Homogenization of a quasilinear elliptic problem in a two-component domain with L^1 data

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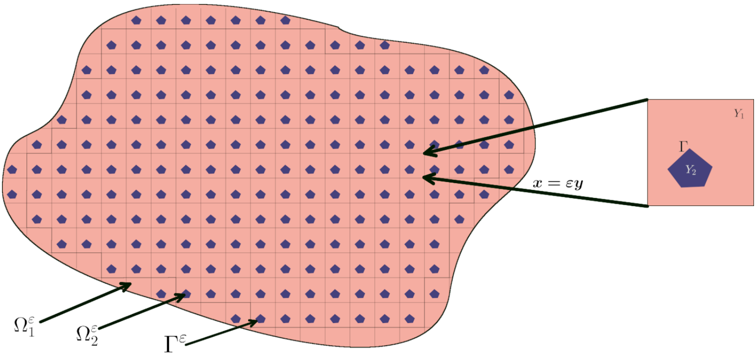
Outline

The two-component domain and the reference cell



Statement of the Problem

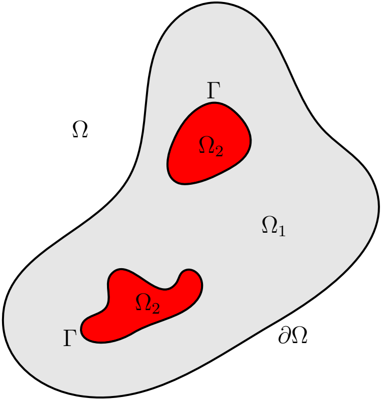
The main goal of this research is to study the homogenization of the following quasilinear problem in the two-component domain described above:

$$\begin{cases} -\mathrm{div}(A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}(A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -\varepsilon^{-1}h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

with an L^1 data and a not globally bounded matrix field, where u_i^ε, $i = 1, 2$, denotes the restriction of the solution u^ε to the set Ω_i^ε.

To this aim, we combine the notion of renormalized solutions and periodic unfolding method, which was done for the first time, to our knowledge, by P. Donato, O. Guibé, and A. Oropeza (2017). We follow a similar approach in this study.

The fixed two-component domain



The problem in a fixed domain

We consider the following elliptic problem posed in the two-component domain Ω described above:

$$\begin{cases} -\operatorname{div}(B(x,u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x,u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ (B(x,u_1)\nabla u_1)\nu_1 = (B(x,u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x,u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

- ν_1 is the unit outward normal to Ω_1;
- $u_i = u\big|_{\Omega_i}$ is the restriction to Ω_i of a function u defined in Ω.

Assumptions

We set the following assumptions:

(B1) $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ such that $0 < h_0 < h(y)$ a.e. on Γ;

(B2) $f \in L^1(\Omega)$;

(B3) B is a Carathéodory function with the following properties:

- (B3.1) $B(x,t)\xi \cdot \xi \geq \alpha|\xi|^2$, a.e. $x \in \Omega, \forall t \in \mathbb{R}, \forall \xi \in \mathbb{R}^N$; and
- (B3.2) $\forall k > 0$, $B(x,t) \in L^\infty(\Omega \times (-k,k))^{N \times N}$.

(B4) $B(x,r)$ is locally Lipschitz with respect to r, that is, for any compact subset K of $\mathbb{R}$, there exists $M_K > 0$ such that

$$|B(x,r) - B(x,s)| \leq M_K|r-s|, \quad \forall r,s \in K.$$

Note that with the assumptions on f and B (specifically (B2) and (B3.2)), a weak solution may not exist. We choose the framework of renormalized solution for our problem as it provides existence, uniqueness, and stability results.

Renormalized solutions

The notion of renormalized solutions was introduced by DiPerna and Lions for first order equations in

- R. J. DiPerna and P. L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Annals of Mathematics. Second Series, 130 (1989), pp. 321–366.

It was then further developed for elliptic equations with Dirichlet boundary conditions, with L^1 data in

- P.-L. Lions and F. Murat, *Sur les solutions renormalisées d'équations elliptiques*, (unpublished manuscript).
- F. Murat, *Soluciones renormalizadas de EDP elipticas no lineales*, Tech. Rep. R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993.

and with measure data in

- G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Annali della Scuola normale superiore di Pisa, Classe di scienze, 28 (1999), pp. 741–808.

