Lecture 3 Crystal base for $U_q(osp(m|n))$

1. Quantum super algebra $U_q(osp(m|n))$

2. Oscillator representation $V(\lambda)$

3. Parabolic Verma modules. $P(\lambda)$

1. Quantum super algebra $U_q(osp(m|n))$

- Orthosymplectic Lie super alg.

V : a superspace . B : "even" bilinear form $\left(B(v_i, v_j) = 0 \quad (i+j \neq 0) \right)$

B : non-deg. super symmetric

$$\left(B|_{V_0 \times V_0} : \text{symm.} \quad B|_{V_1 \times V_1} : \text{skew-symm} \right)$$

$$osp(V) := osp(V)_0 \oplus osp(V)_1$$

$$osp(V)_\varepsilon := \{ x \in gl(V)_\varepsilon \mid B(xv, w) = -(-1)^{\varepsilon |v|} B(v, xw) \text{ for } v, w \in V \}$$

$$osp(l|2m) := osp(\mathbb{C}^{l|2m}) \qquad osp(l|2m)_0 = so(l) \oplus sp(2m)$$

B' : non-deg super skew symm

$\mathfrak{spo}(V)$ can be defined in the same way w.r.t. B'.

<u>Rmk</u> S.Vec : cat. of superspace

$$\Pi : \mathrm{S.Vec} \longrightarrow \mathrm{S.Vec} \qquad (\text{parity change functor})$$

$$V \longmapsto \Pi V \qquad \left((\Pi V)_{\varepsilon} = V_{\varepsilon+1} \right)$$

$$\begin{array}{ccc} \mathfrak{gl}(V) & \cong & \mathfrak{gl}(\Pi V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \Pi f \Pi^{-1} \end{array} \qquad \begin{array}{ccc} \mathfrak{osp}(V) & \cong & \mathfrak{spo}(\Pi V) \\ B & & B^{\pi} \end{array}$$

$$B^{\pi}(\Pi u, \Pi v) := (-1)^{|u|} B(u, v)$$

- $\mathfrak{osp}(2m+1|2n)$

$$\mathfrak{osp}(2m+1|2n) \subset \mathfrak{gl}(2m+1|2n)$$

$$B = \begin{bmatrix} 1 & & & & \\ & 0 & I_m & & \\ & I_m & 0 & & \\ & & & 0 & -I_n \\ & & & I_n & 0 \end{bmatrix}$$

$\{ \delta_a \mid a \in I(m|n) \}$: a basis of $\mathfrak{h}^*$

$$(\delta_a | \delta_b) = \begin{cases} 1 & a = b \leq m \\ -1 & a = b > m \\ 0 & \text{otherwise} \end{cases}$$

induced bilinear form

Dynkin diagram (simple roots Π)

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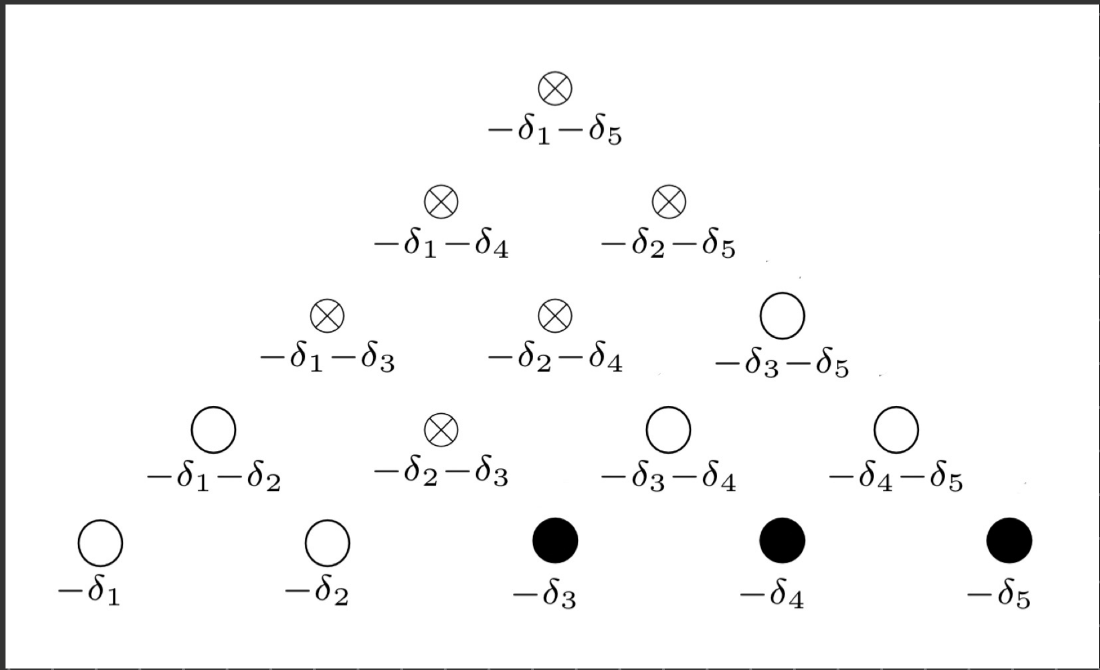
$$\Phi^+ = \Phi_0^+ \cup \Phi_1^+$$

