Divisibility Tests, Recurring Decimals, and Artin
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The photo above, not the one below!
(Credits: Internet.)

(1) Tests for d = 3, 9, 11.
(a) E.g., is 142857 divisible by 3? 9? 11?
(b) *Why do they work?*

Then in my class XI summer holidays, after a MCI-like math-camp (IMO Camp 1996), and on that "mental high", I came across a book by Shakuntala Devi, which mentioned a test for d = 19:

Multiply digits from left to right by powers of 2, and add.
E.g., 361 --> 3x1 + 6x2 + 1x4 = 19. So 19 | 361.

Note that now, the tests for d = 3,9,11 remain the same: *use powers of 1, –1!*

Natural to ask: can one do this for other divisors d?

Yes – the simplest one is d = 29. Now, *use powers of 3:*
87 --> 8 + 3x7 = 29, so 29 | 87.
99 --> 9 + 3x9 = 36 --> not divisible by 29.

(2) Recently, Chika Ofili, a Nigerian-born UK schoolkid re-discovered this test for d = 7. And it made some splash in the media.

But of course, Chika and I (and Shakuntala Devi, and many others I'm sure) were rediscovering what was known to *ancient* Indian mathematicians... in Vedic mathematics! E.g. the book "Vedic Mathematics" by Sri Bharati Tirtha in 1965 outlines this method, using "*eka-adhika*" (1 more).

E.g. the *eka-adhika* for 7 is k = 5, because 5x10 = 1 more than a multiple of 7. So 364 is divisible by 7 because its "osculation" by the *eka-adhika* is 36 + 5x4 = 56, which is divisible by 7.
(**Prove that** 7 | 10n+a <==> 7 | n+5a.)

Apparently, in the Philippines, they teach divisibility by 7 as: 364 --> 36 – 2x4 = 28, which is divisible by 7. Basically, the same thing – because n+5a and n–2a differ by a multiple of 7. So...

And surely Shakuntala Devi knew Vedic mathematics too – hence, in her book.

(3) Here's another nice pattern that one notices. Map each divisor to its *eka-adhika*.
9 ---> 1
19 ---> 2
29 ---> 3
How about 39? (a) Is its *eka-adhika* equal to 4? (b) Using 4, is 351 divisible by 39? (c) Is 391?

Similarly,
1 --> literally any number will work here, so we choose its *eka-adhika* to be... 0 !
11 --> -1
21 --> is it -2 ? Check! E.g. 483.
31 --> -3 ? Check, e.g. 961.

(4) Now let's go from the "how", to the "why".

We want to divide a number *s* by a divisor *d*, using the
decimal representation (in base-10) of *s*.

First, does the above trick work for all *d*?
What was common to 11, 21, 31, and 3, and 7, and 9, 19, 29, 39 ? (Coprime to 10.)

So what if the divisor is not coprime to 10?
This is where you use a property of coprime numbers:
If a,b are coprime (gcd = 1), then $a.b \mid s \iff a|s$ *and* $b|s$.

So we test for powers of 2 and powers of 5 separately.

(5) *Powers of 5:*
A number is divisible by 5 if... ? Why? (Note, there are 2 suffix-strings.)

A number is divisible by 25 if... ? Why? (How many suffix-strings? 10^2/5^2 !)

Similarly, a number is divisible by 5^4 = 625 if... ? (How many suffix-strings?)

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Powers of 2:
A number is even if it ends with... (How many suffix-strings?)

A number is a leap year if it ends with... (How many suffix-strings?)

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Now, how about a test for 250 ?

(6) So dividing a number *s* by powers of 2 or 5 is easy, using the decimal representation of *s*.
How about a divisor *d* coprime to 10 ? I.e., 10*n* + 1, 10*n*+3, 10*n*+7, 10*n*+9.

For each *d* we want a number $\boldsymbol{k} = \boldsymbol{k}\_d$ such that

$s = \underline{s\_m \ldots s\_2\ s\_1\ s\_0} = 10^m\ s\_m + \ldots + 100\ s\_2 + 10\ s\_1 + s\_0.$

(with increasing powers of 10 from *right to left*) is divisible by *d* if (and only if!)