The space V

Let V_1 be the space defined by

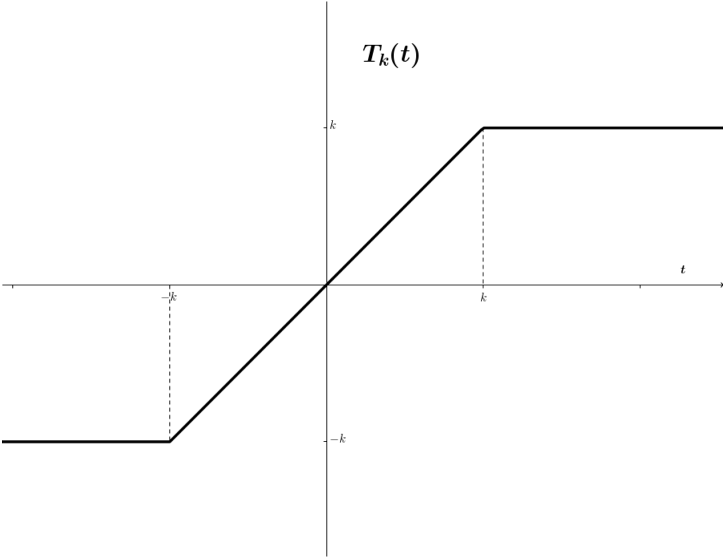
$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Define $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$, equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2.$$

This norm on V takes into account the jump on the interface. It is worth noting that this norm is equivalent to the product norm of $V^1 \times H^1(\Omega_2)$.

The truncation function T_k



Definition of gradient and trace of u

The following proposition is a generalization of Lemma 2.1 of Benilan et al. (1995), and Proposition 2.3 of Guibé and Oropeza (2017):

Proposition

Let $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function such that $T_k(u) \in V$ for every $k > 0$. For $i = 1, 2$,

1. There exists a unique measurable function $v_i : \Omega_i \longrightarrow \mathbb{R}^N$ such that

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i|<k\}} \quad \text{a.e. in } \Omega_i,$$

where $\chi_{\{|u_i|<k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |u_i(x)| < k\}$. We define v_i as the gradient of u_i and write $v_i = \nabla u_i$.

2 If

$$\sup_{k\geq 1} \frac{1}{k} \| T_k(u) \|_V^2 < \infty,$$

then there exists a unique measurable function $w_i : \Gamma \longrightarrow \mathbb{R}$, for $i = 1, 2$, such that

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma,$$

where $\gamma_i : H^1(\Omega_i) \longrightarrow L^2(\Gamma)$ is the trace operator. We define the function w_i as the trace of u_i on Γ and set $\gamma_i(u_i) = w_i$.

Our definition of a renormalized solution of (1)

Definition

Let $u : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$ be a measurable function. We say that u is a renormalized solution of (1) if

$$T_k(u) \in V, \qquad \text{for any } k > 0; \tag{2a}$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \qquad \text{for any } k > 0; \tag{2b}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{|u|<n\}} B(x,u)\nabla u \cdot \nabla u \, dx = 0; \tag{3a}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \tag{3b}$$

and for any $S_1, S_2 \in C^1(\mathbb{R})$ with compact support, u satisfies

$$\begin{aligned}\int_{\Omega_1} S_1(u_1)B(x,u_1)\nabla u_1 \cdot \nabla \psi_1 \, dx + \int_{\Omega_1} S_1'(u_1)B(x,u_1)\nabla u_1 \cdot \nabla u_1 \, \psi_1 \, dx \\ \int_{\Omega_2} S_2(u_2)B(x,u_2)\nabla u_2 \cdot \nabla \psi_2 \, dx + \int_{\Omega_2} S_2'(u_2)B(x,u_2)\nabla u_2 \cdot \nabla u_2 \, \psi_2 \, dx \\ + \int_{\Gamma} h(x)(u_1 - u_2)(\psi_1 S_1(u_1) - \psi_2 S_2(u_2)) \, d\sigma \\ = \int_{\Omega_1} f \psi_1 S_1(u_1) \, dx + \int_{\Omega_2} f \psi_2 S_2(u_2) \, dx, \end{aligned} \tag{4}$$

for all $\psi \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

Remark

Observe that (4) can be obtained by formally choosing

$$S(u)\psi = (S_1(u_1)\psi_1, S_2(u_2)\psi_2)$$

as test function in (1), then integrating the terms. However, this does not justify how the integrals in (4) make sense. The regularity assumptions (2a) and (2b) assure that all the terms in (4) are well defined.