$$\Phi_0^+ = \{ \delta_a - \delta_b \mid a < b \leq m,\ m < a < b \}$$

$$\cup \{ \underbrace{-\delta_a,\ -\delta_b - \delta_c}_{so(2m+1)},\ \underbrace{-\delta_s - \delta_t,\ -2\delta_u}_{sp(2n)} \mid 1 \leq a \leq m,\ b < c \leq m,\ m < s < t,\ m < u \}$$

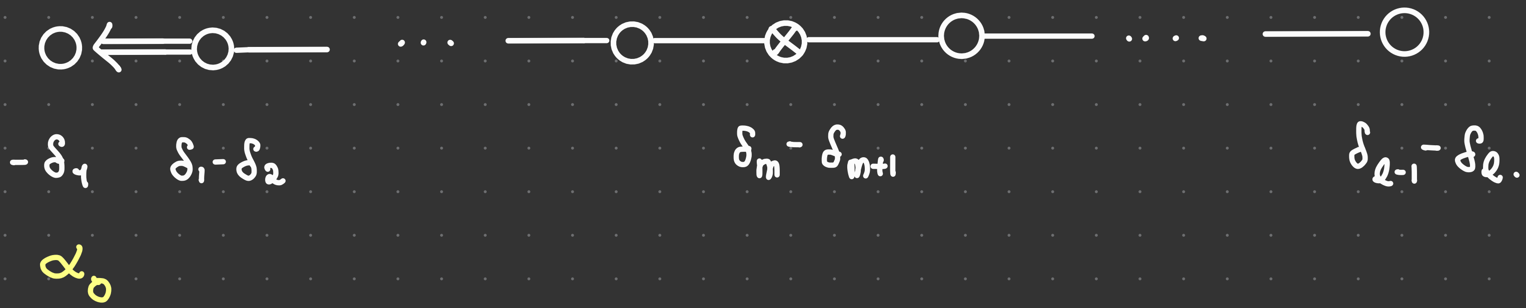
$$\Phi_1^+ = \{ \underbrace{\delta_a - \delta_b}_{\text{isotropic}},\ -\delta_c \mid a \leq m < b, c \}$$



the roots not of type $\mathfrak{gl}(m|n)$

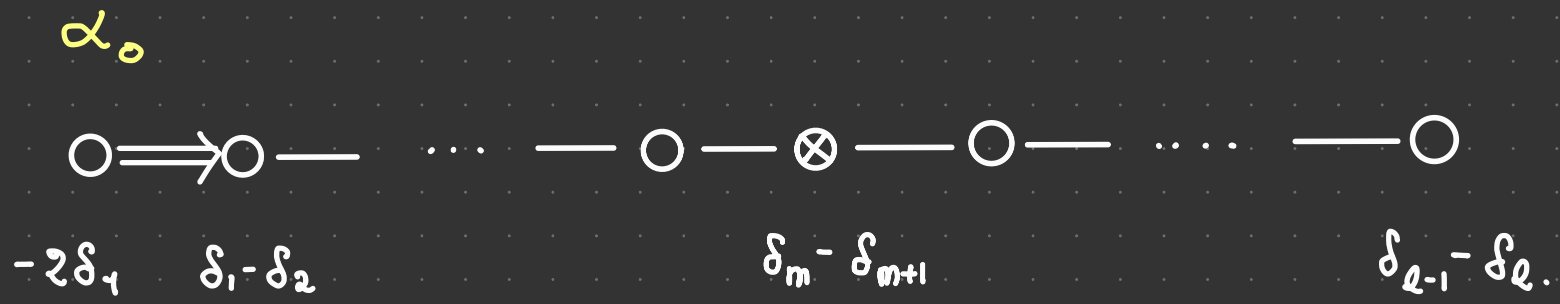
● (black) : non-isotropic odd root. $2\delta_i$: even root

$b_{m|n}$:

