Understanding Bell and Kochen-Specker Theorem

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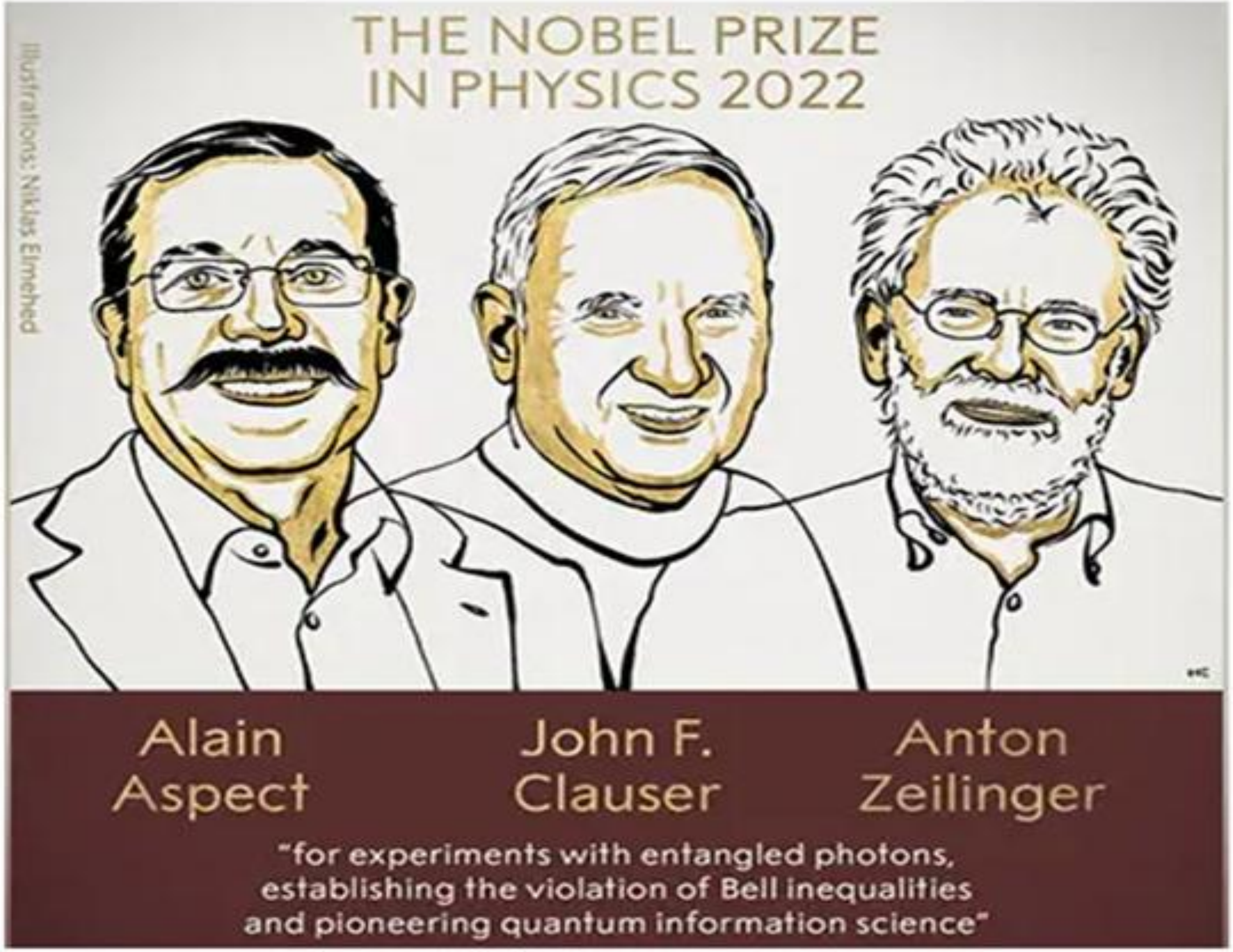


Their experimental results show that microscopic world behaves in a way that is in contradiction with Einstien's conception about nature.

Quantum Rules:

* Nature does not allow joint perfect spin measurement along different directions.