Existence and uniqueness results

Theorem (R.F. and O. Guibé (2021))

Suppose assumptions (B1)-(B3) hold. Then there exists a renormalized solution for (1). *Moreover, if (B4) also holds, then the renormalized solution of* (1) *is unique.*

To prove the existence result, we consider an approximate problem with a data f^ε belonging to $L^2(\Omega)$ and show that the pointwise limit u of the sequence of solutions $\{u^\varepsilon\}$ (up to a subsequence) is a renormalized solution of (1).

Concerning the uniqueness, we use the method developed by Blanchard et al. (2005), and Feo and Guibé (2017). Here, we also need to prove a sign lemma on the interface Γ, to deal with the difficulties that arise from the boundary integral related to the jump of the solution.

Back to the periodic problem

Let us discuss now the homogenization results for the problem presented above

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (A^\varepsilon(x,u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x,u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -\varepsilon^{-1}h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

with an L^1 data and a not globally bounded matrix field, where u_i^ε, $i=1,2$, denotes the restriction of the solution u^ε to the set Ω_i^ε.

In general, the proportionality of the jump of the solution and the flux on the interface depends on ε^γ, where $\gamma \leq 1$. We chose to study the case $\gamma = -1$ due to its particularity that the corresponding cell problem presents a jump on the reference interface.

Assumptions

We prescribe the following assumptions:

(A1) $f \in L^1(\Omega)$;

(A2) h is a $Y-$periodic function in $L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ such that $0 < h_0 < h(y)$ a.e. on Γ, and set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \Gamma^\varepsilon;$$

(A3) $A : (y,t) \in Y \times \mathbb{R} \mapsto A(y,t) \in \mathbb{R}^{N\times N}$ is a real matrix field such that $A(\cdot,t) = \{a_{ij}\}_{i,j=1,\ldots,N}$ is $Y-$periodic for every t, A is a Carathéodory function with the following properties:

(A3.1) $A(y,t)\xi\cdot\xi \geq \alpha|\xi|^2$, a.e. $y \in Y$, $\quad \forall t \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N$;

(A3.2) $A(y,t) \in L^\infty(\Omega \times (-k,k))^{N\times N}, \quad \forall k > 0,$

and set

$$A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right),$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

(A4) The matrix field $A(y,t)$ is locally Lipschitz continuous with respect to the second variable, that is, for every $r > 0$, there exists a positive constant M_r such that

$$|A(y,s) - A(y,t)| < M_r |s-t|, \quad \forall s,t \in [-r,r], \quad \forall y \in Y.$$

In order to apply the existence and uniqueness results from the first part, we prescribe similar assumptions to problem (P), in addition to periodicity conditions.

The space H^ε

The functional space H^ε is defined by

$$H^\varepsilon := \{u = (u_1, u_2) : u_1 \in V^\varepsilon \text{ and } u_2 \in H^1(\Omega_2^\varepsilon)\},$$

equipped with the norm

$$\|u\|_{H^\varepsilon}^2 := \|\nabla u_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla u_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^{-1}\|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2,$$

where $V^\varepsilon = \{u \in H^1(\Omega_1^\varepsilon) : u = 0 \text{ on } \partial\Omega\}$ is endowed with the norm

$$\|u\|_{V^\varepsilon} := \|\nabla u\|_{L^2(\Omega_1^\varepsilon)}.$$

Definition of renormalized solution of (P)

Definition

The function $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a renormalized solution of (P) if

$$T_k(u^\varepsilon) \in H^\varepsilon \quad \text{and} \quad (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \in L^1(\Gamma^\varepsilon), \quad \forall k > 0;$$

$$\lim_{k\to\infty} \frac{1}{k} \int_{\{|u^\varepsilon|<k\}} A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx = 0;$$

$$\lim_{k\to\infty} \frac{1}{k} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \, d\sigma = 0;$$

and for any $\psi \in C^1(\mathbb{R})$ (or equivalently for any $\psi \in W^{1,\infty}(\mathbb{R})$) with compact support, u^ε satisfies

$$\sum_{i=1}^{2} \int_{\Omega_i^\varepsilon} \psi(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \cdot \nabla v_i \, dx + \sum_{i=1}^{2} \int_{\Omega_i^\varepsilon} \psi'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon \, v_i \, dx$$
$$+ \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(v_1\psi(u_1^\varepsilon) - v_2\psi(u_2^\varepsilon)) \, d\sigma = \int_\Omega f v \psi(u^\varepsilon) \, dx,$$

for all $\psi = (\psi_1, \psi_2) \in H^\varepsilon \cap (L^\infty(\Omega_1^\varepsilon) \times L^\infty(\Omega_2^\varepsilon))$.

