Lecture 2. Crystal base for $\mathcal{U}_q(\mathfrak{gl}(m|n))$

1. Quantum super algebra $\mathcal{U}_q(\mathfrak{gl}(m|n))$

2. Polynomial representation $V(\lambda)$

3. Kac module $K(\lambda)$

1. Quantum superalgebra

- Assume that the base field = $\mathbb{C}$

A super space = a $\mathbb{Z}_2$-graded space $V = \underset{\text{even}}{\underline{V_0}} \oplus \underset{\text{odd}}{\underline{V_1}}$

$\mathfrak{gl}(V) := \mathrm{End}_{\mathbb{C}}(V)$

: a super space $\mathfrak{gl}(V)_\varepsilon \ni f : V_k \longrightarrow V_{k+\varepsilon}$

a Lie superalgebra w.r.t. $[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$

called a general linear Lie super algebra

- $m, n \geq 0$ $\quad I(m|n) = \{ \underbrace{1 < 2 < \cdots < m}_{\text{even}} < \underbrace{m+1 < \cdots < m+n}_{\text{odd}} \}$

$$\mathbb{C}^{m|n} = \underbrace{\mathbb{C}^{m|0}}_{\text{even}} \oplus \underbrace{\mathbb{C}^{0|n}}_{\text{odd}} = \bigoplus_{I(m|n)} \mathbb{C} v_a$$

- $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$

$$= \text{the set of } (m+n) \times (m+n) \text{ matrices} \qquad \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \end{array}$$

A, D : even

B, C : odd

$$\mathfrak{gl}(m|n)_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$$

$$\mathfrak{h} = \text{span of } E_{aa} \quad (\text{Cartan subalg})$$

- $(X,Y) := \mathrm{str}(XY)$: non-deg inv super symm. bilinear form

$\{ \delta_a \mid a \in I(m|n) \}$: a basis of $\mathfrak{h}^*$ dual to $\{ E_{aa} \}$

$$(\delta_a | \delta_b) = \begin{cases} 1 & a = b \leq m \\ -1 & a = b > m \\ 0 & \text{otherwise} \end{cases}$$

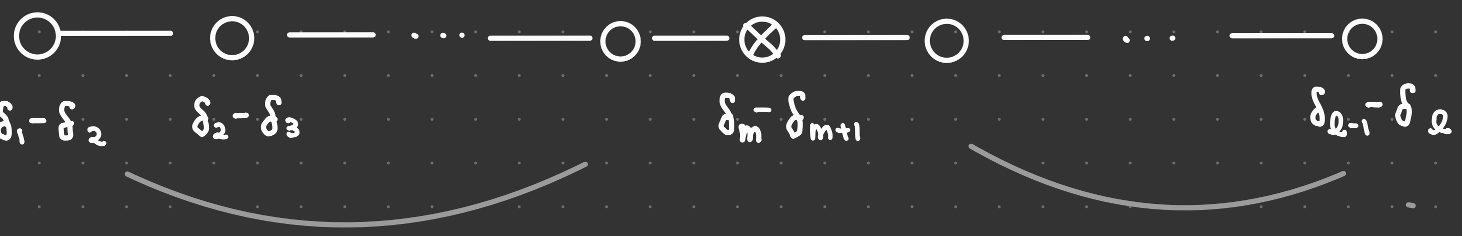
induced bilinear form

- $\Phi^+ = \{ \delta_a - \delta_b \mid \underset{a<b}{a \neq b} \}$: the set of positive roots

$\Phi_0^+ = \{ \delta_a - \delta_b \mid a < b \leq m \,,\; m < a < b \}$ even

$\Phi_1^+ = \{ \delta_a - \delta_b \mid a \leq m < b \}$ odd (isotropic)

$\Pi = \{ \delta_a - \delta_{a+1} \mid 1 \leq a < m+n \}$: the set of simple roots



$(\alpha|\alpha) > 0$ $\quad$ $(\alpha|\alpha) = 0$ $\quad$ $(\alpha|\alpha) < 0$

even $\quad$ odd $\quad$ even

$\ell = m+n$

- $\mathcal{U}(\mathfrak{gl}(m|n))$: the enveloping algebra of $\mathfrak{gl}(m|n)$

$$\mathcal{U}(\mathfrak{gl}(m|n)^+) \cong \mathcal{U}(\mathfrak{gl}(m|n)_0^+) \otimes \mathcal{U}(\mathfrak{gl}(m|n)_1^+)$$

$$\mathcal{U}(\mathfrak{gl}(m|n)_1^+) \cong \Lambda(\mathfrak{gl}(m|n)_1^+) \quad \text{as a } \mathbb{C}\text{-alg.}$$