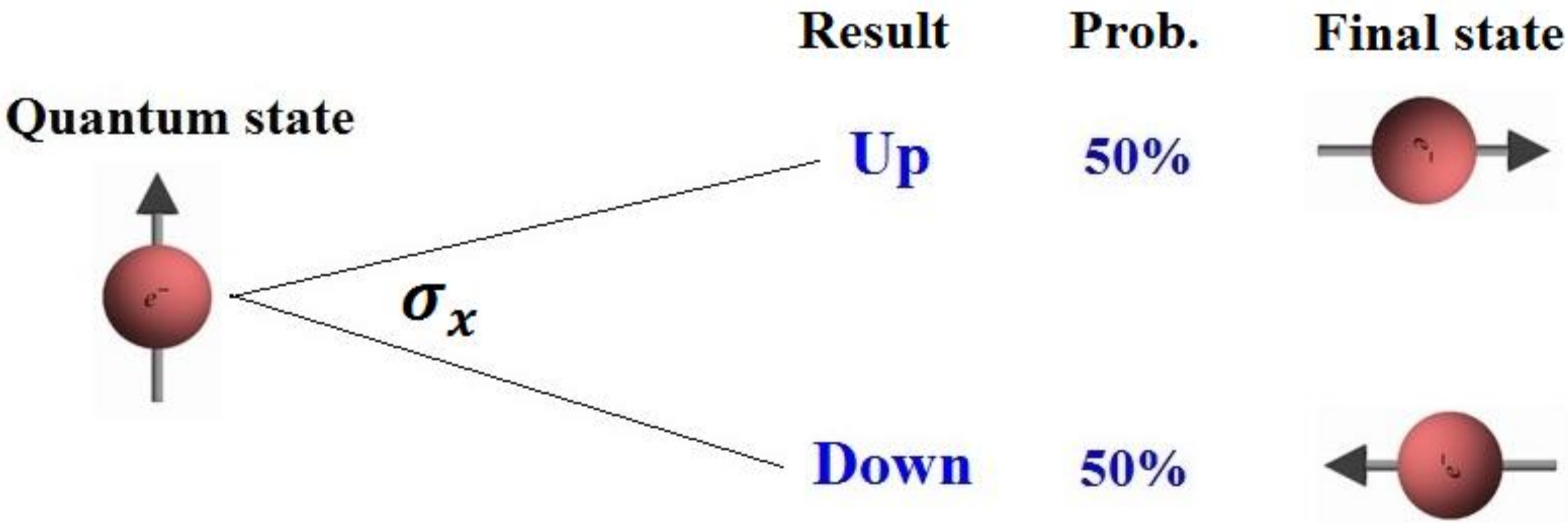
* Measurement result is one of the eigen value of the correspoding S.A. operator associated with observable and it happens with a probability given by Born's rule.

* Measurement, in general disturb the quantum state.

Quantum state $|0\rangle$	up	down
Spin measurement along z:	100%	0%
Spin measurement along x:	50%	50%
Spin measurement along (θ, ϕ) :	$cos^2\frac{\theta}{2}$	$sin^2\frac{\theta}{2}$

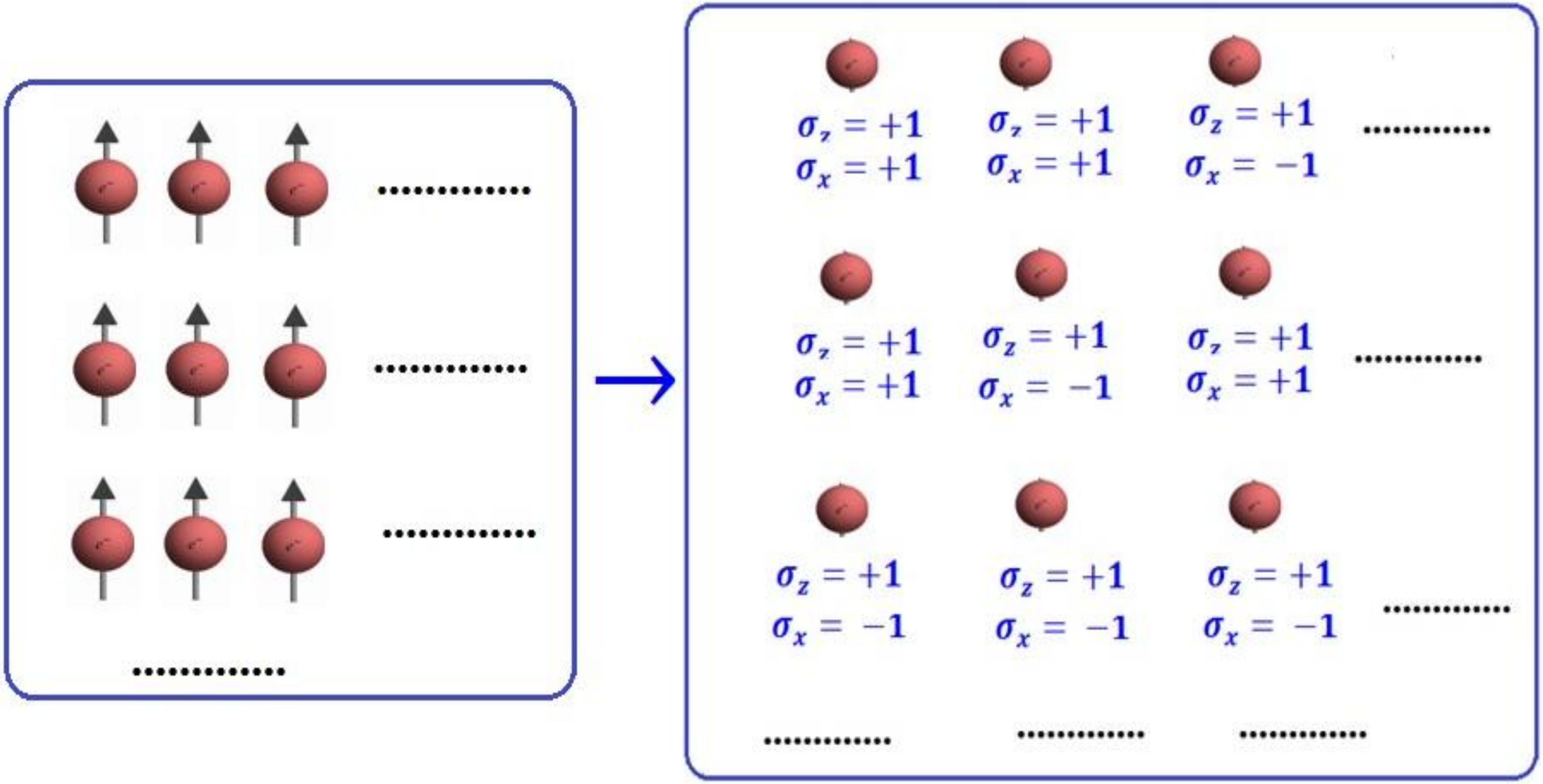
Interpretation of QM:

Measure spin along x:



Each individual particle does not have definite spin value along x. Measurement only creates the results.