$s\_m + \boldsymbol{k}\ s\_{m-1} + \ldots + \boldsymbol{k}^m\ s\_0$

is divisible by *d*.

(7) This is where we use that two positive integers *a,b* have gcd = 1 (are coprime) if one can write 1 = *a.k+b.l* for integers *p,q.* Since 10 is coprime to *d*, we have $10.\boldsymbol{k} + d.l = 1$ for some integers $\boldsymbol{k}$,*l.*

Now check that property #**(6)** holds for this $\boldsymbol{k}$ – using modular arithmetic:

- If s = $\underline{s\_m \ldots s\_2\ s\_1\ s\_0}$ = 10^m s_m + ... + 100 s_2 + 10 s_1 + s_0 equals 0 mod d, then this times $\boldsymbol{k}$^m = 0 mod d. But 10.$\boldsymbol{k}$ = 1 mod d, so we get that s_m + $\boldsymbol{k}$ s_{m-1} + ... + $\boldsymbol{k}$^m s_0 = 0 mod d. Which is our divisibility test!
- **<u>Prove the converse.</u>**

(8) What happens in any other base B>1? Once again, if d divides a power of B then we proceed as above (for divisibility by 4, or 25).

Otherwise d is coprime to B (**<u>prove this</u>**). But then there exists $\boldsymbol{k}$ such that B . $\boldsymbol{k}$ = 1 mod d.
In that case, we proceed as above.

(9)(a) Now for my starting example: **<u>prove that</u>** in any base B, "2024" is always divisible by "11" !

(b) How about "2024" being divisible by *eleven*? **<u>Find</u>** all such bases B – there are infinitely many of them. (A possibly useful tip: B^2 – B + 2 modulo eleven is also B^2 – B – 20.)

(10) How about the "linear growth" phenomenon in #**(3)** above?
We want to examine how $\boldsymbol{k}$_{10n+a} – for a fixed a among {1,3,7,9} – depends on n.
Suppose B . $\boldsymbol{k}$_a = 1 + a . l for some integer l>0. Then check:

1+($B.n$+a)l = 1 + $a.l$ + $B.l.n$ = B . ($\boldsymbol{k}$_a + $l.n$).

Hence we can take $\boldsymbol{k}$_{$B.n$+a} = l . n + $\boldsymbol{k}$_a.
So we have not only proved #**(3)**, but we have also found out the common increment in the arithmetic progression!

(11) As a final note on divisibility tests: they don't hold only for numbers! They hold whenever you can perform the Euclidean algorithm (under a few more assumptions). What is another such system? Here is one: *the set of polynomials, with real (or rational, or complex) coefficients.*

(a) In this system, suppose d =X – c for some scalar c,
and we want to check if a polynomial $s(X)$ is divisible by (X – c) or not, where c is nonzero.
What is $\boldsymbol{k}$_d here? [<u>Hint:</u> Write X coprime to (X–c) via: $l.X$ + k.(X–c) = 1.]

(b) Note that the base B is in fact X ! (To do the Euclidean algorithm.) And,
X . $(1/c)$ – 1 = $(1/c)$ $(X – c)$ is divisible by d, i.e., B . $(1/c)$ = 1 mod d.
Hence we can take $\boldsymbol{k}$_d = $1/c$.

So, the <u>polynomial</u> $s(X) = s\_m\ X$^m + ... + $s\_1\ X$ + s_0 is divisible by X – c, if and only if
the <u>scalar</u> s_m + s{m-1} $(1/c)$ ... + $s\_0\ (1/c)$^m is divisible by X – c.

(c) But how can a scalar be divisible by X – c ? If and only if it is zero!
If and only if c^m times this scalar is zero.
But c^m times this scalar is exactly... $s\_m\ c$^m + ... + $s\_1\ c$ + s_0 = $s(c)$!

(d) So we just showed that (X – c) divides $s(X)$ if and only if $s(c)$ = 0.
In other words, our divisibility test, adapted to polynomials, is simply the <u>Factor Theorem</u> from high school.