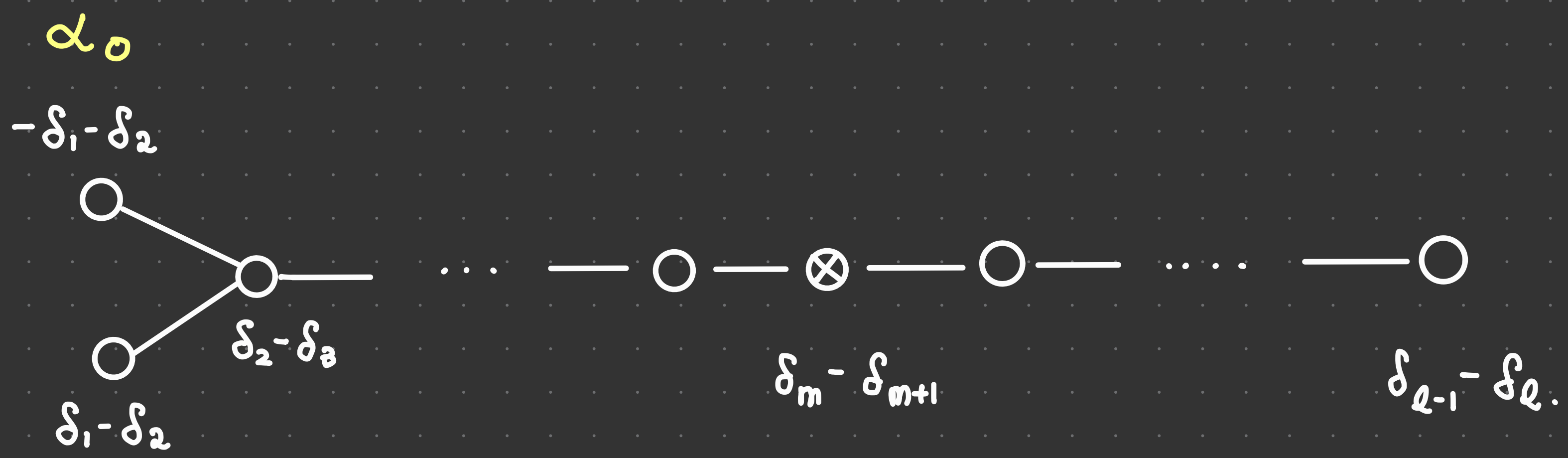


Similary. we may have

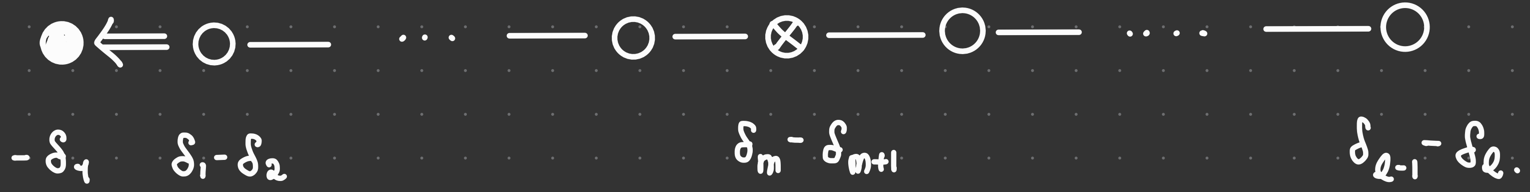
$\mathfrak{c}_{m|n}$:



$\mathfrak{d}_{m|n}$:

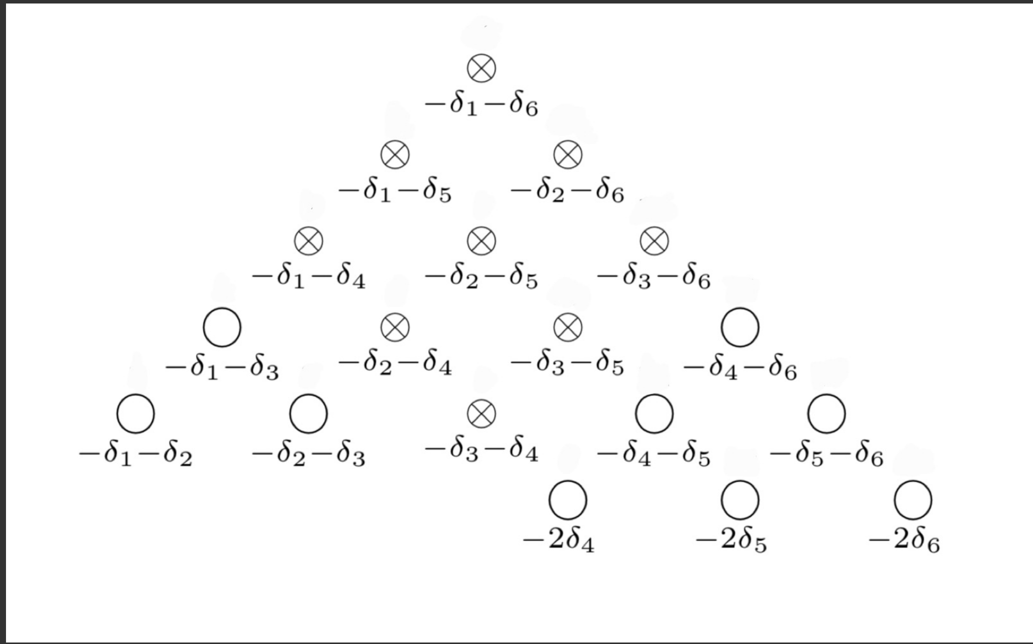


$\mathfrak{b}^\bullet_{m|n}$:

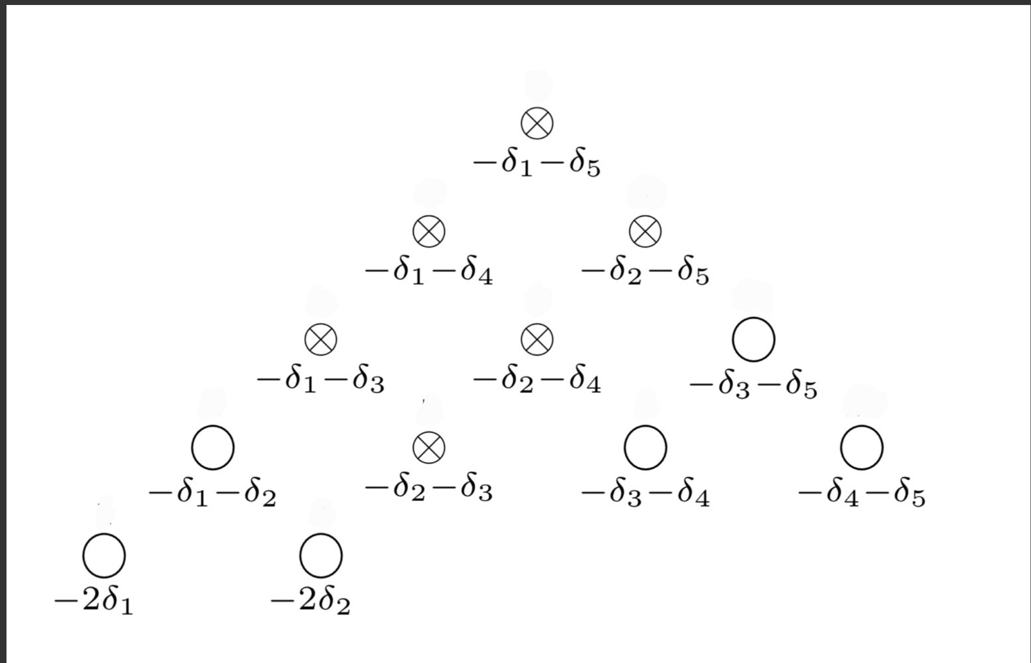


#7

$c_{2|3}$



$d_{3|3}$



the roots not of type $\mathfrak{gl}(m|n)$

- Suppose $\mathfrak{g}_{m|n} = \mathfrak{b}_{m|n}, \mathfrak{c}_{m|n}, \mathfrak{d}_{m|n}$ (not $\mathfrak{b}^{\circ}_{m|n}$ for simplicity)

$\Pi \subset \Phi^+$: the set of simple roots

$\mathcal{U}_q(\mathfrak{g}_{m|n})$: the q-analogue of $\mathcal{U}(\mathfrak{g}_{m|n})$

generators : $E_\alpha, F_\alpha, K_a, K_a^{-1}$ ($\alpha \in \Pi$, $a \in I(m|n)$)

relations : $\mathcal{U}_q(\mathfrak{gl}(m|n)) + \mathcal{U}_q(\mathfrak{g}_{m|0})$

$\mathcal{U}_q(\mathfrak{l}) \subset \mathcal{U}_q(\mathfrak{g}_{m|n})$: subalg. $\cong \mathcal{U}_q(\mathfrak{gl}(m|n))$

* We use the version of K-O-S.'s quantum group.

2. Oscillator representation $V(\lambda)$

- A fin-dim'l $U_q(\mathfrak{g}_{m|n})$-module is not semisimple in general.

- There is also a good family of semisimple repn's of $U_q(\mathfrak{g}_{m|n})$ closed under $\otimes$ (infinite-dimensional, in general)

- Recall (Howe duality)

$$\bigwedge(\mathbb{C}^n)^{\otimes \ell} = \bigoplus_{\lambda} V_{so_{2n}}^{\lambda} \otimes V_{O_\ell}(\lambda) \qquad S(\mathbb{C}^n)^{\otimes \ell} = \bigoplus_{\lambda} \underbrace{V_{sp_{2n}}^{\lambda}}_{\text{inf. dim'l}} \otimes V_{O_\ell}(\lambda)$$

oscillator repn

- $\exists$ a super analog of oscillator repn. of $osp(m|\ell)$

for example, two dualities can be merged into a single duality

$$\bigwedge(\mathbb{C}^{m|n})^{\otimes \ell} = \bigoplus_{\lambda} \underline{V^{\lambda}_{osp(2m|2n)}} \otimes V_{O_\ell}(\lambda)$$

- We introduce / construct

a q-analog of oscillator repn's with crystal base

$$W = \bigoplus_{X_{m|n}} \Bbbk\, \mathbb{m} \qquad X_{m|n} = \mathbb{Z}_2^m \times \mathbb{Z}_{\geq 0}^n \ni (m_1, \cdots, m_m, m_{m+1}, \cdots, m_{m+n})$$