What is the problem with the following picture?



For 50% $\sigma_x = +1$ **and for another 50%** $\sigma_x = -1$

Individual particles have pre-existing properties and measurement reveal those properties.

* Bohr and Heisenberg and their co-workers used to believe that such picture cannot exist.

* Von Neumann, 1932, even claimed to have proved that dispersion free state is not possible.

  (Mathematical Foundation of Quantum Mechanics, Von Neumann, 1932)

* Einstein for some time challanged the impossibility of joint measurement of some observables like position and momentum. But at one point of time he accepted this impossibility imposed by nature but never accepted the impossibility of quantum system having definite values for observables to be revealed in measurement.

Pauscal Jordan:

Observations not only disturb what has to be measured, they produce it ... .

We compel the electron to take a definite position. ...We ourselves produce the result of measurement.

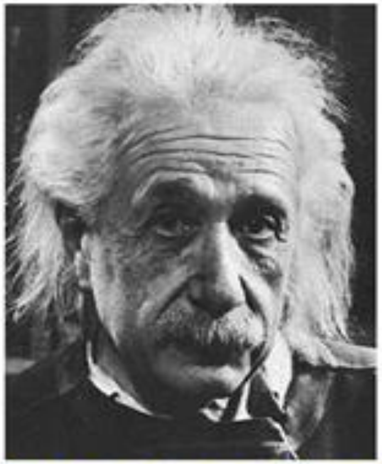
Neils Bohr:

It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we say about nature.

Einstein's response:

"God does not play dice with the universe."

Quantum mechanics is a probabilistic theory.



Fundamental theory of nature

Non-probabilistic
(Realistic)

Local
(no action at a distance)

Measurement does not create but reveals predetermined values for observable.

In 1935, EPR using the idea of reality and locality try to show that QM is incomplete.

EPR argument:

Reality:

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.

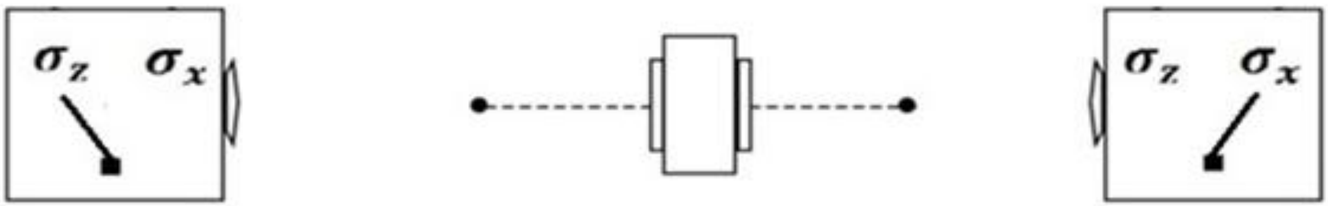
Locality:

If systems are spatially separate, the disturbance on one system does not instantaneously affect the reality that pertains to the others.

Criterion of completeness:

Every element of physical reality must have a counterpart in the physical theory.

Two spin-1/2 particles in an entangled state



Property: Results are completely correlated if both maesure spin along same direction

Due to correlation, by measuring σ_z on particle 1, one can predict the result of σ_z measurement on paricle 2.

Measurement on 1, can not instantaneously change the reality of 2.

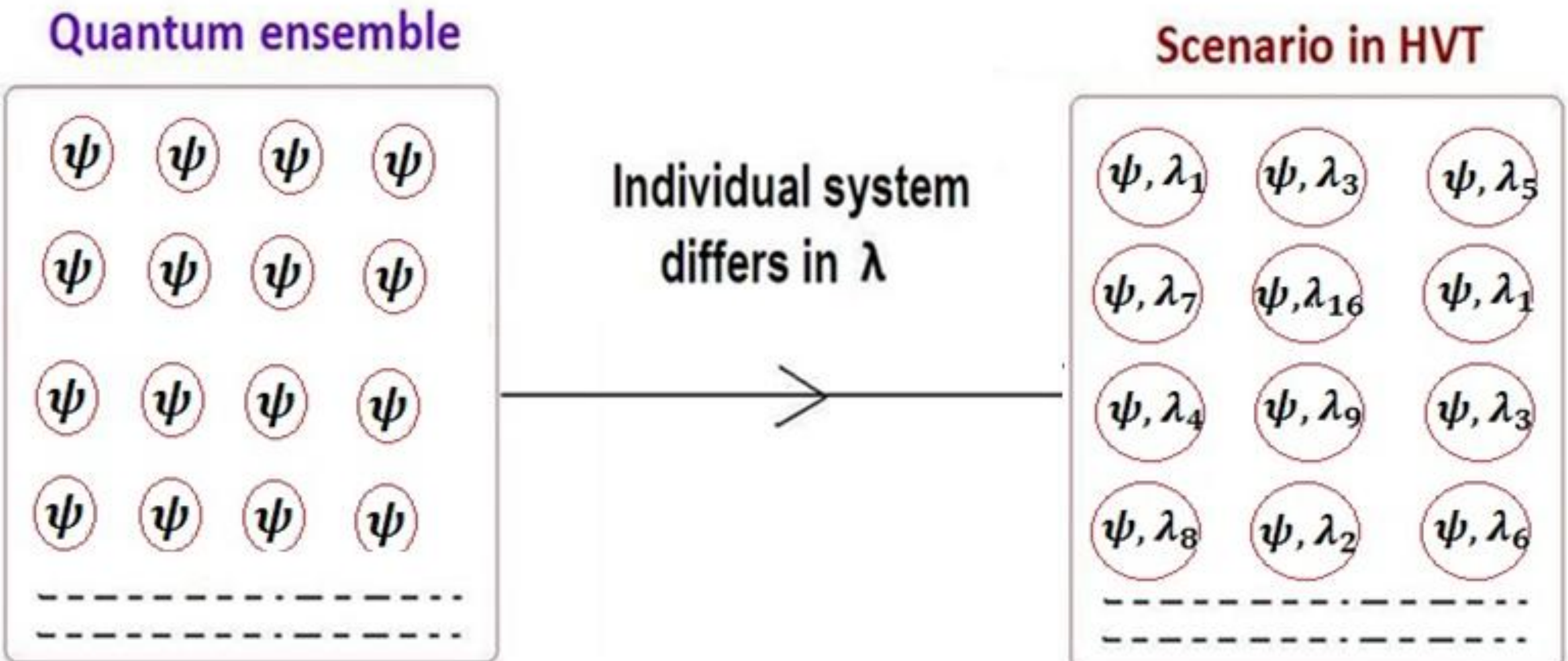
Conclusion: The reality for σ_z for particle 2 was already there.

