Entropy and domination for quasi-Hitchin representations

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Geometric structures and stability, ICTS
17 February 2026

Context

S : closed oriented surface of genus $g \geq 2$.

The $\mathrm{PSL}_n(\mathbb{R})$-representation variety:

$$\chi_n^{\mathbb{R}}(S) = \{\, \rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{R})\}/\mathrm{PSL}_n(\mathbb{R})$$

The Hitchin component $\mathrm{Hit}_n(S)$ is the connected component of $\chi_n^{\mathbb{R}}(S)$ homeomorphic to a ball of dimension $(n^2-1)(2g-2)$.

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For $n > 2$, $\mathrm{Hit}_n(S)$ includes n-Fuchsian representations, obtained by composing with the irreducible representation $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$.

Q. Do these parametrize "geometric structures" on S?

Context

Labourie showed any $\rho \in \mathrm{Hit}_n(S)$ is a Anosov representation:

- geodesic flow on $\mathbb{R}^n$-bundle over S has Anosov dynamics
- discrete and faithful
- there is a ρ-equivariant limit map

$$\xi : \partial_\infty \tilde{S} \to \mathcal{F}(\mathbb{R}^n)$$

where the target is the space of complete flags in $\mathbb{R}^n$.

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This talk: Quasi-Hitchin representations – the connected set of Anosov representations in the $\mathrm{PSL}_n(\mathbb{C})$-representation variety $\chi_n(S)$ containing $\mathrm{Hit}_n(S)$.

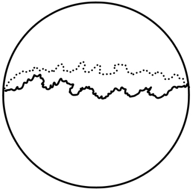
$$\textit{limit map} \;\; \xi : \partial_\infty \tilde{S} \to \mathcal{F}(\mathbb{C}^n)$$

Quasi-Fuchsian representations

A Fuchsian representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{R})$ is a discrete faithful representation.

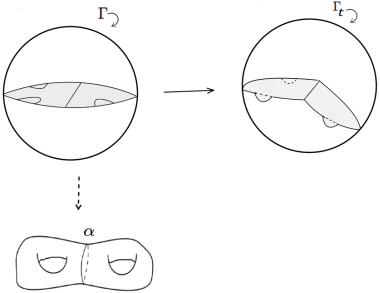
A quasi-Fuchsian representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ is a convex-cocompact representation obtained by a deformation of a Fuchsian representation.

- form an open set containing Fuchsian reps
- orbit map qi-embeds in $\mathbb{H}^3$
- limit map $\Lambda : \partial_\infty \tilde{S} \to \mathbb{CP}^1$



Bending deformation

Start with Fuchsian ρ_0 that preserves the equatorial plane in $\mathbb{H}^3$.
Deform by bending along lifts of a geodesic α, by an angle t.



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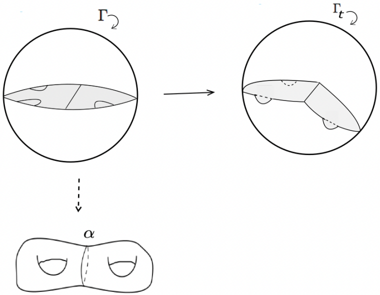

If α is separating such that $S \setminus \alpha = S_1 \sqcup S_2$:

$$\rho_t = \begin{cases} \rho_0 & \text{on } \pi_1(S_1) \\ E_t \rho_0 E_t^{-1} & \text{on } \pi_1(S_2) \end{cases}$$

where E_t is an elliptic commuting with $\rho_0(\alpha)$.

Entropy

Given a quasi-Fuchsian representation

$$\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$$

we define its *entropy* to be

$$H(\rho) = \lim_{L\to\infty} \frac{\log \#\{\gamma \in \pi_1 S \mid l_\rho(\gamma) \leq L\}}{L}$$

where $l_\rho(\gamma)$ is the translation distance of $\rho(\gamma)$ in $\mathbb{H}^3$.