The periodic unfolding method

The periodic unfolding method was introduced by Cioranescu et al. in

- D. Cioranescu, A. Damlamian and G. Griso, *Periodic unfolding and homogenization*, C.R. Acad. Sci. Paris, Ser. I 335 (2002), 99-104.
- D. Cioranescu, A. Damlamian and G. Griso, *The Periodic Unfolding Method in Homogenization*, SIAM J. Math. Analysis, 40 (2008) 1585-1620.

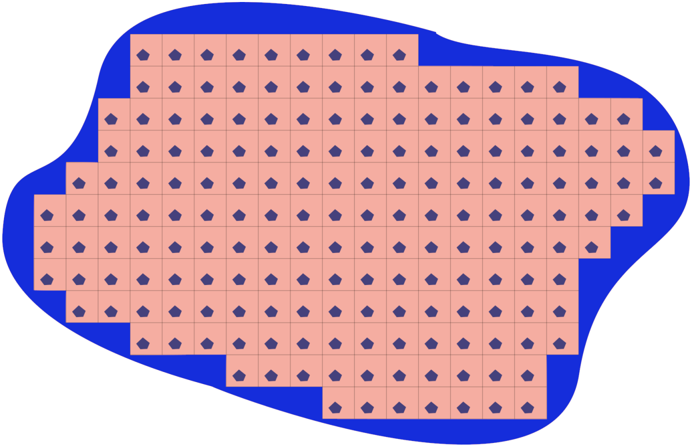
This method is then extended for perforated domains in

- D. Cioranescu, A. Damlamian, P. Donato, G. Griso and R. Zaki, *The Periodic Unfolding Method in Domains with Holes*, SIAM Journal on Mathematical Analysis, 44(2) (2012), 718-760.

and for two-component domains in

- P. Donato, H. Le Nguyen and R. Tardieu, *The periodic unfolding method for a class of imperfect transmission problems*, Journal of Mathematical Sciences, 176 (2011), 891-927.
- P. Donato and K. H. Le Nguyen, *Homogenization of diffusion problems with a nonlinear interfacial resistance*, Nonlinear Differential Equations and Applications NoDEA, 22 (2015), 1345-1380.

The sets $\widehat{\Omega}_\varepsilon$ and $\widehat{\Lambda}_\varepsilon$



The sets $\widehat{\Omega}_\varepsilon$ (orange and purple) and $\widehat{\Lambda}_\varepsilon$ (blue)

The periodic unfolding operator

Suppose the reference cell Y is defined as

$$Y = \prod_{i=1}^{N} [0, l_i),$$

for some $l_i > 0$, $i = 1, \ldots, N$. For $z \in \mathbb{R}^N$, the notation $[z]_Y$ denotes the integer part of z, i.e., $(k_1 l_1, k_2 l_2, \ldots, k_N l_N)$, for $k_i \in \mathbb{Z}$, $i = 1, \ldots, N$ such that $z - [z]_Y \in Y$.

Definition

For $i = 1, 2$, and for any function φ_i Lebesgue measurable on Ω_i^ε, the periodic unfolding operator $\mathcal{T}_i^\varepsilon$ is defined by

$$\mathcal{T}_i^\varepsilon(\varphi_i)(x,y) = \begin{cases} \varphi_i\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y_i \\ 0 & \text{a.e. } (x,y) \in \widehat{\Lambda}_\varepsilon \times Y_i. \end{cases}$$

Proposition (P. Donato, H. Le Nguyen, and R. Tardieu (2011))