- **The same argument can be made for σ_x measurement also.**
  - **Then for a single particle, both σ_z and σ_x will be real.**

QM can not assign values for both and hence incomplete.

Programme of HVT is to construct a theory that provide:

1) **Dispersion free state** 2) **Reproduce QM.**



Knowledge of ψ, λ provides definite values for all possible observables.

$$v_{\psi,\lambda}(A) = \textit{one of the eigenvalue of } A$$

$$\int v_{\psi,\lambda}(A)\, \rho(\lambda) d\lambda = \langle\psi|A|\psi\rangle \;\text{ with }\; \rho(\lambda) \geq 0, \int \rho(\lambda) d\lambda = 1$$

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On the Problem of Hidden Variables in Quantum Mechanics*

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Quote from J.S.Bell:

"Yet the von Neumann proof, if you actually come to grips with it, falls apart in your hands! There is nothing to it. It's not just flawed, it's sillyl !. . .. When you translate [his assumptions] into terms of physical disposition, they're nonsense. You may quote me on that: The proof of von Neumann is not merely false but foolish!"

(Interview in Omni, May, 1988, p. 88.)

Statistics for spin measurement

Every qubit state is eigen state of some spin observable $\boldsymbol{n}.\boldsymbol{\sigma}$.

Let $|\psi\rangle$ ***is up eigen state of*** $\boldsymbol{n}.\boldsymbol{\sigma}$.

$$|\psi\rangle\langle\psi| = \frac{1}{2}[I + \boldsymbol{n}.\boldsymbol{\sigma}], \ |\boldsymbol{n}| = 1$$

Then for spin measurement along vector $\boldsymbol{m}$:

$$p(+1|\psi) = \frac{1}{2}(1 + \boldsymbol{n}.\boldsymbol{m})$$

$$p(-1|\psi) = \frac{1}{2}(1 - \boldsymbol{n}.\boldsymbol{m})$$

(Born rule)

$$\langle\psi|\boldsymbol{m}.\boldsymbol{\sigma}|\psi\rangle = \boldsymbol{n}.\boldsymbol{m}$$

Bell Model for qubit

Completed state: $(\psi, \lambda) \equiv (n, \lambda)$

$\lambda \in [-\frac{1}{2}, \frac{1}{2}]$ *and distribution is uniform.*

Definite value for completed state:

$$v_{n,\lambda}[m.\sigma] = Sign(\lambda + \frac{1}{2}|n.m|\,Sign(n.m))$$

$Sign(x) = 1$ **when** $x \geq 0$

and

$Sign(x) = -1$ **when** $x < 0$

Case*1: *Let* $(n.m)$ *is negetive

$$\langle m.\sigma\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho(\lambda) v_{n,\lambda}[m.\sigma]\, d\lambda$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} Sign(\lambda + \frac{1}{2}|n.m|\, Sign(n.m))]\, d\lambda$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}|n.m|} (-1)\, d\lambda + \int_{\frac{1}{2}|n.m|}^{\frac{1}{2}} (+1)\, d\lambda$$

$$= -\frac{1}{2}[1 + |n.m|] + \frac{1}{2}[1 - |n.m|] \;= n.m$$

Hence QM statistics is reproduced.

Non-Contextuality

Projective measurement in 3 dimension : $\sum_{i=1}^{3} P_i = I$

$$P_i = |\varphi_i><\varphi_i|$$

$\{|\varphi_1>, |\varphi_2>, |\varphi_3>\}$ being an orthogonal basis B_1.

Another projective measurement $P_1 + Q_2 + Q_3 = I$

Q_2 : Projector on $\frac{1}{\sqrt{2}}(|\varphi_2> + |\varphi_3>)$

Q_3 : Projector on $\frac{1}{\sqrt{2}}(|\varphi_2> - |\varphi_3>)$

$\left\{|\varphi_1>, \frac{1}{\sqrt{2}}(|\varphi_2> + |\varphi_3>), \frac{1}{\sqrt{2}}(|\varphi_2> - |\varphi_3>)\right\}$ being an orthogonal basis B_2.