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(orbital counting function, critical exponent)

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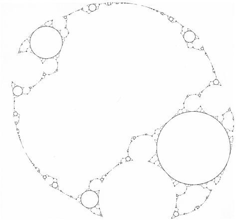
- If ρ is *Fuchsian*, then $H(\rho) = 1$ and is the topological entropy of the geodesic flow.
- The entropy equals the *Hausdorff dimension* of the limit set Λ. *(Patterson, Sullivan, Bishop-Jones, . . .)*

Rigidity theorem

Theorem. (Bowen, 1978)
Entropy strictly increases off the Fuchsian locus.



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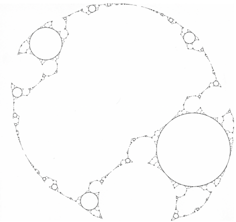

- *In fact he proved:* Hausdorff dimension of the limit set is strictly greater than 1.
- Proof uses the dynamics of $\pi_1 S$ on the limit set.
- Several different proofs since: e.g. Bishop-Jones, Bridgeman-Taylor, Deroin-Tholozan.

Domination

Theorem. (Deroin-Tholozan)
For any representation $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ *there is a Fuchsian representation* $j : \pi_1 S \to \mathrm{PSL}_2(\mathbb{R})$ *that dominates* ρ*,*
i.e. the lengths $l_\rho(\gamma) \leq \lambda \cdot l_j(\gamma)$ *for any* $\gamma \in \pi_1 S$*, for some* $\lambda \leq 1$.

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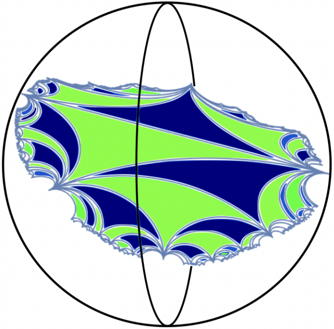
- The $\mathrm{PSL}_2(\mathbb{R})$-case was proved by *Gueritaud-Kassel-Wolff*.
- Bowen's result is a corollary.

A "GEOMETRIC" PROOF

Suppose $\rho : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$ is quasi-Fuchsian. Consider a ρ-equivariant pleated plane in $\mathbb{H}^3$

$$\Psi : \widetilde{S} \to \mathbb{H}^3$$



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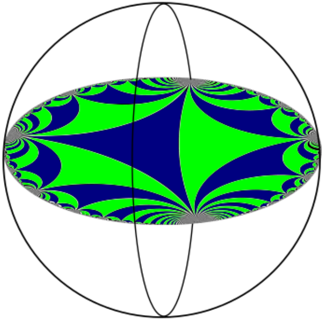
Straightening increases distances



$$d_{\mathbb{H}^3}(\Psi(x), \Psi(y)) \leq d_{\mathbb{H}^3}(\Psi_0(x), \Psi_0(y))$$

Ψ_0 is j-equivariant for a Fuchsian representation j.

A "GEOMETRIC" PROOF

Consequence:

For all $\gamma \in \pi_1 S$ we have

$$l_\rho(\gamma) \leq l_j(\gamma)$$

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Geodesic flow is Anosov $\implies$ $k \sim \alpha l_j(\gamma)$ for a *typical* curve, so

$$l_\rho(\gamma) \leq \lambda l_j(\gamma)$$

for some $\lambda < 1$ for any such curve $\implies$ $\boxed{H(j) < H(\rho)}$

Q. What happens for quasi-Hitchin representations
$\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})$?

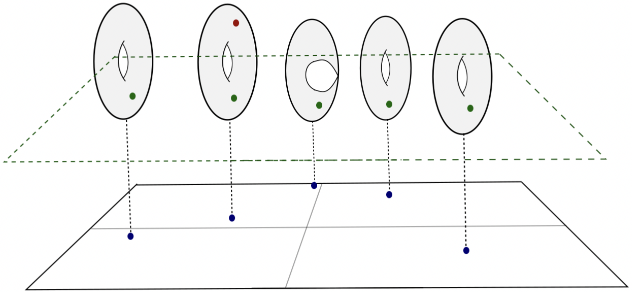
Isometries of the symmetric space $\mathbb{X}_n = SL_n(\mathbb{C})/SU(n)$

not CAT(-1)
no analogue of a pleated plane

Idea: Use *complexified* Bonahon-Dreyer parameters.
(Maloni-Martone-Mazzoli-Zhang)