Let $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$ be a bounded sequence in H^ε. Then there exist a subsequence (still denoted by ε), and three functions $v_1 \in H_0^1(\Omega)$, $\widehat{v}_1 \in L^2(\Omega, H_{per}^1(Y_1))$ with $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$ a.e. in Ω, $\widehat{v}_2 \in L^2(\Omega, H^1(Y_2))$ such that as $\varepsilon \to 0$,

$$\begin{cases} \mathcal{T}_i^\varepsilon(v_i^\varepsilon) \longrightarrow v_1 & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1,2, \\ \mathcal{T}_i^\varepsilon(\nabla v_i^\varepsilon) \rightharpoonup \nabla v_1 + \nabla_y \widehat{v}_i & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1,2. \end{cases}$$

Furthermore,

$$\frac{\mathcal{T}_1^\varepsilon(v_1^\varepsilon) - \mathcal{T}_2^\varepsilon(v_2^\varepsilon)}{\varepsilon} \rightharpoonup \widehat{v}_1 - \widehat{v}_2 \quad \text{weakly in } L^2(\Omega \times \Gamma), \quad \text{as } \varepsilon \to 0.$$

Remark

When the data f in (P) belongs to $L^2(\Omega)$ and $A(y,t)$ is bounded, one can show that the solution u^ε is bounded in H^ε (see Beltran (MS Thesis 2014)). Then by the previous proposition, we have

$$\begin{cases} \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i=1,2 \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_i & \text{weakly in } L^2(\Omega \times Y_i), \quad i=1,2, \end{cases}$$

for some $u_1 \in H_0^1(\Omega)$ and $\widehat{u}_i \in L^2(\Omega, H^1(Y_i))$, $i=1,2$.

However, in our case, the renormalized solution u^ε does not necessarily belong to H^ε. We instead work on the truncates of the solution, i.e., $T_n(u^\varepsilon)$ for $n \in \mathbb{N}$.

We achieve this by proving suitable a priori estimates.

A priori estimates

Proposition

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a renormalized solution of (P). Then there exists a positive constant C, independent of ε and k, such that for every $k > 0$ and $\varepsilon > 0$,

$$\| T_k(u^\varepsilon) \|_{H^\varepsilon}^2 \leq Ck.$$

Proposition

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a renormalized solution of (P). Then, for any $k > 0$,

$$\| T_k(u_i^\varepsilon) \|_{L^2(\Omega_i^\varepsilon)}^2 \leq C_1 k, \quad i = 1, 2,$$

for some positive constant C_1 independent of k and ε. Moreover, for $\varepsilon < 1$, we have for any $k > 0$

$$\| T_k(u_i^\varepsilon) \|_{L^2(\Gamma^\varepsilon)}^2 \leq C_2 k \varepsilon^{-1}, \quad i = 1, 2,$$

where C_2 is a positive constant independent of k and ε.

Corollary

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *be a renormalized solution of* (P). *We can find a subsequence (still denoted by* ε*) such that for any* $n \in \mathbb{N}$*, there exist* $u_1^n \in H_0^1(\Omega)$*,* $\widehat{u}_1^n \in L^2(\Omega, H^1_{per}(Y_1))$ *with* $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$ *a.e. in* $x \in \Omega$*, and* $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$ *such that the following convergences hold:*

$$\begin{cases} \mathcal{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow u_1^n & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1,2, \\ \mathcal{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla u_1^n + \nabla_y \widehat{u}_i^n & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1,2, \\ \widetilde{T_n(u_i^\varepsilon)} \rightharpoonup \theta_i u_1^n & \text{weakly in } L^2(\Omega), \quad i = 1,2, \end{cases}$$

as ε *goes to* 0*, where* $\theta_i = \dfrac{|Y_i|}{|Y|}$.

Furthermore, for any $n \in \mathbb{N}$*, as* ε *tends to* 0*,*

$$\frac{\mathcal{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathcal{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \rightharpoonup \widehat{u}_1^n - \widehat{u}_2^n \quad \text{weakly in } L^2(\Omega \times \Gamma).$$

Theorem

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a renormalized solution of (P). *Then there exists a measurable function $u_1 : \Omega \longrightarrow \mathbb{R}$, finite almost everywhere, such that (up to a subsequence)*

$$\mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 \quad \text{a.e. in } \Omega \times Y_i \text{ and on } \Omega \times \Gamma, \quad i = 1, 2,$$

with

$$T_n(u_1) = u_1^n, \quad \forall n \in \mathbb{N},$$

where u_1^n is given in the previous corollary, and

$$\mathcal{T}_i^\varepsilon(A^\varepsilon(x, T_n(u_i^\varepsilon))) \longrightarrow A(y, T_n(u_1)) \quad \text{a.e. in } \Omega \times Y_i, \quad i = 1, 2.$$