- q : indeterminate , $\Bbbk = \mathbb{Q}(q)$

$\mathcal{U}_q(\mathfrak{gl}(m|n))$: the q-analogue of $\mathcal{U}(\mathfrak{gl}(m|n))$ introduced by Yamane (94)

generators : $E_\alpha, F_\alpha, K_a, K_a^{-1}$ $\quad (\alpha \in \Pi ,\ a \in I(m|n))$

relations :

$K_a^{\pm 1}$: commutative $\qquad K_a E_\alpha K_a^{-1} = q^{(\alpha, \varepsilon_a)} E_\alpha \qquad K_a F_\alpha K_a^{-1} = q^{-(\alpha, \varepsilon_a)} F_\alpha$

$$E_\alpha F_\beta - (-1)^{|\alpha||\beta|} F_\beta E_\alpha = \delta_{\alpha\beta} \cdot \mathrm{sgn}(\alpha|\alpha) \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}} \qquad (K_\alpha = K_a K_{a+1}^{-1})$$

usual Serre relations for E_α, F_α $\qquad (\alpha|\alpha) > 0, < 0$

\+ odd Serre relation for E_α, F_α $\qquad (\alpha|\alpha) = 0$

Rmk

① $(\alpha|\alpha) > 0 \qquad \langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \rangle \cong \mathcal{U}_q(sl_2)$

$\qquad\quad < 0 \qquad \langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \rangle \cong \mathcal{U}_{q^{-1}}(sl_2)$

② $\mathcal{U}_q(gl(m|n))$: $\mathbb{Z}_2$-graded $\qquad \deg(E_\alpha) = \deg(F_\alpha) = 1 \qquad (\alpha|\alpha) = 0$

③ Instead of super representations, we consider a repn of

$$\mathcal{U}_q(gl(m|n))[\sigma] = \mathcal{U}_q(gl(m|n)) \oplus \mathcal{U}_q(gl(m|n))\sigma$$

$$\sigma^2 = 1.$$

$$\sigma K_a^{\pm 1} = K_a^{\pm 1}\sigma \qquad \sigma X_\alpha = (-1)^{|\alpha|} X_\alpha \sigma \quad (X = E, F)$$

V : a $U_q(\mathfrak{gl}(m|n))[\sigma]$-module if

- $V = V_0 \oplus V_1$: $\mathbb{Z}_2$-graded $U_q(\mathfrak{gl}(m|n))$-module
- $\sigma|_{V_\varepsilon} = (-1)^\varepsilon \mathrm{id}_{V_\varepsilon}$.

(= a super representation of $U_q(\mathfrak{gl}(m|n))$)

Hopf algebra structure of $U_q(\mathfrak{gl}(m|n))[\sigma]$

$$\Delta K_a^{\pm 1} = K_a^{\pm 1} \otimes K_a^{\pm 1}$$

$$\Delta E_\alpha = E_\alpha \otimes K_\alpha^{-1} + \sigma^{|\alpha|} \otimes E_\alpha \, , \quad \Delta F_\alpha = F_\alpha \otimes 1 + \sigma^{|\alpha|} K_\alpha \otimes F_\alpha$$

- $P = \bigoplus_a \mathbb{Z}\,\delta_a$

- A weight space of V with $\lambda \in P$

$$V_\lambda = \{ v \mid K_\mu v = q^{(\mu|\lambda)} v \ \text{ for } \mu \in P \}$$

We assume V has a wt. space decomposition

Rmk We may consider another version of QSA

(due to Kuniba-Okado-Sergeev '15)

$$q_a = \begin{cases} q & (1 \leq a \leq m) \\ -q^{-1} & (m < a \leq m+n) \end{cases}$$

$\mathbb{q}(\lambda,\mu) = \prod_a q_a^{\lambda_a \mu_a}$ for $\lambda = \sum_a \lambda_a \delta_a$, $\mu = \sum_a \mu_a \delta_a$

defining relations : $q^{(\lambda|\mu)} \rightsquigarrow \mathbb{q}(\lambda,\mu)$

weight space

$q \rightsquigarrow -q^{-1}$

(on odd space)