PART 2 – RECURRING DECIMALS – I. The length of the recurring part of 1/*d*

(1) That same summer, I also saw a "magical" property of 1 4 2 8 5 7 – in "*Figuring: The Joy of Numbers*", also by Shakuntala Devi:

142857 x 1 = 142857
x 2 = 285714
x 3 = 428571
x 4 = 571428
x 5 = 714285
x 6 = 857142

(a) (Why is this so?) And then she mentioned the all-important fact: 142857 x 7 = 999999.

(b) Can you see why this is the key? Because now we can firstly write 1/7 in decimal form:
1/7 = 0. (999999) (999999) ... / 7 = 0. (142857) (142857) ...

So 142857 is just the repeating part of 1/7! But then, what is 10/7 ?

- On the one hand, 10/7 = 1 + 3/7,
- and on the other, 10/7 is the repeating decimal 1. (428571) (428571) ...

And so we obtain: 3 x 142857 = 428571.
Similarly, 100/7 = 14 + 2/7 = 14. (285714) (285714) ..., so 2 x 142857 = 285714. And so on.

(2) Let us write, for each digit in 142857, the multiple of 142857 that starts with that digit:
1 4 2 8 5 7 1 4 2 ...
(1 3 2 6 4 5) 1 3 2 ...

On the other hand, recall the divisibility test for 7: we had ***k*** = 5. So, the sequence of powers of 5 (taken modulo 7) to multiply would be:
1, then 5, then 5^2 = 4, then ? (Check:) 6, 2, 3.

This sequence 1 5 4 6 2 3 is *exactly* the reverse of the above sequence of multiples of 142857!
Why? (From left to right we have powers of 3, or of 10, modulo 7,
which means that from right to left we have powers of 1/10 modulo 7, i.e. of ***k*** = 5.)

(3) From now on, we will focus only on repeating decimals.
First of all, why is it that the decimal expansion is *unique*? Well, the first digit of 1/*d* is the integer part of 10/*d*. (**Think of** what the other digits are, yourselves.)

Second, why is it that 1/7 has 6 digits in its repeating part? This has two aspects:

- Having six digits means that 999999 is divisible by 7.
- (We'll see this later:) *Exactly* 6 digits means – *no smaller sequence of nines is divisible by 7.*

So now one can ask: does something similar hold for other prime numbers – or any numbers *d* – that are coprime to 10? (Because 1/5 or 1/2 have terminating decimal expansions.)

(4) All this was also during that same summer in high school.
Without any intuition for this, or mathematical knowledge, I went after this question via "brute force". Namely, I worked out 1/*d* for *d* = 1, 3, 7, 9, 11, 13, 17, 19, ..., 101, 103, 107, 109

– including for the non-prime *d* – in 1996. So I only had an 8-digit calculator!

For example: 1/17 ? (Now we know that there are at most 16 digits.) When I did this, my calculator gave: 1/17 = 0.0588235 (with the last digit possibly rounded up). How to get the remaining digits?

- Get the decimal without the final digit, times 17: 058,823 * 17 = 999,991.
  This leaves 9, so do 9/17 next = 0.5294118.
- Get the decimal without the final digit, times 17: 529,411 * 17 = 8,999,987.
  This leaves 13, so do 13/17 next = 0.7647059.
- Now combine the strings obtained (without the final digits), and look at the first 16 digits:
  1/17 = 0.(05882352 94117647)...

(5) 1/17 was still ok. But how about e.g. 1/109 = ? Now we can simply type it into Wolfram Alpha, which says: https://www.wolframalpha.com/input?i=1%2F109
0.009174311926605504587155963302752293577981651376146788990825688073394495412844036697247706422018348623853211...0092 (period 108).
In full? Just type into Wolfram Alpha: "1/109 to 112 decimal places" to get:
1/109 = 0.009174311926605504 587155963302752293 577981651376146788
990825688073394495 412844036697247706 422018348623853211
0092

But I only had an 8-digit calculator. So I kept going, 6-7 digits at a time...
I truly... *worked it till my fingers bled.*
Was the Summer of... '96!
(Those were the best days of my life?)

(6) And those experiments did make me see some patterns.
Suppose $d < 110$ is coprime to 10. Then:
the rational number $1/d$ has a recurring decimal expansion, say $1/d = 0.(a_1 \ldots a_n)(a_1 \ldots a_n)\ldots$;
moreover, the string of n nines (same n) is divisible by d: $d * (a_1 \ldots a_n) = 99\ldots9 = 10^n - 1$.