An important tool

Theorem

Let $\widehat{u}_1^n \in L^2(\Omega, H^1_{per}(Y_1))$ *and* $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$, $n \in \mathbb{N}$, *be the functions given by the previous corollary with* $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$. *Then there exists a unique measurable function*

$$\widehat{u}_i : \Omega \times Y_i \longrightarrow \mathbb{R}, \quad i = 1, 2,$$

such that for every $\mathcal{R} \in C^0(\mathbb{R})$ *with compact support, verifying*

$$\operatorname{supp} \mathcal{R} \subset [-m, m], \qquad \text{for some } m \in \mathbb{N},$$

we have

$$\mathcal{R}(u_1)\widehat{u}_i^n = \mathcal{R}(u_1)\widehat{u}_i \quad \text{a.e. in } \Omega \times Y_i,$$

for all $n \geq m$, *where* u_1 *is the function given by the previous theorem.*
Moreover, we have

$$\widehat{u}_i(x, \cdot) \in H^1(Y_i), \quad i = 1, 2, \text{ with } \mathcal{M}_\Gamma(\widehat{u}_1) = 0, \quad \text{for a.e. } x \in \Omega.$$

A suitable convergence lemma

Lemma

Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *be a renormalized solution of* (P). *We can find a subsequence (still denoted by* ε*) such that for any* $n \in \mathbb{N}$ *the following convergences hold as* ε *tends to* 0*:*

$$\begin{cases} \widetilde{T_n(u_i^\varepsilon)} \rightharpoonup \theta_i T_n(u_1) & \textit{weakly in } L^2(\Omega), \quad i = 1,2, \\ \mathcal{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow T_n(u_1) & \textit{strongly in } L^2(\Omega, H^1(Y_i)),\ i = 1,2, \end{cases}$$

and for any $S \in C^0(\mathbb{R})$ *with compact support and* $\operatorname{supp} S \subset [-n, n]$*, for* $i = 1,2$*,*

$$\begin{cases} \mathcal{T}_i^\varepsilon(S(u_i^\varepsilon)\nabla T_n(u_i^\varepsilon)) \rightharpoonup S(u_1)(\nabla T_n(u_1) + \nabla_y \widehat{u}_i), & \textit{weakly in } L^2(\Omega \times Y_i), \\ \mathcal{T}_i^\varepsilon(S(u_i^\varepsilon)) \dfrac{\mathcal{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathcal{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} & \\ \qquad\qquad \rightharpoonup S(u_1)(\widehat{u}_1 - \widehat{u}_2), & \textit{weakly in } L^2(\Omega \times \Gamma), \end{cases}$$

where $u_1 : \Omega \longrightarrow \mathbb{R}$ *is measurable and finite a.e., and* $\widehat{u}_i : \Omega \times Y_i \longrightarrow \mathbb{R}$*, for* $i = 1,2$ *with* $\widehat{u}_i(x, \cdot) \in H^1(Y_i)$ *for a.e.* $x \in \Omega$*, and* $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$.

The unfolded homogenized problem

Theorem (The unfolded homogenized problem)

Let u_1, $\widehat{u}_1$, and $\widehat{u}_2$ be functions as in the previous lemma. Let ψ_1, ψ_2 be functions in $C^1(\mathbb{R})$ (or equivalently, $\psi_1, \psi_2 \in W^{1,\infty}(\mathbb{R})$) with compact supports. Then the triple $(u_1, \widehat{u}_1, \widehat{u}_2)$ satisfies

$$\begin{cases} \displaystyle\sum_{i=1}^{2} \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_i)(\nabla(\psi_1(u_1)\varphi) + \psi_2(u_1)\nabla_y \Phi_i)\, dx\, dy \\ \displaystyle + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2)\, dx\, d\sigma_y = \int_\Omega f(x)\psi_1(u_1)\varphi(x)\, dx \\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \Phi_i \in L^2(\Omega, H^1_{per}(Y_i)), \quad i = 1, 2. \end{cases}$$

In addition, for $k > 0$, the following limits hold:

$$\lim_{k\to\infty} \frac{1}{k} \int_{\{|u_1|<k\}\times Y_i} A(y,u_1)(\nabla T_k(u_1) + \nabla_y \widehat{u}_i)(\nabla T_k(u_1) + \nabla_y \widehat{u}_i)\, dx\, dy = 0,$$

for $i = 1, 2$, and

$$\lim_{k\to\infty} \frac{1}{k} \int_{\{|u_1|<k\}\times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2\, dx\, d\sigma_y = 0.$$

