading to a uniform and constant director field. However, this configuration may be distorted by imperfections, boundary conditions, and other external forces. The energetic cost of such deformations is given by the Frank free energy density:

$$F = \tfrac{1}{2}K_s(\nabla\cdot\hat{\mathbf{n}})^2 + \tfrac{1}{2}K_b\left((\hat{\mathbf{n}}\cdot\nabla)\hat{\mathbf{n}}\right)^2.$$

The first term is called the splay term, and the second is called the bending term (often the constants are numbered such that $K_s \equiv K_1$ and $K_b \equiv K_3$). In three dimensions the above terms naturally extend to their three dimensional forms and an additional term, named twist (or helicity) appears, $\frac{1}{2}K_T(\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}})^2$. An additional term (called the saddle splay) can be written as the divergence of a function, and for simplicity was omitted above.

I will skip the discussion about the simple nematic and jump straight to an exotic phase made by bent-core liquid crystals nematogens. In this case an additional coupling term in the energy arises and reads $\Delta F = -K_b\left((\hat{\mathbf{n}}\cdot\nabla)\hat{\mathbf{n}}\right)b_0$ or alternatively (up to a constant)

$$F = \tfrac{1}{2}K_s(\nabla\cdot\hat{\mathbf{n}})^2 + \tfrac{1}{2}K_b\left((\hat{\mathbf{n}}\cdot\nabla)\hat{\mathbf{n}} - b_0\right)^2.$$

We could name the distortion modes $s = |\nabla\cdot\hat{\mathbf{n}}|$ and $b = |(\hat{\mathbf{n}}\cdot\nabla)\hat{\mathbf{n}}|$ to obtain the energy in compact form:

$$F = \tfrac{1}{2}K_s s^2 + \tfrac{1}{2}K_b(b-b_0)^2.$$

This form transparently favors a zero splay configuration with a constant bend of value b_0, and one may naïvely expect that we should only consider small perturbations about this ground state. However, as we next show, there exists no ground state with vanishing splay and a constant bending.

Setting $\hat{\mathbf{n}} = (\cos(\theta), \sin(\theta))$, the vanishing splay requirement amounts to $\theta_y = \tan(\theta)\theta_x$. This gives for the bending the expression

$$b = \theta_x/\cos(\theta) = \theta_y/\sin(\theta).$$

This can be immediately shown to be incompatible with the no splay condition as

$$\theta_{xy} = -b\sin(\theta)\theta_y = -b^2\sin^2(\theta) \neq b^2\cos^2(\theta) = b\cos(\theta)\theta_x = \theta_{yx}.$$

The uniformly bent ground state is therefore frustrated and the ground state will inevitably contain some splay or display non-uniform bending. This is in fact quite intuitive. If we think of constant bend phases, the director is aligned along sections of circular arcs, yet circular arcs cannot pack equidistantly from each other.

2.5.1 The compatibility conditions

We may now write

$$s = \nabla \cdot \hat{\mathbf{n}} = \partial_x \cos(\theta) + \partial_y \sin(\theta) = -\sin(\theta)\partial_x\theta + \cos(\theta)\partial_y\theta$$

and correspondingly for the bend

$$b = |(\hat{\mathbf{n}}\cdot\nabla)\hat{\mathbf{n}}| = |(\cos(\theta)\partial_x + \sin(\theta)\partial_y)\hat{\mathbf{n}}| = |(\cos(\theta)\partial_x\theta + \sin(\theta)\partial_y\theta)\hat{\mathbf{n}}_\perp| = \cos(\theta)\partial_x\theta + \sin(\theta)\partial_y\theta$$

This means that we may write

$$\begin{pmatrix} b \\ s \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \partial_x\theta \\ \partial_y\theta \end{pmatrix}$$

Inverting this relation is straightforward

$$\begin{pmatrix} \partial_x\theta \\ \partial_y\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} b \\ s \end{pmatrix}$$