Are these two events related? And why should such a string of nines exist in the first place?

(7) For *d* a prime, we know why this is so: due to *Fermat.* What he showed in 1640, was that if *p* is any prime number coprime to 10, then a sequence of $(p - 1)$-many nines is always divisible by *p.*
I.e., $p \mid (10^{p-1} - 1)$. More generally:

Fermat's *l*ittle Theorem. If a prime *p* does not divide an integer *B*, then *p* divides $B^{p-1} - 1$.

[I won't prove "F*l*T" here (nor prove "F*L*T"!) since (a) you might know it already, but also
(b) we won't need it because we will work more generally than for *d* a prime.]

So in any base *B*, if *p* does not divide *B* then $1/p$ has at most $p - 1$ digits in its recurring part.
We can say more: the number of digits is in fact a *factor* of $p - 1$. (Shown more generally, below.)

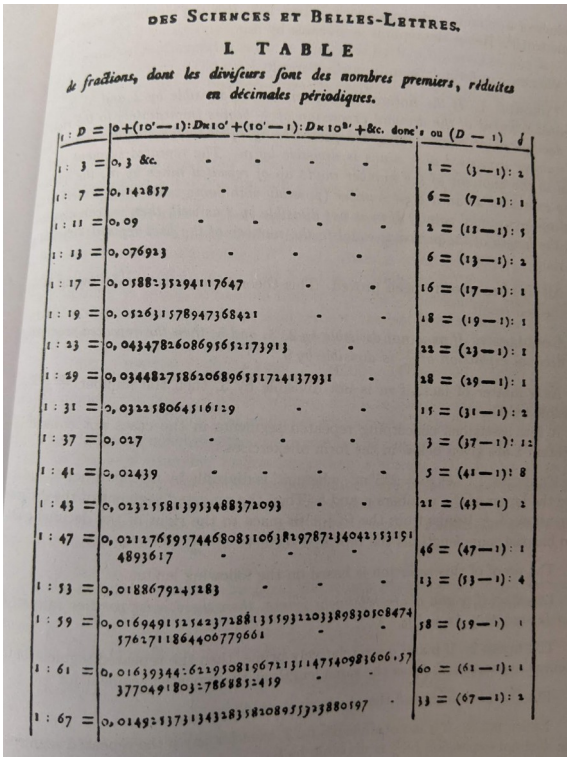
Lemma 1. *The number n_p of digits in the repeating part of $1/p$ (for a prime p not dividing 10) is a factor of $p - 1$. And for $p>5$, it is the length of the smallest string of ones* 11...1 *that is divisible by p.*

The last line was observed as early as... 1773! By Johann Bernoulli III, in *Proc. of Berlin Academy*.

(8) Now let's prove Lemma 1, not only for primes, but for any integer *d* coprime to 10. Or to *B*.

Theorem 2. *Suppose B,d* >1 *are coprime integers.*
(a) *The number n_d of digits in the recurring base-B expansion of* 1/*d is at most d. (In fact, at most φ(d).)*
(b) *This number n_d equals the length of the smallest string of "nines" that is divisible by d.*
[Where by "nines" we mean the digit (*B* – 1).]
(c) *If a string of N "nines" in base-B is divisible by d, then N has to be a multiple of n_d. (And conversely.)*

(**<u>Now you can see</u>** why Bernoulli used "1"s – if *p* is not 3, and some string of nines is divisible by *p*, then so is the string after dividing by nine.)


DES SCIENCES ET BELLES-LETTRES.

I. TABLE

de fractions, dont les diviseurs sont des nombres premiers, réduites en décimales périodiques.