W : a $U_q(\mathfrak{l})$-module

$$e_i\, \mathbb{m} = [m_{i+1}] \, (\cdots \; m_i + 1, \; m_{i+1} - 1, \; \cdots)$$

$$f_i\, \mathbb{m} = [m_i] \, (\cdots \; m_i - 1, \; m_{i+1} + 1, \; \cdots)$$

$$q^{\delta_i^\vee} \mathbb{m} = q^{m_i} \, \mathbb{m}$$

W_k = the subspace spanned by $\mathbb{m}$ of degree k

$\cong V_{m|n}(1^k)$ poly. repn. of $U_q(\mathfrak{l})$

① Suppose first $\mathfrak{g} = \mathfrak{b}, \mathfrak{d}$

$\exists$ $\mathcal{U}_q(\mathfrak{g}_{m|n})$-action on W such that

$$W : \text{irreducible if } \mathfrak{g}_{m|n} = \mathfrak{b}_{m|n}$$

$$W = W_+ \oplus W_- \quad \text{if } \mathfrak{g}_{m|n} = \mathfrak{d}_{m|n}$$

where

$$\begin{cases} e_0 \mathbf{m} = (m_1 - 1, m_2, \cdots) \\ f_0 \mathbf{m} = (m_1 + 1, m_2, \cdots) \end{cases} \quad \text{if } \mathfrak{g} = \mathfrak{b}$$

$$\begin{cases} e_0 \mathbf{m} = (m_1 - 1, m_2 - 1, m_3, \cdots) \\ f_0 \mathbf{m} = (m_1 + 1, m_2 + 1, m_3, \cdots) \end{cases} \quad \text{if } \mathfrak{g} = \mathfrak{d}$$

※ a super analog of spin repn

② Suppose $\mathfrak{g} = \mathfrak{c}$

$\exists$ $U_q(\mathfrak{g}_{m|n})$-action on $W^{\otimes 2}$ given by

$$e_0\, \mathbf{m} \otimes \mathbf{m}' = a\,(m_1 - 1, m_2, \cdots) \otimes (m_1' - 1, m_2, \cdots)$$

$$f_0\, \mathbf{m} \otimes \mathbf{m}' = b\,(m_1 + 1, m_2, \cdots) \otimes (m_1' + 1, m_2, \cdots)$$

for some a, b.

※ a super analog of fundamental repn $\subset W^{\otimes 2}$

$\because$ $n = 0$

$$W^{\otimes 2} = V_{\mathfrak{c}_m}(\varpi_m) \oplus V_{\mathfrak{c}_m}(\varpi_{m-1})^{\oplus 2} \oplus V_{\mathfrak{c}_m}(\varpi_{m-2})^{\oplus 3} \oplus \cdots$$

ϖ_k : kth fundamental weight.

Put $\mathcal{F} = \begin{cases} W & \text{if } \mathfrak{g} = b.d \\ W^{\otimes 2} & \text{if } \mathfrak{g} = c. \end{cases}$

$\mathcal{F}$ has a crystal base $(\mathcal{L}, \mathcal{B})$ where

$$\mathcal{L} = \bigoplus_{X_{m|n}} A_0 m \qquad \mathcal{B} = \{ m \ (\mathrm{mod}\ q\mathcal{L}) \}$$

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pf.) $\tilde{e}_i, \tilde{f}_i$ $i \in \{1, 2, \cdots, m+n-1\}$: crystal operators for $U_q(\mathfrak{l})$

$\tilde{e}_0, \tilde{f}_0$: usual crystal operators. for $U_q(sl_2)$

$\mathcal{F}^{\otimes \ell}$ is completely reducible. w/ crystal base $(\mathcal{L}^{\otimes \ell}, \mathcal{B}^{\otimes \ell})$

$\because$ $\mathcal{F}$ has a polarization, a non-deg. symm bilinear form

$$(x\, m, m') = (m, \eta(x) m')$$

$^{\exists}\eta : U_q(\mathfrak{g}_{m|n}) \longrightarrow U_q(\mathfrak{g}_{m|n})$ anti-auto. $(\eta \otimes \eta) \circ \Delta = \Delta \circ \eta$

$(\ ,\ )$: positive definite on $\mathcal{L}/_{q\mathcal{L}}$ $(\Rightarrow \mathcal{F} : \text{semi simple})$

$\mathcal{F}^{\otimes \ell}$ has a polarization $\Rightarrow$ $\mathcal{F}^{\otimes \ell}$: semisimple