HIGGS BUNDLES VIEWPOINT

$\mathcal{M}_{Higgs}(S) \cong \chi_n(S)$ *(Non-abelian Hodge)*



Hitchin fibration $p : \mathcal{M}_{Higgs}(S) \to \oplus_{i=2}^{n} H^0(S, K^{\otimes i})$

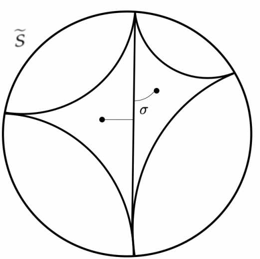
Conjecture. (Deroin-Tholozan) *In any fibre the Hitchin representation strictly dominates.*

Dai-Li: true for n-Fuchsian fibers

Generalizing shear-bend parameters

SHEAR COORDINATES FOR $\mathcal{T}(S)$

hyperbolic surface $\rightsquigarrow$ *gluing ideal triangles*



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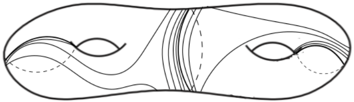

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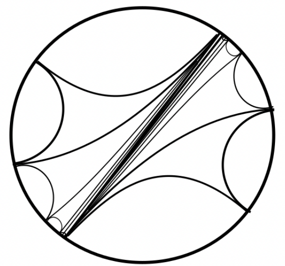

Shear coordinates for $\mathcal{T}(S)$

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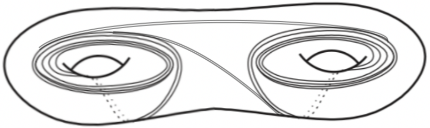

λ: maximal lamination on S

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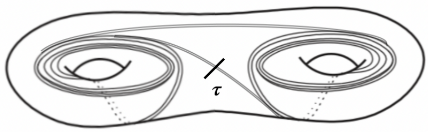

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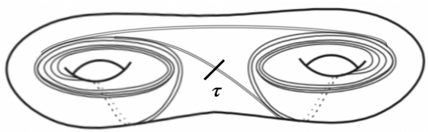

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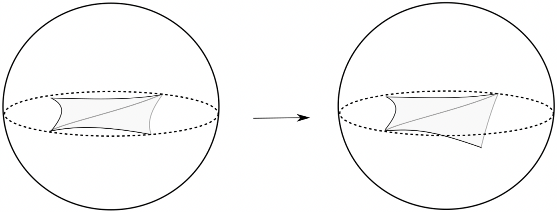
Theorem. (Bonahon) $\mathcal{T}(S)$ is diffeomorphic to a finite-sided cone $\mathcal{C}(\lambda) \subset \mathcal{H}(\lambda; \mathbb{R})$.

SHEAR-BEND COORDINATES FOR $\mathrm{PSL}_2(\mathbb{C})$

$\mathcal{R}(\lambda)$: *representations pleated along a maximal lamination* λ



shear-bend: $\sigma + i\theta$

Shear-bend coordinates for $\mathrm{PSL}_2(\mathbb{C})$

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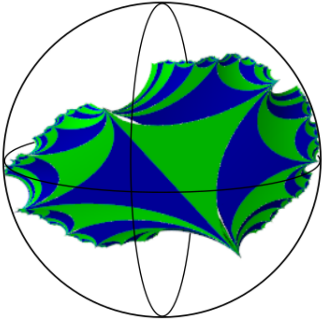