Measuements in B_1 basis and B_2 basis are different.

A HVT is called non-contextual if it assigns value to observables in a context independent way i.e. independent of other observables along with which it is measured.

In this case non-contextuality implies,

$$v_{B_1}(P_1) = v_{B_2}(P_1)$$

Bell- Kochen-Specker Theorem

QM is in cotradiction with non-contextual HVT.

18 vectors in 4-dimension:

$\varphi_1 = (0,0,0,1)$	$\varphi_2 = (0,0,1,0)$	$\varphi_3 = (1,1,0,0)$
$\varphi_4 = (1,-1,0,0)$	$\varphi_5 = (0,1,0,0)$	$\varphi_6 = (1,01,0)$
$\varphi_7 = (1,0,-1,0)$	$\varphi_8 = (1,-1,1,-1)$	$\varphi_9 = (0,0,1,1)$
$\varphi_{10} = (1,1,1,1)$	$\varphi_{11} = (0,1,0,-1)$	$\varphi_{12} = (1,0,0,1)$
$\varphi_{13} = (1,0,0,-1)$	$\varphi_{14} = (0,1,-1,0)$	$\varphi_{15} = (1,1,-1,1)$
$\varphi_{16} = (1,1,1,-1)$	$\varphi_{17} = (-1,1,1,1)$	$\varphi_{18} = (1,-1,-1,1)$

Rule of value assignment:

1) $v(\varphi_i) \equiv v(|\varphi_i><\varphi_i|)$ = 0 or 1

2) $\sum v(\varphi_i) = 1$, $\{\varphi_i\}$ form a orthogonal basis.

$$v(\varphi_1) + v(\varphi_2) + v(\varphi_3) + v(\varphi_4) = 1$$

$$v(\varphi_1) + v(\varphi_5) + v(\varphi_6) + v(\varphi_7) = 1$$

$$v(\varphi_8) + v(\varphi_{18}) + v(\varphi_3) + v(\varphi_9) = 1$$

$$v(\varphi_8) + v(\varphi_{10}) + v(\varphi_7) + v(\varphi_{11}) = 1$$

$$v(\varphi_2) + v(\varphi_5) + v(\varphi_{12}) + v(\varphi_{13}) = 1$$

$$v(\varphi_{18}) + v(\varphi_{10}) + v(\varphi_{13}) + v(\varphi_{14}) = 1$$

$$v(\varphi_{15}) + v(\varphi_{16}) + v(\varphi_4) + v(\varphi_9) = 1$$

$$v(\varphi_{15}) + v(\varphi_{17}) + v(\varphi_6) + v(\varphi_{11}) = 1$$

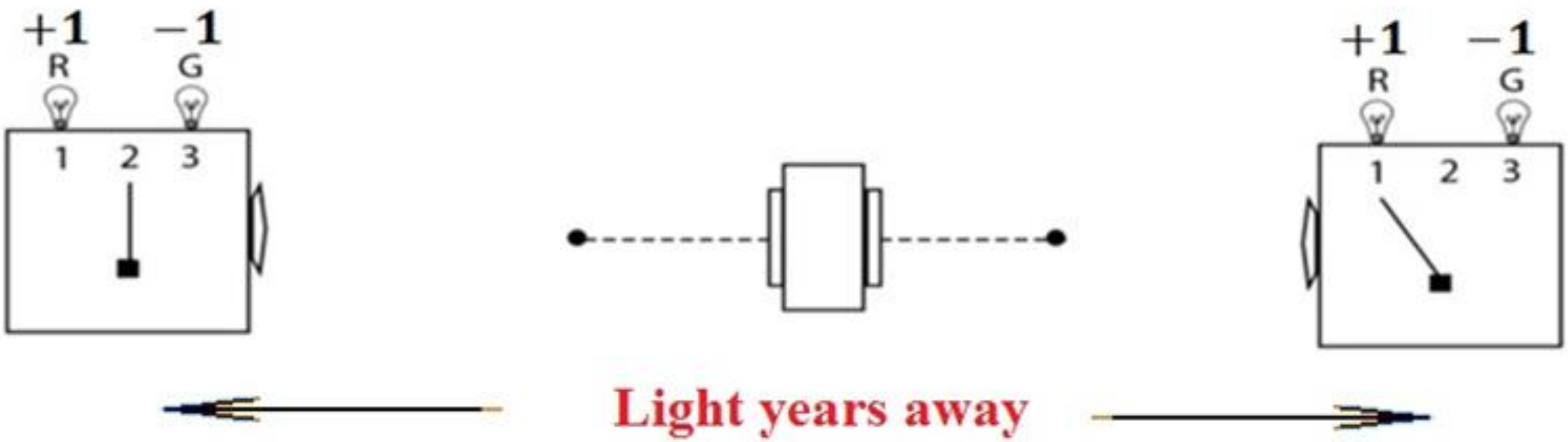
$$v(\varphi_{16}) + v(\varphi_{17}) + v(\varphi_{12}) + v(\varphi_{14}) = 1$$

If added, the L.H.S. is even as every vector has appeared twice and the R.H.S. is odd.

It shows that non-contextual HVT, in general can not reproduce quantum mechanics.