It is almost isomorphic to $\mathcal{U}_q(\mathfrak{gl}(m|n))$ in the sense ;

$$\mathcal{U}_q(\mathfrak{gl}(m|n))\,[\sigma_a] = \langle\, \mathcal{U}_q(\mathfrak{gl}(m|n)),\ \sigma_a \,\rangle$$

$\sigma_a \ (a \in I(m|n))$: $\sigma_a \sigma_b = \sigma_b \sigma_a$ $\quad \sigma_a^2 = 1$

$\sigma_a K_b = K_b \sigma_a$ $\quad \sigma_a X_\alpha = (\delta_a|\delta_a)^{(\delta_a|\alpha)} X_\alpha \sigma_a$ $\quad (X = E, F)$

$\exists$ $\mathcal{U}_q^{KOS}[\sigma_a] \xrightarrow{\cong} \mathcal{U}_q^{Y}[\sigma_a]$ as a $\Bbbk$-alg.

$X_\alpha \longmapsto X_\alpha \times (\text{product of } \sigma_a\text{'s})$ $\quad X = E, F, K$

$\mathfrak{gl}^{(\lambda|\mu)}$ $\longleftrightarrow$ $\mathfrak{q}^{(\lambda|\mu)}$

μ-wt sp $\qquad\qquad$ μ-wt sp

From now on, we use $\mathcal{U}_{m|n} = \underline{\mathcal{U}_q(\mathfrak{gl}(m|n)) \text{ by KOS}}$ w/

$$\Delta K_a^{\pm 1} = K_a^{\pm 1} \otimes K_a^{\pm 1}$$

$$\Delta E_\alpha = E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha, \quad \Delta F_\alpha = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha$$

Many arguments in usual QG can be applied directly w/ above change of convention

2. Polynomial representations

- Unlike $\mathcal{U}_q(\mathfrak{gl}(m+n))$, a fin-dim'l $\mathcal{U}_{m|n}$-module is not semisimple in general.

- But, there is a good family of semisimple repn's closed under $\otimes$ (due to Schur-Weyl-Jimbo duality)

- $\mathcal{P}_{\geq 0} = \bigoplus_{a \in I(m|n)} \mathbb{Z}_{\geq 0}\, \delta_a$: the set of polynomial weights.

- $\mathcal{O}_{\geq 0}$: the category of $\mathcal{U}_{m|n}$-modules w/ wt's in $\mathcal{P}_{\geq 0}$

- $V_{m|n} = \bigoplus_{a \in I(m|n)} \Bbbk v_a$: the natural representation of $\mathcal{U}_{m|n}$

$$v_a \underset{E_\alpha}{\overset{F_\alpha}{\rightleftarrows}} v_{a+1} \qquad (\alpha = \delta_a - \delta_{a+1})$$

$$\delta_a \qquad\qquad \delta_{a+1}$$

$V_{m|n}^{\otimes d} \in \mathcal{O}_{\geq 0}$

Moreover, $\exists$ analog of Schur-Weyl-Jimbo duality on $V_{m|n}^{\otimes d}$

$\mathcal{U}_{m|n} \curvearrowright V_{m|n}^{\otimes d} \curvearrowleft \mathcal{H}_d$: Heck alg. of type A_{d-1}

$\rightsquigarrow$ semisimple

- $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$: a partition w/ $\lambda_{m+1} \leqslant n$ $(\in \mathcal{P}_{m|n})$

$$\Lambda_\lambda = \lambda_1 \delta_1 + \cdots + \lambda_m \delta_m + \mu_1 \delta_{m+1} + \cdots + \mu_n \delta_{m+n}$$

where $\mu = (\lambda_{m+1} \geqslant \lambda_{m+2} \geqslant \cdots)'$

1	1	1	1	1	1
2	2	2	2	2	
3	4	5	6	7	
3	4	5	6		
3	4				
3					
3					

$m = 2$

$n = 5$

- $V_{m|n}(\lambda)$: the irreducible h.w. module w/ h.w. Λ_λ

Then

① $V_{m|n}^{\otimes d} = \bigoplus_{\substack{\lambda \\ |\lambda| = d}} V_{m|n}(\lambda) \otimes S^{\lambda}$

② Every irreducible module $\in \mathcal{O}_{\geq 0} \cong V_{m|n}(\lambda)$

(Benkart - Kang - Kashiwara 00)

- $\exists$ a combinatorial model for $\mathrm{ch}\, V_{m|n}(\lambda)$

$SST_{m|n}(\lambda)$ = the set of $(m|n)$-hook semistandard tableaux

$$\mathrm{ch}\, V_{m|n}(\lambda) = \sum_{T \in SST_{m|n}(\lambda)} x^{T} = hs_{\lambda}(x)$$

hook Schur poly. (super)

- Theorem (Benkart-Kang-Kashiwara 00)

$V_{m|n}(\lambda)$ has a "crystal base" w/ a cnn. crystal str. on $SST_{m|n}(\lambda)$

What is a "crystal base" here?