1 : D =	0 + (10′ − 1) : D × 10′ + (10′ − 1) : D × 10^8′ + &c. donc s ou (D − 1)	d
1 : 3 =	0, 3 &c.	1 = (3−1) : 2
1 : 7 =	0, 142857	6 = (7−1) : 1
1 : 11 =	0, 09	2 = (11−1) : 5
1 : 13 =	0, 076923	6 = (13−1) : 2
1 : 17 =	0, 0588235294117647	16 = (17−1) : 1
1 : 19 =	0, 052631578947368421	18 = (19−1) : 1
1 : 23 =	0, 0434782608695652173913	22 = (23−1) : 1
1 : 29 =	0, 0344827586206896551724137931	28 = (29−1) : 1
1 : 31 =	0, 0322580645161229	15 = (31−1) : 2
1 : 37 =	0, 027	3 = (37−1) : 12
1 : 41 =	0, 02439	5 = (41−1) : 8
1 : 43 =	0, 0232558139534883720093	21 = (43−1) 2
1 : 47 =	0, 0212765957446808510638297872340425531914893617	46 = (47−1) : 1
1 : 53 =	0, 0188679245283	13 = (53−1) : 4
1 : 59 =	0, 0169491525423728813559322033898305084745762711864406779661	58 = (59−1) 1
1 : 61 =	0, 0163934426229508196721311475409836065573770491803278688524590	60 = (61−1) : 1
1 : 67 =	0, 0149253731343283582089552238805970	33 = (67−1) : 2

Johann Bernoulli III, 1773
(Credit: Kvant Selecta)

Proof (only in base 10; **<u>prove this for other bases</u>**)**:**

(a) Consider the remainders modulo *d*, when dividing
0 = 10^0 – 1,
9 = 10^1 – 1,
99 = ...,
and so on. In the first *d*+1 of these, some two sequences of nines leave the same remainder, by "<u>Dirichlet's box principle</u>" (what do you know it as?).
So their difference is divisible by *d* – and it is of the form 99...900...0, with at most *d* nines (why?).
But this is of the form 99...9 x 10^*j*, and since *d* is coprime to 10, it must divide the string of nines.

Now take the **shortest** string of nines divisible by *d.*
Say it has minimum length *m*, and denote the quotient by <u>a_1 ... a_*m*</u>
(where we may have initial 0s, e.g. 1/13). But then, 1/*d* = 0. (a_1 ... a_*m*) (a_1 ... a_*m*) ...
So (i) firstly, 1/*d* does have a repeating expansion, i.e. *n_d* exists / is finite!
And (ii) secondly, *n_d* is at most this minimum length *m*, which is at most *d* from above.
(And if you know Euler's theorem, *n_d* is at most *φ(d)* < *d.*)

(b) We now want to show that in fact *n_d* = *m.*
What do we know? 1/*d* = 0. (a_1 ... a_*m*) and also 1/*d* = 0. (b_1 ... b_{*n_d*})
with the second expression being the *smallest*-length recurring-string.

<u>Convince yourself</u> that *m* must be an integer multiple of *n_d.*
(Remember from above that the decimal expansion is unique!)

Now let *r* = "repeating part" = the (shortest) string with digits <u>b_1 ... b_{*n_d*}</u>. Then the number
<u>a_1 ... a_*m*</u> equals (*r r* ... *r*) = *r* (1 + 10^{*n_d*} + 10^{2.*n_d*} + ... + 10^{*K*.*n_d*})
= *r* (10^*m* – 1) / (10^{*n_d*} – 1).

But on the other hand, <u>a_1 ... a_*m*</u> = (10^*m* – 1) / *d* ! Equating these two,
r = <u>b_1 ... b_{*n_d*}</u> = (10^{*n_d*} – 1) / *d*,
i.e., the string of *n_d* nines is also divisible by *d.* But *n_d* was a factor of *m*. **<u>Hence</u>**(?) *n_d* = *m.*

(c) Now suppose a string *A* of *N*-many nines is divisible by *d* – as is the string of *n_d* nines.
Then *A* – *r**10^{*N* – *n_d*} = 99..99<u>99...99</u> – 99..99<u>00...00</u> is also divisible by *d*,
and this is a smaller string of nines, specifically, *N* – *n_d* nines.
Again subtract from it, *r* times a suitable power of 10, to get a still smaller string of *N* – 2*n_d* nines.
And keep going.