Simplest proof of Bell theorem

QM violates Local-Realism



- Three possible settings on both sides
- Outcomes are +1 or -1 for all of them
- Choices of setting on each side are completely random
- Statistics will be collected by running the expt. many times

The statistics generated from the measurement satisfy two properties:

(1) ***For same measurement, the outcomes are same.***

(2) ***The probability of having same outcomes on both sides for different measurements is*** $\frac{1}{4}$.

Bell theorem:

These two properties are in contradiction with the following assumptions:

a) **Locality**

b) **Reality**

c) **Free will to choose among the settings**

Local deterministic value assignment:

Alice 1	Alice 2	Alice 3	Bob 1	Bob 2	Bob 3
+1	+1	+1	+1	+1	+1
+1	+1	−1	+1	+1	−1
+1	−1	+1	+1	−1	+1
−1	+1	+1	−1	+1	+1
+1	−1	−1	+1	−1	−1
−1	+1	−1	−1	+1	−1
−1	−1	+1	−1	−1	+1
−1	−1	−1	−1	−1	−1

Among the 64 *possibilities, only these* 8 *are allowed due to property* (1).

Let these $\mathbf{8}$ *possible value assignment occur with probability* $p_1, p_2 \dots . p_8.$

$$p_{same}(1,2) = p_1 + p_2 + p_7 + p_8$$

$$p_{same}(2,3) = p_1 + p_4 + p_5 + p_8$$

$$p_{same}(1,3) = p_1 + p_3 + p_6 + p_8$$

$$p_{same}(1,2) + p_{same}(2,3) + p_{same}(1,3)$$

$$= p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + 2\,(p_1 + p_8)$$

$$= 1 + 2(p_1 + p_8)$$

$$p_{same}(1,2) + p_{same}(2,3) + p_{same}(1,3) \geq 1$$

This is violated by the second property which tells:

$$p_{same}(1,2) + p_{same}(2,3) + p_{same}(1,3)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Quantum Mechanical Example with this property

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|a_0a_0\rangle + |a_1a_1\rangle)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|b_0b_0\rangle + |b_1b_1\rangle)$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|c_0c_0\rangle + |c_1c_1\rangle$$

$$Setting1: |a_0\rangle = |0\rangle, |a_1\rangle = |1\rangle$$

$$Setting2: |b_0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, |b_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

$$Setting3: |c_0\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, |c_0\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Expression of ME in various basis

$$|\Phi^+\rangle = \frac{|a_0\rangle(|b_0\rangle + \sqrt{3}|b_1\rangle) + |a_1\rangle(\sqrt{3}|b_0\rangle - |b_1\rangle)}{2\sqrt{2}}$$

$$|\Phi^+\rangle = \frac{|a_0\rangle(|c_0\rangle + \sqrt{3}|c_1\rangle) - |a_1\rangle(\sqrt{3}|c_0\rangle - |c_1\rangle)}{2\sqrt{2}}$$

$$|\Phi^+\rangle = \frac{(|b_0\rangle + \sqrt{3}|b_1\rangle)(|c_0\rangle + \sqrt{3}|c_1\rangle)}{4\sqrt{2}}$$

$$- \frac{(\sqrt{3}|b_0\rangle - |b_1\rangle)(\sqrt{3}|c_0\rangle - |c_1\rangle)}{4\sqrt{2}}$$

$$p_{same}(1,2) + p_{same}(2,3) + p_{same}(1,3)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Towards understanding Bell's inequality:

An Interesting two-party Game

They are not allowed to communicate after the game starts.

$a_1 = ?$ or $a_2 = ?$

$b_1 = ?$ or $b_2 = ?$

Alice

Bob

Answers can be $+1$ *or* -1.

Bell game

Alice **Bob**

Winning Condition

The answers have to satisfy:

a_1	$V(a_1)\ V(b_1) = +1$	b_1
a_1	$V(a_1)\ V(b_2) = +1$	b_2
a_2	$V(a_2)\ V(b_1) = +1$	b_1
a_2	$V(a_2)\ V(b_2) = -1$	b_2

or

$$\langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle = 4$$

Can you suggest some winning strategy for this game?

Impossibility of winning in local world

Consider the following answers decided by Alice and Bob in one turn:

Question	Alices's answers	Question	Bob's answers
a_1	$V_{Alice}(a_1)$	b_1	$V_{Bob}(b_1)$
a_2	$V_{Alice}(a_2)$	b_2	$V_{Bob}(b_2)$

Alice's answer, in no way, depends on Bob's input and vice versa.

Now the answers have to satisfy all the winning conditions as pair of question in each turn are random.

$$V_{Alice}(a_1)\ V_{Bob}(b_1) = +1$$
$$V_{Alice}(a_1)\ V_{Bob}(b_2) = +1$$
$$V_{Alice}(a_2)\ V_{Bob}(b_1) = +1$$
$$V_{Alice}(a_2)\ V_{Bob}(b_2) = -1$$

if one take the product of these equations, the R.H.S. will be positive but L.H.S. will be negetive, showing the impossibility.

First try with all possible local deterministic correlations

a_1	a_2
+1	+1
+1	−1
−1	+1
−1	−1

b_1	b_2
+1	+1
+1	−1
−1	+1
−1	−1

There are 16 possible local correlations.

Most general deterministic correlation is a probabilistic mixture of these 16 correlations.