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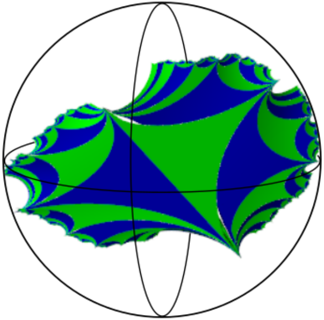

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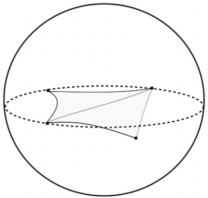
Theorem. (Bonahon) $\mathcal{R}(\lambda) \cong \mathcal{C}(\lambda) + i\mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$.

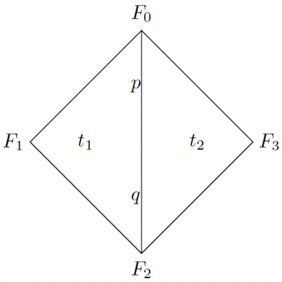
Fock-Goncharov coordinates



quadruples in $\mathbb{C}\mathrm{P}^1$:

shear-bend = log(complex cross-ratio)

FOCK-GONCHAROV COORDINATES



triples and quadruples of flags in $\mathbb{C}^n$:

$\frac{(n-1)(n-2)}{2}$ *triangle invariants*
$(n-1)$ *edge invariants*

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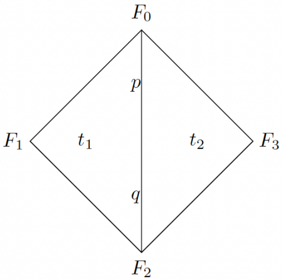

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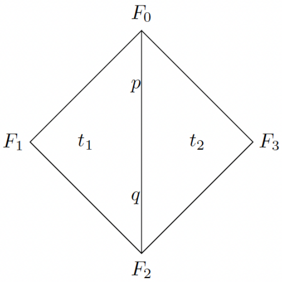

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Theorem. (Bonahon-Dreyer) For λ a maximal lamination, $\mathrm{Hit}_n(S) \cong \mathcal{C}(\lambda, n)$, a finite-sided cone in $\mathbb{R}^T \times \mathcal{H}(\lambda; \mathbb{R}^{n-1})$, where T is the total number of triangle invariants.

n-PLEATED REPRESENTATIONS

complexify Bonahon-Dreyer parameters

Definition. (Martone-Maloni-Mazzoli-Zhang)
$\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})$ is pleated along a lamination λ if it has

- the Anosov property on $\mathbb{R}^n$-bundle over λ, and
- a ρ-equivariant λ-limit map

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The set of n-pleated representations

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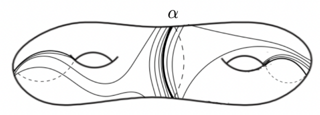
bending deformations

BENDING DEFORMATIONS

Example (for $n = 3$): Start with a Hitchin representation ρ_0.



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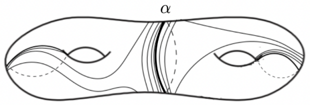

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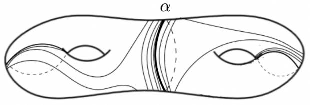

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$$\rho = \begin{cases} \rho_0 & \text{on } \pi_1(S_1) \\ A\rho_0 A^{-1} & \text{on } \pi_1(S_2) \end{cases} \quad \text{where } A = \begin{pmatrix} e^{i(\theta+\phi)} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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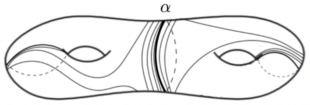

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Q. What happens to the entropy?

Entropy and domination results

Length functions and entropy

$$\boxed{\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})}$$

- The Hilbert length of a closed curve γ on S is
$$l_\rho(\gamma) = \ln \frac{\lambda_n}{\lambda_1}$$
where $\lambda_1, \lambda_2, \dots \lambda_n$ are the moduli of eigenvalues of $\rho(\gamma)$ in increasing order.