Eventually, we would get a string of *j*-many nines, where *j* = remainder of *N* divided by *n_d*.
And this string is divisible by *d*. Since *n_d* = *m* = length of smallest string of nines, and *j* < *n_d*,
hence *j* = 0. But this means that *N* is a multiple of *n_d*.
(E.g. in Lemma 1, F<u>l</u>T implies that N = p – 1 works! Hence n_p is a factor of p – 1, as claimed.)
(**<u>Prove the converse part.</u>** E.g., 999,999 = 999 * 1001, 999,999,999 = 999 * 1001001, etc.)
(QED)

(9) This was the main result. Now for a small diversion – for *d* a *prime*. Two facts on 142857:

- Break it up into two halves and add: 142+857 = 999!
- Break it up into *thirds* and add: 14+28+57 = 99!

Similarly, 1/13 = 0. (076923) (076923) ... --> again, 076 + 923 = 999 and 07+69+23 = 99 !
(Notice, this is *not* true for non-prime *d*. E.g., 1/33 = 0.030303...)

What is the general fact underlying this? Recall, the length *n_p* of the recurring part of 1/*p* is called the *order* of 10 modulo *p*. (Note that the order of 10 modulo 3 is 1, which is odd.)

Lemma 3. Say the order of 10 modulo a prime *p*>5 is even: *n_p* = 2*n*. If the recurring part of 1/*p* has digits <u>a_1 a_2 ... a_{2*n*}</u>, then <u>a_1 a_2 ... a_*n*</u> + <u>a_{*n*+1} ... a_{2*n*}</u> = 99...9 (*n* nines).

Proof: (First work out for 142857?) This is equivalent to saying that the two recurring decimals
0. (<u>a_1 a_2 ... a_*n* a_{*n*+1} ... a_{2*n*}</u>) and 0. (<u>a_{*n*+1} ... a_{2*n*} a_1 ... a_*n*</u>)
add up to 0. 999... = 1, if and only if their sum has no fractional part.

But the first decimal is precisely 1/*p*, and the second is precisely (the fractional part of) (10^*n*)/*p*,
so we need to show: 1 + 10^*n* is divisible by *p*.

Now *p* divides 10^{2*n*} – 1 = (10^*n* – 1).(10^*n* + 1),
and it doesn't divide the first factor, since *2n* is the least such integer / order of 10 modulo *p*,
so *p* must divide the second factor, as desired.
(QED)

(10) What about generalizing 14+28+57 = 99? Try a larger prime, to formulate the correct claim:
1/17 = 0. (05882352 94117647) ... --> 16 digits, and if we break it into four parts and add, we get:
0588 + 2352 + 9411 + 7647 = 9999 + 9999. So in general, we claim:

Lemma 4. Suppose 1/*p* has *m* . *n* digits in its recurring part, with *m* > 1.
Split it into groups of *n* and add them up. Then we obtain a multiple of 99...9 (*n* nines).

Proof: By the same logic as above, we need to show that the sum of the recurring parts starting with the *m* different *n*-length strings, has no fractional part. But this is precisely the fractional part of
(1 + 10^*n* + ... + 10^{(*m* – 1)*n*}) / *p* = *X*/*p*, say;
and via geometric series, *X* . (10^*n* – 1) = 10^{*m*.*n*} – 1, which is a multiple of *p*.
<u>Hence</u>(?) so is *X*.
(QED)

(11) Many of these facts can be found in a nice article in a Russian periodical for students: *Kvant*.
I obtained a copy of it in 2003 when I was in USA for my PhD.
But where did I buy it? Coincidentally: in the bookstore in... IISc Bangalore!

(12) Finally, the main question: what is the *period* n_p = length of the recurring part of $1/d$?
We have seen what happens for $d = p$ = prime ---> n_p is a factor of $p - 1$, and equals the order of 10 modulo p. As we now see, the answer for general n and base B depends *only* on the shortest string of "nines" ($B^{n\_p} - 1$) for prime divisors $p|d$!

To illustrate, let us take an example: $d = 27 . 11 = 297$.

Now 1/27 = 0.037 037 ... --> 27*37 = 999. So, 999 is divisible by 27, so is 999,999, etc.

Similarly, 1/11 = 0.09 09 ... --> 11*9 = 99. So, 99 is divisible by 11, so is 9999, etc.