For each correlation:

$$v_k(a_1)v_k(b_1) + v_k(a_1)v_k(b_2) + v_k(a_2)v_k(b_1) - v_k(a_2)v_k(b_2)$$

$$= v_k(a_1)[v_k(b_1) + v_k(b_2)] + v_k(a_2)[v_k(b_1) - v_k(b_2)] = \pm 2$$

$$\langle a_i b_j \rangle = \sum_{k=1}^{16} p_k v_k(a_i) v_k(b_j) \ , i,j = 1,2$$

Consider the quantity:

$$\langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle$$

$$= \sum_{k=1}^{16} p_k v_k(a_1) v_k(b_1) + \sum_{k=1}^{16} p_k v_k(a_1) v_k(b_2)$$

$$+ \sum_{k=1}^{16} p_k v_k(a_2) v_k(b_1) - \sum_{k=1}^{16} p_k v_k(a_2) v(b_2)$$

$\left(p_k \textit{ is independent of } a_i \textit{ and } b_j \right)$

$$= \sum_{k=1}^{16} p_k \left[v_k(a_1) v_k(b_1) + v_k(a_1) v_k(b_2) + v_k(a_2) v_k(b_1) - v_k(a_2) v_k(b_2) \right]$$

$$= \sum_{k=1}^{16} p_k \ (\pm 2)$$

$$-2 \leq \langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle \leq 2$$

(Famous Bell inequality)

The winning probability by using any local deterministic correlation cannot go beyond 75%.

$$\langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle \leq 2$$

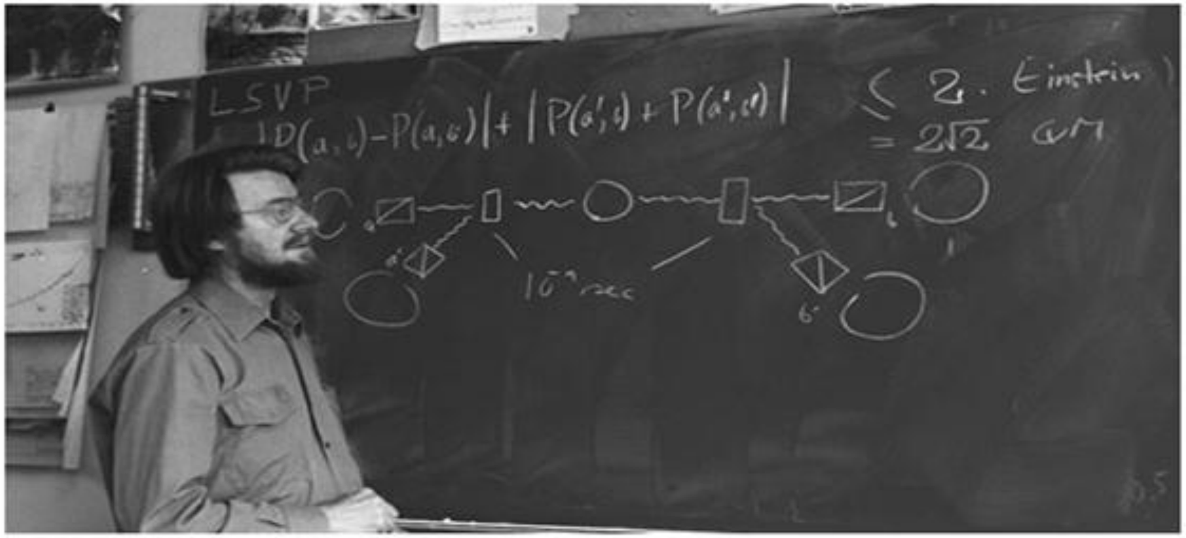
$$\langle a_i b_j\rangle = p(a_i = b_j) - p(a_i \neq b_j)$$

$$p(a_i = b_j) + p(a_i \neq b_j) = 1$$

$$\langle a_i b_j\rangle = 2p(a_i = b_j) - 1 = 1 - 2p(a_i \neq b_j)$$

$$p(a_1 = b_1) + p(a_1 = b_2) + p(a_2 = b_1) + p(a_2 \neq b_2) \leq 3$$

$$\frac{1}{4}[p(a_1 = b_1) + p(a_1 = b_2) + p(a_2 = b_1) + p(a_2 \neq b_2)] \leq \frac{3}{4}$$



J.S.Bell derived this inequality in 1964.

Bell's inequality implies that for any correlation for which

$$|\langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle| \geq 2$$

can not be expressed as mixture of deterministic local correlation.