Theorem

Using the assumptions and notations of the previous theorem, the function $\widehat{u}_i$, $i = 1, 2$, can be expressed as

$$\widehat{u}_i(x,y) = -\sum_{j=1}^{N} \chi_i^j(y, u_1(x)) \frac{\partial u_1}{\partial x_j}(x), \quad i = 1, 2,$$

where $\chi^j = (\chi_1^j, \chi_2^j)$ is the unique solution of the cell problem presented below written for $\lambda = e_j$ with $\{e_j\}_{j=1}^N$ being the canonical basis.

The cell problem

Define the space $W_{per}(Y_1)$, by

$$W_{per}(Y_1) = \{u \in H^1_{per}(Y_1) \mid \mathcal{M}_\Gamma(u) = 0\} \quad \text{with} \quad \|u\|_{W_{per}(Y_1)} = \|\nabla u\|_{L^2(Y_1)}.$$

The cell problem that corresponds to the homogenization of (P) is the following:

$$\begin{cases} \text{Find } \chi^\lambda(\cdot,t) = (\chi_1^\lambda(\cdot,t), \chi_2^\lambda(\cdot,t)) \in W_{per}(Y_1) \times H^1(Y_2) \text{ such that} \\ \displaystyle\sum_{i=1}^{2} \int_{Y_i} A(y,t)\nabla_y \chi_i^\lambda(y,t) \nabla_y v_i \, dy + \int_\Gamma h(y)(\chi_1^\lambda(y,t) - \chi_2^\lambda(y,t))(v_1 - v_2)\, d\sigma \\ \displaystyle= \sum_{i=1}^{2} \int_{Y_i} A(y,t)\lambda \nabla_y v_i \, dy \\ \text{for any } v = (v_1, v_2) \in H^1_{per}(Y_1) \times H^1(Y_2), \end{cases}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$.

The homogenized matrix A^0

Define the homogenized matrix $A^0(t)$, for every $t \in \mathbb{R}$, by

$$A^0(t) = A_1^0(t) + A_2^0(t),$$

where

$$A_i^0(t)\lambda = \frac{1}{|Y|}\int_{Y_i} A(y,t)\nabla_y w_i^\lambda(y,t)\,dy, \quad i = 1,2, \quad \forall \lambda \in \mathbb{R}^N,$$

with

$$w_i^\lambda(y,t) = \lambda \cdot y - \chi_i^\lambda(y,t),$$

and $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$ the solution of the cell problem.

Theorem (P. Donato and R.F. (2020))

Suppose that the matrix field $A(y,t)$ satisfies the assumptions (A3) and (A4). Then the homogenized matrix A^0 is locally Lipschitz, that is, for every $r > 0$, there exists a positive constant C_r such that

$$|A^0(s) - A^0(t)| \le C_r |s - t| \quad \forall s,t \in (-r,r).$$

Theorem (The homogenized problem in Ω)

Let u_1 be a cluster point of the sequence $\{\widetilde{\mathcal{T}_i^\varepsilon}(u_i^\varepsilon)\}$, $i = 1, 2$, as above. Then u_1 is a renormalized solution of

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla u_1) = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

that is,

$$T_k(u_1) \in H_0^1(\Omega), \quad \text{for any } k > 0,$$
$$\lim_{k\to\infty} \frac{1}{k} \int_{\{|u_1|<k\}} A^0(u_1)\nabla u_1 \nabla u_1 \, dx = 0,$$

and for every $\psi \in C^1(\mathbb{R})$ (or equivalently, $\psi \in W^{1,\infty}(\mathbb{R})$) with compact support,

$$\int_\Omega \psi(u_1) A^0(u_1) \nabla u_1 \nabla \varphi \, dx + \int_\Omega \psi'(u_1) A^0(u_1) \nabla u_1 \nabla u_1 \, \varphi \, dx = \int_\Omega f \psi(u_1) \varphi \, dx,$$

for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

If in addition, (A4) holds, then u_1 is the unique renormalized solution of the equation above and all of the sequences in the convergence lemma above converge (not just a subsequence).

Thank you very much for your attention.