What is the smallest string of nines that is divisible by 297?
It is the one that is divisible by *both* 27 and by 11, since they are coprime.
So by **(8)** above, it is the shortest string whose length is divisible by both $n\_27 = 3$ and $n\_11 = 2$!
I.e., their LCM. This implies:

Proposition 5. *Suppose $d = p\_1^{a\_1} ... p\_r^{a\_r}$ is coprime to 10.*
If n_i = period $(1 / p\_i^{a\_i})$, then period(1/d) = LCM($n\_1, n\_2, ..., n\_r$).

That is,$1/d$ has LCM($n\_1, n\_2, ..., n\_r$) digits in its recurring part.

(13) Finally, we want to answer: what is the period of $1/p$, or $1/p^2$, or $1/p^3$, ... ?

Lemma 6. Fix a prime p coprime to 10, and let n_1, n_2, ... be the number of digits in the recurring part of $1/p$, $1/p^2$, ... respectively. Then $n\_\{i+1\} = n\_i$ or $p . n\_i$ (for all i).

Proof: Let q = quotient $(10^{n\_i} - 1)/p^i$. If this is divisible by p then $n\_\{i+1\} = n\_i$.
If it is not divisible, then as we saw above, the string of $n\_\{i+1\}$ nines is divisible in particular by p^{n_i}, so $n\_\{i+1\}$ is a multiple of n_i.

Suppose $n\_\{i+1\} = N . n\_i$, with N the smallest. Then,
$10^{N.n\_i} - 1 = (10^{n\_i} - 1) * (1 + 10^{n\_i} + 10^{2.n\_i} + ... + 10^{(N-1).n\_i})$,
so the prime p must divide the second factor.
But looking at the first factor, 10^{n_i} is 1 mod p,
so the sum in the second factor is N mod p. And N is the smallest, so $N = p$.

(14) Given Lemma 6, we see that
1/3 = 0.(3), 1/9 = 0.(1), and since 1 is not divisible by 3, 1/27 has 3 repeating digits.
(In fact, 1/27 = 0.(037).)
But what about 1/81 – does it have 3 digits or 9 digits?

Similarly, 1/7 = 0. (142857), and since 142857 is not divisible by 7 (check),
1/49 has 42 repeating digits.
But what about 1/343 – does it have 42 digits or 42*7 digits?

The final step is: Once the digits start to increase (not from 1/3 to 1/9, but from 1/9 to 1/27), they will *always* increase!

Proposition 7. *Fix a prime p coprime to 10, and let n_i = period(1 / p^i).*
If n_{i+1} = p . n_i for some i, then n_{j+1} = p . n_j for all j>i.

Proof: Suppose not. Then there is some $j>i$ such that $n_{j+1} = n_j = p \,.\, n_{j-1}$.
Let $n_{j-1} = D$, say. Then $(10^{p.D} - 1)$ is divisible by p^{j+1} and $(10^D - 1)$ is divisible by p^{j-1} *but not by p^j*, so:
p^2 divides $(1 + 10^D + ... + 10^{(p-1)D})$.

We already know that $10^D = 1$ modulo p. Suppose it $= 1 + p.c$. Then using the binomial theorem,
$1 + 10^D + ... + 10^{(p-1)D}$
$= 1 + (1 + p.c) + (1 + p.c)^2 + ... + (1 + p.c)^{p-1}$
$= 1 + (1 + p.c) + (1 + 2.p.c) + ... + (1 + (p-1).p.c)$ + a multiple of p^2
$= p + p \,.\, c \,.\, (1 + 2 + ... + (p-1))$ + a multiple of p^2
$= p + p \,.\, c \,.\, p(p-1)/2$ + a multiple of p^2,

and since $p - 1$ is even, this is p modulo p^2 --> contradicts the underlined sentence above.
(QED)

(15) More questions (from the abstract): Is $p = 3$ the only prime for which $1/p$ and $1/p^2$ have the same recurring period in base 10?
I didn't know – so I asked some fellow PhD students.
And one of them actually wrote and ran some computer code, and it yielded... $p = 487$ also works.

(**I don't know:** Are there infinitely many of these "special" primes?
Meaning, infinitely many primes p for which the smallest positive power of 10 that is 1 modulo p, is also 1 modulo p^2 ?