- Irreducible component in $\mathcal{F}^{\otimes \ell}$

$$P = \mathbb{Z}\Lambda \oplus \bigoplus_{a \in I(m|n)} \mathbb{Z}\delta_a$$: the weight lattice w/ $(\cdot\,,\cdot)$

$$(\delta_a, \delta_b) = (-1)^{p(a)} \delta_{ab}\, r \qquad r = \begin{cases} 2 & \text{for } \mathfrak{g} = b \\ 4 & \text{for } \mathfrak{g} = c, d \end{cases}$$

$$(\Lambda, \delta_a) = -4$$

$$W \ni \mathbb{0} = (0, \cdots, 0)$$

$\mathrm{wt}(\mathbb{0}) = \Lambda$ for $\mathfrak{g} = b, d$

$\mathrm{wt}(\mathbb{0}^{\otimes 2}) = 2\Lambda$ for $\mathfrak{g} = c$

: the fundamental weight for α_0

say Λ_0

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$\lambda \in \mathcal{P}_{m|n}$: $(m|n)$-hook partition $\quad (\lambda_{m+1} \leqslant n)$

$V_{m|n}(\lambda)$ poly. repn. of $U_q(\mathfrak{gl}(m|n))$ w/ h.w. Λ_λ

1	1	1	1	1	1
2	2	2	2	2	
3	4	5	6	7	
3	4	5	6		
3	4				
3					
3					

$m = 2$

$n = 5$

$$\mathcal{P}_{m|n}(\mathfrak{g}) = \left\{ (\lambda, \ell) \mid \ell \geqslant \lambda_1 \ (\mathfrak{g} = b, d),\ \lambda_1/2 \ (\mathfrak{g} = c) \right\} \subset \mathcal{P}_{m|n} \times \mathbb{Z}_{>0}$$

$$\Lambda_{(\lambda,\ell)} := \ell \Lambda_0 + \Lambda_\lambda \quad \text{for } (\lambda, \ell) \in \mathcal{P}_{m|n}(\mathfrak{g})$$

$V(\lambda, \ell)$: the irreducible h.w. $U_q(\mathfrak{g}_{min})$-module w/ h.w. $\Lambda_{(\lambda,\ell)}$

Then

$$V \subset \mathcal{F}^{\otimes \ell} : \text{irreducible} \iff V \cong V(\lambda, \ell) \quad \text{for } (\lambda, \ell) \in P_{min}(\mathfrak{g})$$

---

pf.) It is done by taking classical limit of $\mathcal{F}^{\otimes \ell}$

applying the associated Howe dualities to characterize the h.w. of V ▨

Rmk

① q-Howe duality is naturally expected

non-super case (Sartori - Tubbenhauer 22)

in terms of co-ideal subalg

② $V(\lambda, \ell)$: infinite dimensional

③ $(\mu, \ell), (\nu, \ell') \in \mathcal{P}_{m|n}(\mathfrak{g})$

$$V(\mu, \ell) \otimes V(\nu, \ell') = \bigoplus_{(\lambda, \ell+\ell')} V(\lambda, \ell+\ell')^{c(\lambda, \ell+\ell')}$$

where $c(\lambda, \ell+\ell')$ = the usual LR coeff. of type $\mathfrak{g}$

④ $V(\lambda, \ell)$ naturally corresponds to an integrable h.w. module of type $\mathfrak{g}$ from a view point of "super duality"

(Cheng-Lam-Wang)

Theorem (K15) For $(\lambda, \ell) \in P_{\min}(\mathfrak{g})$

$V(\lambda, \ell)$ has a unique crystal base $(\mathcal{L}(\lambda), B(\lambda))$ where

$B(\lambda, \ell)$ is connected.

Rmk $\mathcal{L}(\lambda, \ell) = \sum A_0 \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \qquad (i_1, \cdots, i_r, \; x = e, f)$

$B(\lambda, \ell) = \{\pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \pmod{q\mathcal{L}(\lambda, \ell)}\} \setminus \{0\}.$

Sketch of proof.)

① Consider the crystal base of $\mathcal{F}^{\otimes \ell}$ $(\mathcal{L}^{\otimes \ell}, B^{\otimes \ell})$

② Find a h.w. vector $v_{(\lambda,\ell)}$ of wt. $\Lambda_{(\lambda,\ell)}$ s.t. $v_{(\lambda,\ell)} \in B^{\otimes \ell}$
$(\mathrm{mod}\ q\mathcal{L}^{\otimes \ell})$

③ Compute char. of $C(v_{(\lambda,\ell)}) \subset B^{\otimes \ell}$

④ Show $\mathrm{ch}\, C(v_{(\lambda,\ell)}) = \mathrm{ch}\, \overline{V(\lambda,\ell)}$

classical limit (repn of $U(\mathfrak{g}_{m|n})$)

This implies that $(\mathcal{L}(\lambda,\ell), B(\lambda,\ell))$ is a crystal base ▨

Rmk

① The main step in the proof is to compute $\mathrm{ch}\, C(v_{(\lambda,\ell)}) \subset B^{\otimes \ell}$

② $B = \bigsqcup_{k \geq 0} SST_{m|n}(1^k)$ as a crystal over $U_q(\mathfrak{gl}(m|n))$

$B^{\otimes \ell} \ni T_1 \otimes \cdots \otimes T_\ell$ where T_i : $(m|n)$-hook tableau of single column

③ $v_{(\lambda,\ell)} = H_{(1^{\lambda'_1})} \otimes H_{(1^{\lambda'_2})} \otimes \cdots =: b$

We give an explicit combinatorial description of $C(b)$

so called "spinor model"

: a natural super analog of classical BCD tableaux (Kashiwara - Nakashima)

$\exists$ a h.w. module (not necessarily irreducible)
w/ a crystal base ?