We may now treat this equation as a PDE for the reconstruction of the phase from knowledge of b and s. When we do so we obtain the compatibility condition in the form of

$$\partial_x\partial_y\theta - \partial_y\partial_x\theta = 0.$$

Expanding the above we have

$$\partial_x(\sin(\theta)b + \cos(\theta)s) = \sin(\theta)\partial_x b + b\cos(\theta)\theta_x + \cos(\theta)\partial_x s - s\sin(\theta)\partial_x\theta$$

$$\partial_y(\cos(\theta)b - \sin(\theta)s) = \cos(\theta)\partial_y b - b\sin(\theta)\partial_y\theta - \sin(\theta)\partial_y s - s\cos(\theta)\partial_y\theta$$

When subtracted from each other these equations give

$$\begin{pmatrix}\cos(\theta), & \sin(\theta)\end{pmatrix}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} s + \begin{pmatrix}\sin(\theta), & -\cos(\theta)\end{pmatrix}\begin{pmatrix}\partial_x \\ \partial_y\end{pmatrix} b + b^2 + s^2 = 0$$

The equation

$$\hat{\mathbf{n}} \cdot \nabla s - \hat{\mathbf{n}}_\perp \cdot \nabla b + s^2 + b^2 = 0,$$

is called the compatibility condition for planar two dimensional liquid crystals. Through this equation it becomes transparent that the only uniform distortion phase (in which s and b are constant) is the trivial one $s = b = 0$.

2.5.2 Obtaining the mechanical equilibrium equations

Here again it is important to perform the variation in the configuration rather than in the local distortion fields b and s. We start from a general energy of the form $F = K_s(s - s_0)^2 + K_b(b - b_0)^2$. We consider the perturbation $\theta \to \theta + \delta\theta$ and expand the splay and bend to first order in the perturbation

$$\begin{aligned}
\delta s &\approx \delta\theta\,(\sin(\theta)\partial_y\theta + \cos(\theta)\partial_x\theta) - \sin(\theta)\partial_x\delta\theta + \cos(\theta)\partial_y\delta\theta + \mathcal{O}(\delta\theta^2) \\
&\approx -\,b\,\delta\theta + (\hat{\mathbf{n}}_\perp \cdot \nabla)\delta\theta + \mathcal{O}(\delta\theta^2), \\
\delta b &\approx \delta\theta\,(\cos(\theta)\partial_y\theta - \sin(\theta)\partial_x\theta) + \cos(\theta)\partial_x\delta\theta + \sin(\theta)\partial_y\delta\theta + \mathcal{O}(\delta\theta^2) \\
&\approx s\,\delta\theta + (\hat{\mathbf{n}} \cdot \nabla)\delta\theta + \mathcal{O}(\delta\theta^2).
\end{aligned}$$

The functional variation therefore reads

$$\begin{aligned}\delta F =&2K_s(s-s_0)\delta s + 2K_b(b-b_0)\delta b\\ =&2K_s(s-s_0)(b\,\delta\theta + (\hat{\mathbf{n}}_\perp\cdot\nabla)\delta\theta) + 2K_b(b-b_0)(s\,\delta\theta + (\hat{\mathbf{n}}\cdot\nabla)\delta\theta)\\ =&2K_s(s-s_0)(\hat{\mathbf{n}}_\perp\cdot\nabla)\delta\theta + 2K_b(b-b_0)(\hat{\mathbf{n}}\cdot\nabla)\delta\theta\\ &-2K_s(s-s_0)b\,\delta\theta + 2K_b(b-b_0)s\,\delta\theta\\ =&B.T. - \delta\theta\,\nabla\cdot(\hat{\mathbf{n}}_\perp 2K_s(s-s_0)) - \delta\theta\,\nabla\cdot(\hat{\mathbf{n}}\,2K_b(b-b_0))\\ &-2K_s(s-s_0)b\,\delta\theta + 2K_b(b-b_0)s\,\delta\theta,\end{aligned}$$

where again $B.T.$ stands from boundary terms and the last equality holds only under the integral after intergration by parts. We can expand the divergence terms recalling that $\nabla\cdot\hat{\mathbf{n}} = s$and $\nabla\cdot\hat{\mathbf{n}}_\perp = -b$,