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- Translation length in the symmetric space $\mathbb{X}_n$ defines $l^X_\rho(\gamma) = (\ln^2 \lambda_1 + \ln^2 \lambda_2 + \cdots + \ln^2 \lambda_n)^{1/2}$

Length functions and entropy

For each, we can define the corresponding entropy

$$H(\rho) = \lim_{L\to\infty} \frac{\log \#\{\gamma \in \pi_1 S \mid l_\rho(\gamma) \leq L\}}{L}$$

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- This entropy rigidity also holds for the translation length entropy, but the simple root entropies are constant on Hit_n. *(Potrie-Sambarino)*

Domination result

joint work with Pabitra Barman

Theorem A. *If* $\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})$ *is obtained by a bending deformation of* $j \in Hit_n(S)$*, then* j *dominates* ρ*, i.e*

$$l_\rho(\gamma) \leq l_j(\gamma) \text{ for all } \gamma \in \pi_1 S$$

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- It is possible to strictly dominate one Hitchin representation in $\mathrm{PSL}_3(\mathbb{R})$ by another. *(Zhang)*

Entropy result

joint work with Pabitra Barman

Theorem B. *An Anosov representation* $\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})$ *obtained by a bending deformation of an n-Fuchsian representation has strictly larger Hilbert entropy and translation length entropy.*

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- For ρ a hyperconvex representation, the Hausdorff dimension of the limit set
  - equals the maximum of the simple root entropies *(Pozzetti-Sambarino-Wienhard)*
  - strictly increases off Hitchin representations *(Farre-Pozzetti-Viaggi)* *(small bending: Bridgeman-Pozzetti-Sambarino-Wienhard)*

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- The Hilbert entropy of a $\rho \in \mathrm{Hit}_3(S)$ that is not 3-Fuchsian, equals the Hausdorff dimension of the set of non-differentiable points of ξ. *(Pozzetti-Sambarino)*

Idea of the proofs

Proof of Theorem A

Issue: No "pleated plane" in the symmetric space $\mathbb{X}_n$ to straighten!

$$\rho \in \mathcal{C}(\lambda, n) + i\mathcal{Y}(\lambda; n, \mathbb{R}/2\pi\mathbb{Z}) \qquad \rightsquigarrow \qquad j \in \mathcal{C}(\lambda, n)$$

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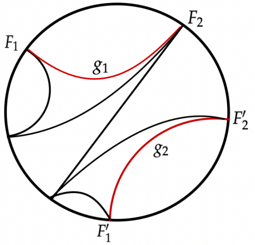
- A formula for $\rho(\gamma)$.
- Relation with weighted planar networks.
- Entrywise domination of $\rho(\gamma)$ and $j(\gamma)$.

Proof of Theorem A

$$\rho(\gamma) = \overrightarrow{\prod_{\Delta}} M_\Delta$$

Slithering map $\Sigma : \tilde{\lambda} \times \tilde{\lambda} \to \mathrm{SL}_n(\mathbb{C})$

- $\Sigma(g_1, g_2)$ takes flags at the endpoints of g_1 to those for g_2.
- $\Sigma(g_1, g_2) = \Sigma(g_1, g_3) \circ \Sigma(g_3, g_2)$.



$$\Sigma(g_1, g_2) = \overrightarrow{\prod_{g_1 < \Delta < g_2}} M_\Delta$$

Proof of Theorem A

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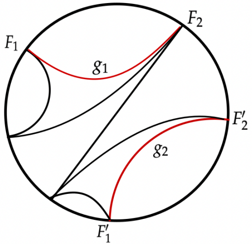

$$\Sigma(g, \gamma \cdot g) = \overrightarrow{\prod_{\Delta}} M_\Delta = \rho(\gamma)$$