It is defined in a similar way w.r.t. crystal operators $\tilde{f}_\alpha$

$$\tilde{f}_\alpha = \begin{cases} \text{lower crystal operator} & (\alpha|\alpha) > 0 \\ \text{upper crystal operator} & (\alpha|\alpha) < 0 \\ \text{multiplication by } F_\alpha & (\alpha|\alpha) = 0 \end{cases} \qquad (\alpha \in \Pi)$$

Rmk (Upper crystal base)

- $v = \sum_{n \geq 0} f_i^{(n)} v_n \in M_\lambda \qquad (e_i v_n = 0)$

$$\tilde{e}_i^{up} v = \sum_{n \geq 1} q_i^{-l_n + 2n - 1} f_i^{(n-1)} v_n \qquad \tilde{f}_i^{up} v = \sum_{n \geq 0} q_i^{l_n - 2n - 1} f_i^{(n+1)} v_n$$

$$l_n = \langle h_i, wt(v_n) \rangle$$

- $V(\lambda)$ has a crystal base $(\mathcal{L}^{up}(\lambda), B^{up}(\lambda))$ w.r.t. $\tilde{e}_i^{up}, \tilde{f}_i^{up}$

- Tensor product theorem holds w.r.t. Δ_+: upper co-mult.

$$\Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i$$

$$\Delta_+(f_i) = 1 \otimes f_i + f_i \otimes t_i^{-1}$$

$$\Delta_+(q^h) = q^h \otimes q^h$$

Rmk $\alpha \in \Pi$ $\quad (\alpha|\alpha) < 0$ $\quad$ As $\mathbb{Q}$-alg.'s we have

$$\begin{array}{ccccc}
\langle E_\alpha, F_\alpha, K_\alpha^{\pm 1} \rangle & = & \mathcal{U}_{-q^{-1}}(\mathfrak{sl}_2) & \overset{-}{\cong} & \mathcal{U}_p(\mathfrak{sl}_2) \\
E_\alpha & \longleftarrow\!\shortmid & e & \longleftarrow\!\shortmid & e \\
F_\alpha & \longleftarrow\!\shortmid & f & \longleftarrow\!\shortmid & f \\
K_\alpha & \longleftarrow\!\shortmid & k & \longleftarrow\!\shortmid & k \\
q & \longleftarrow\!\shortmid & -q^{-1} = p & \longleftarrow\!\shortmid & p^{-1}
\end{array}$$

Δ	Δ : lower co-mult.	$\overline{\Delta}$: upper (flipped)
CB at $q = 0$	CB at $p = \infty$	CB at $p = 0$

$\tilde{f}_\alpha \leftsquigarrow \tilde{f}$: upper crystal operator + tensor product rule (in reverse order) $\leftsquigarrow$ upper crystal operator

With respect to these crystal operators.

- **Def.** $V \in \mathcal{O}_{\geq 0}$

$(\mathcal{L}, B)$: a crystal base of V if

① $\mathcal{L}$: A_0-lattice of V + wt sp. decomp.

② $\mathbf{B} = B \cup (-B)$ $\quad B \subset \mathcal{L}/q\mathcal{L}$: $\mathbb{Q}$-basis. + wt. sp. decomp.

③ $\tilde{x}_\alpha \mathcal{L} \subset \mathcal{L}$, $\quad \tilde{x}_\alpha B \subset B \cup \{0\}$ $\quad x = e, f \quad \alpha \in \Pi$

Rmk

As a $U_{0|n}$-module, $(\mathcal{L}, B)$: upper crystal base.

$U_{m|0}$-module, " : lower crystal base.

- **Theorem** (BKK)

$(\mathcal{L}_i, \mathcal{B}_i)$: a crystal base of $V_i \in \mathcal{O}_{\geq 0}$

$\Rightarrow$ $(\mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$: a crystal base of $V_1 \otimes V_2$

for $\alpha \in \Pi$ w/

$(\alpha|\alpha) > 0$ $\quad \tilde{f}_\alpha(b_1 \otimes b_2)$ as in Tensor product theorem/rule.