Probabilistic correlation which does not imply signalling

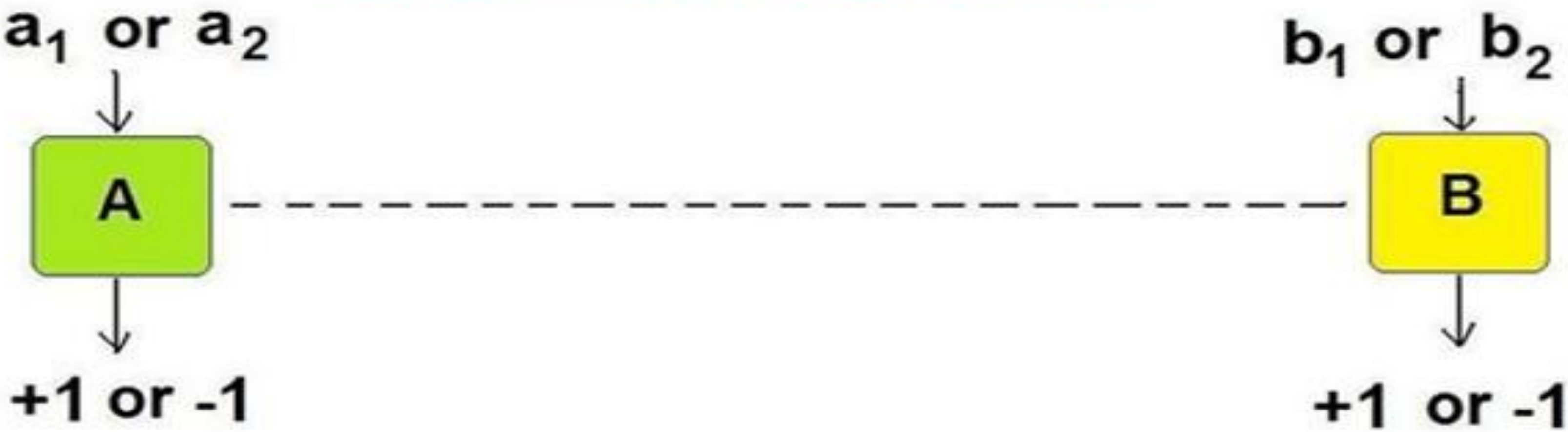
(Popescu-Rorlich box)

Input A	Input B	Output A	Output B	Probability
a_1	b_1	+1	+1	$\frac{1}{2}$
		-1	-1	$\frac{1}{2}$
a_2	b_1	+1	+1	$\frac{1}{2}$
		-1	-1	$\frac{1}{2}$
a_1	b_2	+1	+1	$\frac{1}{2}$
		-1	-1	$\frac{1}{2}$
a_2	b_2	-1	+1	$\frac{1}{2}$
		+1	-1	$\frac{1}{2}$

$$\langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle = 4$$

The game can be won with 100% probability by using this correlation.

Realizing P-R correlation in terms of deterministic correlation



Input		Output	
A	B	A	B
a_1	b_1	+1	+1
a_2	b_1	+1	+1
a_1	b_2	+1	+1
a_2	b_2	+1	-1

Input		Output	
A	B	A	B
a_1	b_1	-1	-1
a_2	b_2	-1	-1
a_1	b_2	-1	-1
a_2	b_2	-1	+1

Violate no-signaling condition.

Input		Output	
A	B	A	B
a_1	b_2	+1	+1
a_2	b_2	+1	-1

India vs Pakisthan
Cricket match

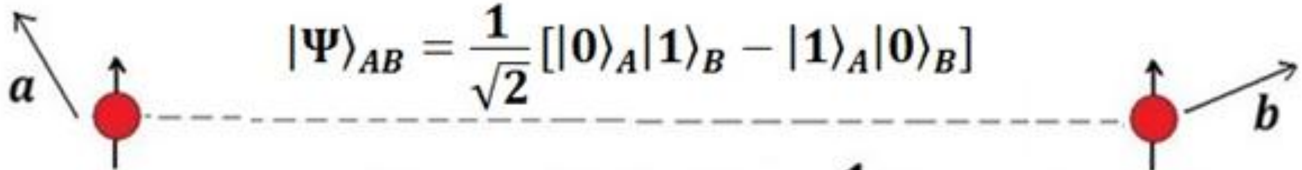
a_1 or a_2 — **A** ← Light years away → **B** — b_2

	Input in Box A	Output in Box B
India Won	a_1	+1
India lost	a_2	-1

So this correlation violates the principle of impossibility of instantaneous signaling.

Quantum Strategy

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]$$

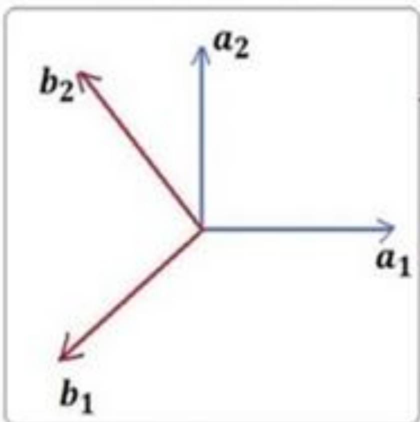


$$p(uu|a,b) = p(dd|a,b) = \frac{1}{4}(1 - a.b)$$

$$p(a = b) = \frac{1}{2}(1 - a.b)$$

$$p(a \neq b) = \frac{1}{2}(1 + a.b)$$

Choice of measurement direction



$$p(a_1 = b_1) = p(a_1 = b_2) = p(a_2 = b_1)$$
$$= p(a_2 \neq b_2) = \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$$

$$\frac{1}{4}[p(a_1 = b_1) + p(a_1 = b_2) + p(a_2 = b_1)$$
$$+ p(a_2 \neq b_2)] = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \approx 0.853$$

Bell's theorem:

If one whishes to realize certain quantum correlations as mixture of deterministic correlations, then some of the deterministic correlations will be necessarily nonlocal.

Conclusion:

Though QM does not allow signaling it is in contradiction with local realism.

Allowed theories:

* **Local but not real (Standard Quantum Mechanics)**
* **Real but nonlocal (Bohmian Quantum Mechanics)**
* **Nonlocal and non-real**

Does the impossibility of joint measurement of some observables have any role to play in Bell's theorem?

$$\textit{Joint measurement} + \textit{no-signaling} \Rightarrow \textit{No violation of Bell's inequality}$$

Violation of Bell's inequality by statistics collected from real experiments establishes that there are indeed some observables whose joint measurement is prohibited by nature.