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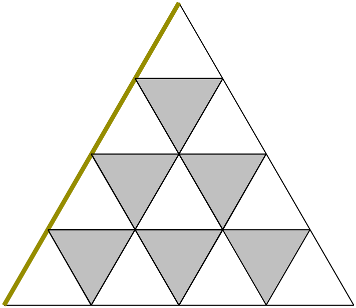
Fock-Goncharov's snakes:



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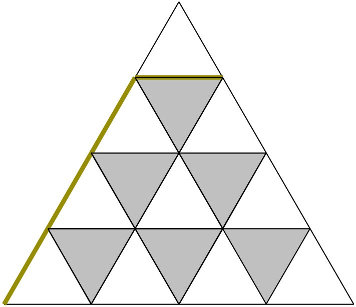

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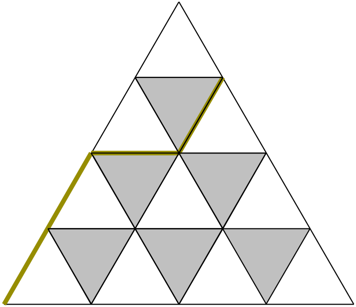

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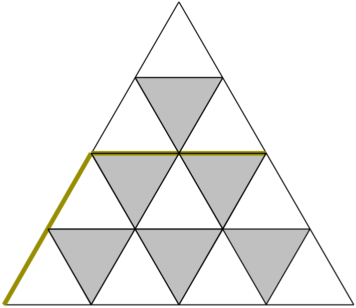

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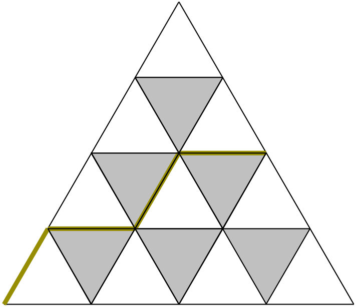

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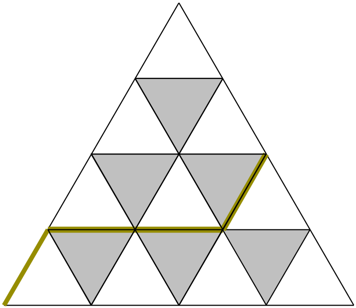

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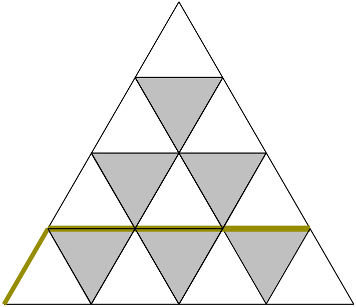

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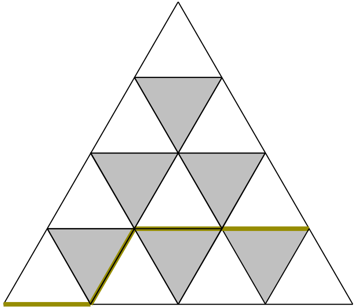

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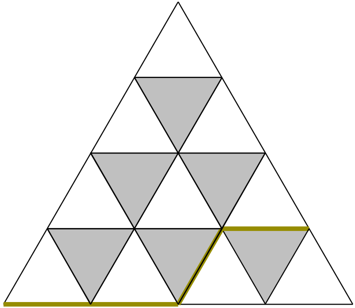

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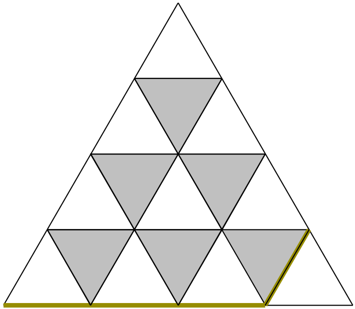