(**I also don't know:** Any (or infinitely many) primes for which p^2 --> p^3, or p^4, or ... ?
Clearly, if $B - 1$ is divisible by p^r, then one has at least one such prime with p^r. What about $>r$?)

(16) Conclusion: Suppose e.g. we want to find the recurring period of 1 / (41 * 81 * 121).
41 divides 99999, so $n_41 = 5$,
for powers of 3, the periods are 1, 1, 3, (9 --> for 81),
for powers of 11, 1/11 = 0.(09), and 09 is not divisible by 11, so the period of 1/121 is 2*11 = 22.
Hence, the recurring period of 1 / (41 * 81 * 121) is:
LCM(5, 81, 22) = 5*22*81 = 8910.

Another example, this time in *binary* – so $B = 2$.
Take $d = 99007599 = 9851 * 9949$.
The periods in base 2 turn out to be 9850 and 9948.
So the recurring period of $1/d$ is their LCM, which is 48,993,900.

Thus the recurring period of 1/99007599 has 48993900 repeating digits. Such facts are useful in generating pseudo-random binary sequences.

(17) Work this out over any natural base $B>1$ (at least for B even?) instead of $B=10$.

PART 3 – RECURRING DECIMALS – II. Artin's conjecture

(18) As we saw, Johann Bernoulli III studied the recurring period of $1/p$ for primes p.

Gauss was also interested in this – specifically, the *full-period primes*.
These are the primes with recurring period $p - 1$, like 7, 17, 19, 23, 29, ...
Gauss studied these in his 1801 work *Disquitiones Arithmeticae*.

(19) In 1927, Emil Artin conjectured: There are infinitely many full-period primes in base 10.
Is this true? *96 years later, we still don't know for sure!*
In fact, Artin talked about full-period primes in base B – not just for $B = 10$:

Artin's primitive root conjecture: *If $B>1$ is not a square, then there are infinitely many full-period primes in base B.* (Made in 1927, following Allan Cunningham in 1914.)

(20) What is known by now?

- In 1967, Christopher Hooley proved Artin's conjecture, *assuming* that a "bigger" conjecture – the Riemann Hypothesis – holds in a *G*eneralised form (related to not just *n*th roots of unity, but *n*th roots of larger *B*).

- But the RH, and then the GRH, is a very big ***if***. That is, assuming it can allow you to prove very powerful results. For example, it allowed Paul Pollack to show in 2014 that for any base $B>1$ which is not a square:
  * There are 2023 *consecutive* primes, all of which are full-period primes in base *B*.
  * Or *808017424794512 875886459904961 710757005754368-billion* consecutive primes.
  * (Maybe you are reminded of a theorem on arbitrarily long strings of primes, say in AP?
  * Also, **can you see** why this is stronger than / implies Artin's conjecture?)

- Returning to Artin's conjecture: without assuming any unproved statements, it wasn't even known for fifty years, if Artin's conjecture holds for *any base at all!*

- In 1984, Rajiv Gupta and M. Ram Murty proved that it holds for *some* number from a list of 13 choices – but, they couldn't tell which number!

(21) In 1985, David Rodney (Roger) Heath-Brown reduced "13" to "7", and thereby showed:

1. *Artin's conjecture holds for every base B that is prime, with at most two exceptions.*
   (So it holds for all but two bases among { 2, 3, 5, 7, 11, ... } – that's infinitely many bases!)
   Or, AYBABT***A***: *All* your (prime)* **base** *are belong to... Artin.* [**save for two*]

2. *Artin's conjecture holds for every base B that is square-free (not divisible by any square >1), with at most three exceptions.*
   So if a computer (2), an emu (6), an ichthyostega (14), and Heath-Brown walk into a bar – named *\Q-bar* – and the bartender only serves those who *know* there are infinitely many full-period primes for them, then *no one gets drinks.* (And the bar shuts down: *\Q-bar is closed.*)

(**HW:** *See his theorem on the hyperlink on his name. From it, try to deduce #1 and #2 yourselves.*)
So almost 100 years after Artin stated his conjecture, we still cannot provide *a single concrete number / base* (or species with fingers) for which there are infinitely many full-period primes.
But we do know that there are infinitely many of these bases. *No construction, but **lots** of existence!*