As in the case of $U_q(\mathfrak{gl}(m|n))$,

- no presentation of $V(\lambda,\ell)$ known
- no natural direct limit of $B(\lambda,\ell)$
- a crystal base of $U_q(\mathfrak{g}_{m|n})^-$ may not be satisfactory.

But like Kac modules, $\exists$ parabolic Verma modules
w/ "good" crystal base

Parabolic Verma modules

$U_q(\mathfrak{p})$: the subalg. gen. by $U_q(\mathfrak{l})$ & e_0

$\lambda \in P_{m|n}$ $\quad P(\lambda) := U_q(\mathfrak{g}_{m|n}) \otimes_{U_q(\mathfrak{p})} V_{m|n}(\lambda)$

Theorem (Jang-K-Uruno 23)

$P(\lambda)$ has a unique crystal base $(\mathcal{L}(P(\lambda)), B(P(\lambda)))$ where $B(P(\lambda))$ is connected.

Rmk In this case,

$P(\lambda)$ is not integrable with respect to $\langle e_0, f_0, t_0^{\pm 1} \rangle \cong U_q(sl_2)$

Instead, we consider the action of

$$B_q(sl_2) = \langle e_0', f_0 \mid e_0' f_0 = q_0^{-2} f_0 e_0' + 1 \rangle$$

q-Boson alg. of rank 1

For $u \in P(\lambda)$: weight vector

$$u = \sum_{k \geq 0} f_0^{(k)} u_k \qquad \text{where } e_0' u_k = 0$$

Define $\tilde{e}_0 u = \sum_{k \geq 1} f_0^{(k-1)} u_k$, $\tilde{f}_0 u = \sum_{k \geq 0} f_0^{(k+1)} u_k$

For the proof, we also need a PBW type basis of $U_q^-(\mathfrak{g}_{m|n})$.

We follow the construction of root vectors (by Leclerc) based on the theory of Lyndon words (by Lalonde-Ram) (Clark-Hill-Wang for super case)

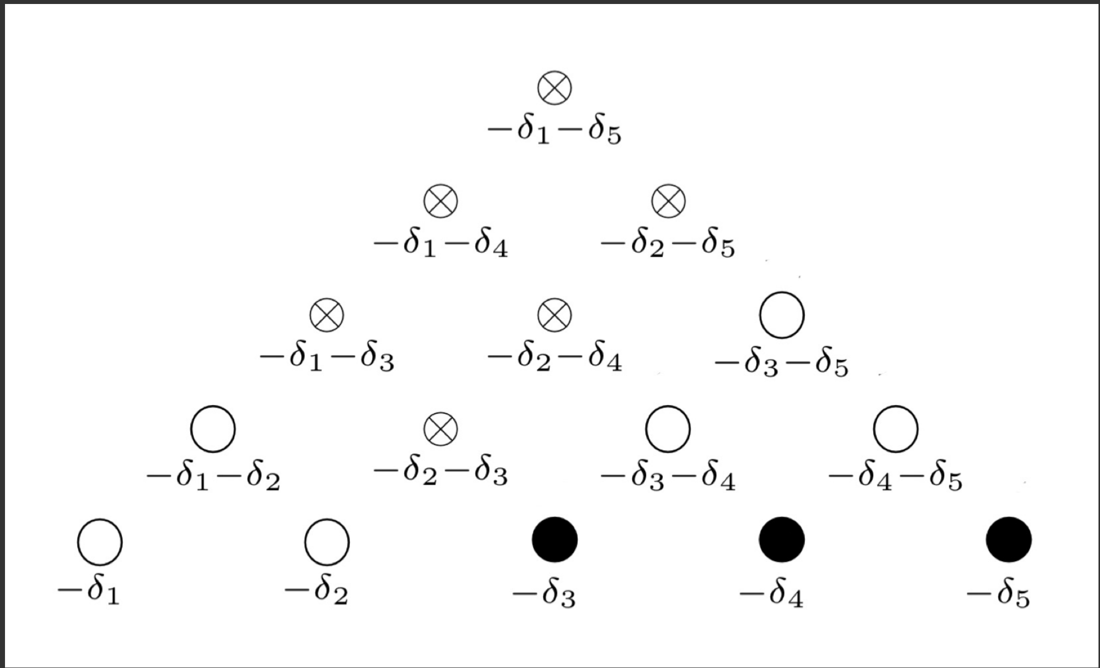
$\Phi^+(\ell)$ = the roots of $\mathfrak{gl}(m|n)$