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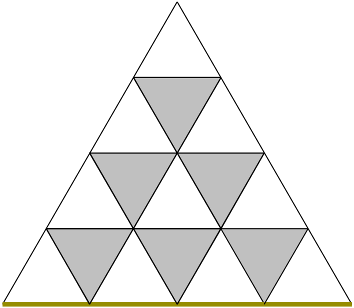

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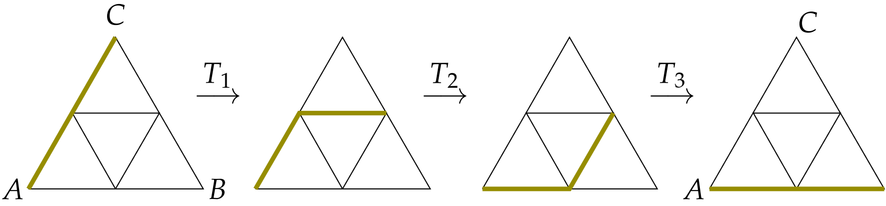
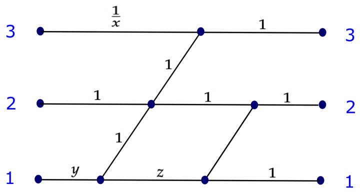


$$M_\Delta = \begin{pmatrix} yz & y(z+1) & y \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{x} \end{pmatrix}$$

Observation: M_Δ *is a weight matrix of a planar network.*

Weight matrices

Planar network: Acyclic directed planar graph Γ *with* n *"sources" and* n *"sinks", and each edge* e *is assigned a complex weight* $w(e)$*.*



The *weight of a directed path* in Γ is the product of the weights of its edges. The *weight matrix* $W(\Gamma)$ is an $n \times n$ matrix where

$$w_{ij} = \text{sum of weights of paths from source } i \text{ to sink } j$$

e.g.

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Fact. (see Fomin-Zelevinksy)
If all edge-weights of Γ *are real and positive, then* $W(\Gamma)$ *is totally positive, that is, all minors are positive.*

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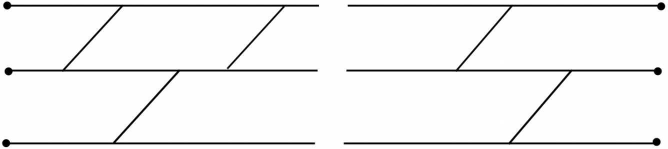

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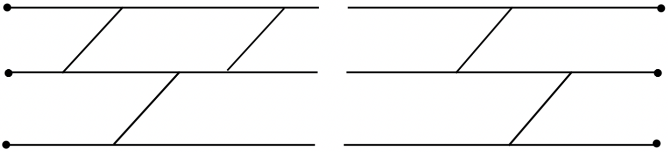

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In our setting, Fock-Goncharov parameters real and positive $\implies$ *weights are real and positive.*

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Let $\rho(\gamma) = W = (a_{ij})$ and $j(\gamma) = |W| = (W_{ij})$ be their weight matrices. Then $|a_{ij}| \leq W_{ij}$ by the triangle inequality.

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Applying this argument to $\rho(\gamma^{-1})$, we obtain $l_\rho(\gamma) \leq l_j(\gamma)$.

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Applying our argument to $W = \rho(\gamma)$ and $W' = j(\gamma)$, we obtain $\prod_{i=1}^k \lambda_{n-i+1} \leq \prod_{i=1}^k \lambda'_{n-i+1}$ for each $1 \leq k \leq n$.

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Applying our argument to $W = \rho(\gamma)$ and $W' = j(\gamma)$, we obtain $(\ln \lambda_1, \ln \lambda_2, \ldots, \ln \lambda_n) \prec (\ln \lambda'_1, \ln \lambda'_2, \ldots, \ln \lambda'_n)$.

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Standard *majorization inequalities* then imply that $l_\rho^X(\gamma) \leq l_j^X(\gamma)$.