$$\nabla\cdot(\hat{\mathbf{n}}_\perp 2K_s(s-s_0)) = -2K_s(s-s_0)b + 2K_s(\hat{\mathbf{n}}_\perp\cdot\nabla)s$$

$$\nabla\cdot(\hat{\mathbf{n}}\,2K_b(b-b_0)) = 2K_b(b-b_0)s + 2K_b(\hat{\mathbf{n}}\cdot\nabla)b$$

which eventually gives the Euler-Lagrange equations

$$K_b\hat{\mathbf{n}}\cdot\nabla b + K_s\hat{\mathbf{n}}_\perp\cdot\nabla s = 0, \quad \text{in} \quad \mathcal{D}.$$

Note that neither b_0 nor s_0 made it to the Euler-Lagrange equations. They do appear in the boundary conditions

$$(K_s(s-s_0)\hat{\mathbf{n}}_\perp + K_b(b-b_0)\hat{\mathbf{n}})\cdot\hat{\boldsymbol{N}} = 0 \quad \text{on} \quad \partial\mathcal{D},$$

where $\hat{\boldsymbol{N}}$ is the outward facing unit normal to the boundary of the domain, $\partial\mathcal{D}$.

Let $\bar{g}$ be a reference metric corresponding to a uniform Gaussian curvature, $\bar{K}$. We now seek an elastic energy minimizing configuration in the space of some other constant Gaussian curvature, $K = const$. We assume the domain of consideration and the parametrization such that both the reference metric and the embedding metric g are uniformly close to the Euclidean metric, $|\bar{g} - I| \leq \delta$, and $|g - I| \leq \delta$.

To leading order in δ the divergence equation reduces to the cartesian divergence equation.

$$\partial_i S^{ij} = 0,$$

which implies the existence of a scalar function Φ (Airy stress potential) such that

$$\partial_1\partial_1\Phi = S_{22}, \quad \partial_2\partial_2\Phi = S_{11}, \quad \partial_1\partial_2\Phi = -S_{12}.$$

For simplicity we now consider a material with a vanishing Poisson ratio and set the Young's modulus to unity. In such a case the Bilaplacian of the scalar function reads

$$\triangle^2\Phi = \bar{K} - K = -\Delta K,$$

where we have made use of the linearized Gaussian curvature (in leading order in δ) where by

$$K = -\tfrac{1}{2}(\partial_1\partial_1 g_{22} + \partial_2\partial_2 g_{11} - 2\partial_1\partial_2 g_{12}).$$

Note the symmetry between embedding a hyperbolic surface in Euclidean space and the embedding of a flat surface on a positively curved space. We now consider a strip of length L and width w such that $w << L << L_{geo}$ where L_{geo} is the smallest geometric lengthcale associated with the curvatures; $\frac{1}{\sqrt{K}}$ and $\frac{1}{\sqrt{\overline{K}}}$. Assuming that away from the boundaries the solution will not depend on the coordinate along the long direction, x^2, we obtain

$$S_{11} = S_{12} = 0, \qquad S_{22} = -\frac{\Delta K}{2}\big((x^1)^2 - \frac{w^2}{12}\big).$$

The value of the constant above was set to zero by requiring the stress to have a vanishing average in the range $-w/2 \leq x^1 \leq w/2$. Upon integration we obtain for the elastic energy

$$E \propto w^5 L(\Delta K)^2.$$

The above scaling reads $E \propto w^4 A$ for strips of constant width and varying area, and scales as $E \propto \alpha^2 A^3$ for strips of constant aspect ratio $\alpha = w/L$. One can also solve the above equations is cylindrical geometry to recover the constant aspect ratio scaling $E \propto A^3$ under the assumption of axial symmetry.

It is important to emphasize how exceptional this result is. It implies uniformly frustrated elastic bodies have super-extensive energy. doubling the size of an object more than doubles its energy. Moreover, the object distinguishes between isotropic and uniaxial growth. A growing body may start growing isotropically, but at a certain size due to frustration alone continue growing along only one direction. This is termed "frustration induced filamentation".