$\Phi^+(u) = \Phi^+ \setminus \Phi^+(\ell)$

F_β : the root vector for $\beta \in \Phi^+$

$$\left\{ \prod^{<}_{\Phi^+(u)} F_\beta^{(c_\beta)} \prod^{<}_{\Phi^+(l)} F_\alpha^{(c_\alpha)} \;\middle|\; c_\gamma \in \mathbb{Z}_{\geq 0} ,\ c_\gamma \leq 1 .\ (\gamma : \text{reduced \& isotropic}) \right\}$$

: a $\Bbbk$-basis of $U_q(\mathfrak{g}_{m|n})^-$



$\Phi^+(u)$ for $\mathfrak{b}_{2|3}$

$$N := \text{the } \Bbbk\text{-span of } \prod_{\Phi^+(u)}^{<} F_\beta^{(c_\beta)}$$

$$N = \text{the subalg. gen. by } F_\beta \ (\beta \in \Phi^+(u))$$

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pf.) It follows from Levendorskii - Soibelman relation.

which have been explicitly computed by using embedding

into a quantum shuffle alg. ∎

$$P(\lambda) = \underbrace{U_q(\mathfrak{g}_{m|n})}_{U_q(\mathfrak{p})} \otimes V_{m|n}(\lambda) = N \otimes_{\Bbbk} V_{m|n}(\lambda) \quad \text{as } \Bbbk\text{-space.}$$

Sketch of proof.)

④ (Crucial part)

$$P(\lambda) \cong P(0) \otimes V_{\min}(\lambda) \quad \text{as a } U_q(\mathfrak{l})\text{-module}$$

& $$P(0) \cong N \curvearrowleft U_q(\mathfrak{l}) \quad \text{via quantum adjoint.}$$

$$\mathcal{L}(N) := A_0\text{-span of } \prod_{\Phi^+(u)}^{<} F_\beta^{(c_\beta)}$$

$$B(N) := \text{the image of } \pm \prod_{\Phi^+(u)}^{<} F_\beta^{(c_\beta)} \pmod{q\mathcal{L}(N)}$$

$(\mathcal{L}(N), B(N))$: a crystal base of N as $U_q(\mathfrak{l})$-module.

② $\tilde{x}_0 \mathcal{L}(N) \subset \mathcal{L}(N)$, $\tilde{x}_0 B(N) \subset B(N) \cup \{0\}$

where $\tilde{x}_0 = \tilde{e}_0, \tilde{f}_0$: crystal operators associated to α_0

(induced from a q-boson alg. of rank 1)

Hence $(\mathcal{L}(N), B(N))$: a crystal base over $U_q(\mathfrak{g}_{m|n})$

③ Put $\mathcal{L} = \mathcal{L}(N) \cdot \mathcal{L}(V_{m|n}(\lambda))$ $\quad B = B(N) \cdot B(V_{m|n}(\lambda))$

We show that $(\mathcal{L}, B)$: a crystal base over $U_q(\mathfrak{g}_{m|n})$

④ We prove B is connected. $\blacksquare$

Rmk ① $B(N)$ can be identified with the set of

$$(c_\beta)_{\beta \in \Phi^+(u)} \quad w/ \quad c_\beta \in \mathbb{Z}_{\geq 0} \quad c_\beta \leq 1 \ (\beta : \text{reduced isotropic})$$

$$B(P(\lambda)) = B(N) \times SST_{m|n}(\lambda)$$

$\tilde{e}_i, \tilde{f}_i$ can be described explicitly. in a combinatorial way.

② One can prove that

$$P(\lambda) \xrightarrow{\pi_\lambda} V(\lambda, \ell)$$

$$\mathcal{L}(P(\lambda)) \longrightarrow \mathcal{L}(\lambda, \ell)$$

$$B(P(\lambda)) \longrightarrow B(\lambda, \ell) \cup \{0\}$$

We have an embedding of crystals

$$B(\lambda, \ell) \xhookrightarrow{\iota_\lambda} B(N) \times SST_{m|n}(\lambda)$$

To understand $B(\lambda.\ell)$, it is natural question to describe ι_λ

For $\mathfrak{g} = b, c$, $\exists$ combinatorial algorithm of ι_λ (using symmetric RSK)

For $\mathfrak{g} = d$, it is in progress (using Burge correspondence)

analogue of RSK using domino insertion.

(Jang - K - Kwon, in preparation)

③ We can take a limit of $V_{m|n}(\lambda)$ to have

a crystal base of $U_q^-(\mathfrak{g}_{m|n})$.

$$\begin{array}{ccccc} \mathbf{B}(\lambda,\ell) & \hookrightarrow & \mathbf{B}(N)\times SST_{m|n}(\lambda) & \hookrightarrow & \mathbf{B}(N)\times \mathbf{B}(U_q^-(\ell)) \\ & & & & \parallel \\ & & & & \mathbf{B}(U_q^-(\mathfrak{g}_{m|n})) \end{array}$$

(Jang - K - Uruno , in preparation)

Thank you

Happy 65th birthday to Prof. Vyjayanthi Chari