Proof of Theorem B (entropy rigidity)

Anosov property of the geodesic flow

$\Downarrow$

a typical γ of length $\leq L$ on S traverses a prescribed sequence of ideal triangles $k = O(L)$ times

PROOF OF THEOREM B (ENTROPY RIGIDITY)

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Key Lemma. The corresponding planar network with weights determined by the FG-parameters of ρ and j, have weight matrices satisfying $|a_{ij}| \leq e^{-\alpha k} A_{ij}$.

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Consequence:

$$l_\rho(\gamma) \leq l_{\rho_0}(\gamma) - \alpha k \text{ for a typical curve.}$$

$$\text{Since } k \sim l_{\rho_0}(\gamma)$$

$$\text{we have } l_\rho(\gamma) \leq \lambda l_{\rho_0}(\gamma)$$

SUMMARY OF RESULTS

ongoing joint work with Pabitra Barman

Theorem A. (Domination) *If* $\rho : \pi_1 S \to \mathrm{PSL}_n(\mathbb{C})$ *is obtained by a bending deformation of* $j \in Hit_n$*, then* j *dominates* ρ*, i.e*

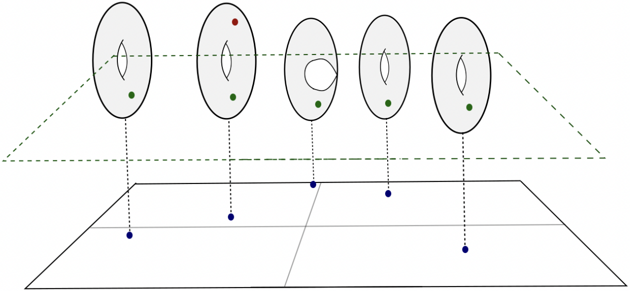
$$l_\rho^X(\gamma) \le l_j^X(\gamma) \text{ for all } \gamma \in \pi_1 S$$

for both the Hilbert length and translation length.

Theorem B. (Entropy rigidity) *Bending deformations of any n-Fuchsian representation strictly increase the Hilbert entropy and translation length entropy.*

Further questions

$$\mathcal{M}_{Higgs}(S) \cong \quad \chi_n(S)$$ *(Non-abelian Hodge)*



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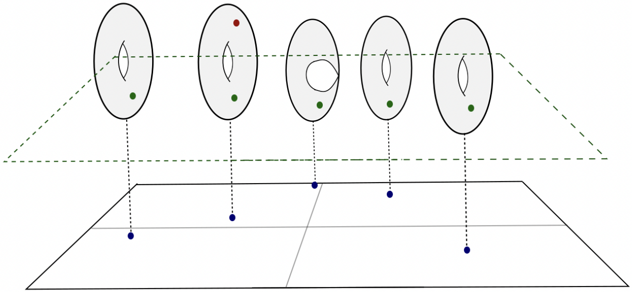

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Ott-Swoboda-Wentworth-Wolf (for $n = 2$)

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Two domination conjectures.

- *For the triangle-group* $\Delta(p, q, r)$ *and the one-parameter family* $\rho_t : \Delta(p, q, r) \to PU(2, 1)$ *introduced by Goldman-Parker, the representation* ρ_t *dominates* ρ_s *if* $t < s$. *(Schwartz)*
- *For any Hitchin representation* ρ *in* $\mathrm{PSL}_n(\mathbb{R})$*, there is a Hitchin representation* j *in* $\mathrm{PSp}_n(\mathbb{R})$ *(if* n *is even) or* $\mathrm{PSO}(\frac{n-1}{2}, \frac{n+1}{2})$ *(if* n *is odd) such that* $l_j \leq l_\rho$. *(Lee-Zhang,* $n = 3$ *by Tholozan)*

Quasi-Hitchin limit maps

$$\xi : \partial_\infty \tilde{S} \to \mathcal{F}(\mathbb{C}^n)$$

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THANK YOU FOR YOUR ATTENTION!