$(\alpha|\alpha) < 0$ $\quad \tilde{f}_\alpha(b_1 \otimes b_2)$ applying " to $b_2 \otimes b_1$

$$(\alpha|\alpha) = 0 \qquad \tilde{f}_\alpha(b_1 \otimes b_2) = \begin{cases} \tilde{f}_\alpha b_1 \otimes b_2 & \text{if } (\mathrm{wt}(b_1)|\alpha) \neq 0 \\ b_1 \otimes \tilde{f}_\alpha b_2 & \text{if } (\mathrm{wt}(b_1)|\alpha) = 0 \end{cases}$$

Rmk

① $B_{m|n}(\lambda)$ can be realized as a subgraph of $B_{m|n}^{\otimes|\lambda|}$

where $B_{m|n}$: crystal of $V_{m|n}$

$$\Rightarrow B_{m|n}(\lambda) \cong SST_{m|n}(\lambda) \subset B_{m|n}^{\otimes|\lambda|}$$

② $B_{m|n}(\lambda)$ may have an element b s.t

$b \neq v_\lambda$ but $\tilde{e}_\alpha v_\lambda = 0$ for all $\alpha \in \Pi$

③ From the connectedness of $SST_{m|n}(\lambda)$

$$\mathcal{L}_{m|n}(\lambda) = \sum_{\beta_1,\cdots,\beta_r} A_0 \tilde{x}_{\beta_1} \cdots \tilde{x}_{\beta_r} v_\lambda \qquad (r \geq 0,\ \beta_i \in \Pi,\ x = e, f)$$

$$B_{m|n}(\lambda) = \left\{ \pm \tilde{x}_{\beta_1} \cdots \tilde{x}_{\beta_r} v_\lambda \ (\mathrm{mod}\ q\mathcal{L}_{m|n}(\lambda)) \right\} \setminus \{0\}$$

: a crystal base of $V_{m|n}(\lambda)$ $\quad (\lambda \in P_{m|n})$

④ Unlike $B_m(\lambda)$, $\exists$ no natural crystal embedding

$$B_{m|n}(\lambda) \longrightarrow B_{m|n}(\mu) \quad \text{for } \lambda, \mu \in P_{m|n}$$

which yields a direct limit as in the case of $U_q(gl_n)$

Example

$m = 3, \ n = 4$

$$B_{3|4} : \quad 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} 4 \xrightarrow{4} 5 \xrightarrow{5} 6 \xrightarrow{6} 7.$$

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 6 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array} \in SST_{3|4}(5,3,2,2)$$

$\longrightarrow$ row reading

$$w = 6\ 3\ 2\ 1\ 1\ 7\ 5\ 2\ 5\ 3\ 6\ 4 \in B_{3|4}^{\otimes 12}$$

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$\tilde{f}_1 \quad (\alpha_1|\alpha_1) > 0$

$$6\ 3\ 2\ 1\ 1\ 7\ 5\ 2\ 5\ 3\ 6\ 4 = w$$
$$-\ +\ +\qquad -$$

$$\downarrow \tilde{f}_1$$

$$6\ 3\ 2\ 2\ 1\ 7\ 5\ 2\ 5\ 3\ 6\ 4$$
$$-\ -\qquad -$$

$\tilde{f}_4 \quad (\alpha_4|\alpha_4) < 0$

$$6\ 3\ 2\ 1\ 1\ 7\ 4\ 2\ 5\ 3\ 6\ 4$$
$$+\quad -\quad +$$

$$\downarrow \tilde{f}_4$$

$$6\ 3\ 2\ 2\ 1\ 7\ 5\ 2\ 5\ 3\ 6\ 4 = w$$
$$-\quad -\quad +$$

$$\tilde{f}_3 \qquad (\alpha_3|\alpha_3) = 0$$

$$\begin{array}{cccccccccccccc}
6 & 3 & 2 & 1 & 1 & 7 & 4 & 2 & 5 & 3 & 6 & 4 & = & w \\
 & + & & & & & - & & & & & - & & \\
\end{array}$$

$$\Big\downarrow \tilde{f}_3$$

$$\begin{array}{cccccccccccc}
6 & 4 & 2 & 1 & 1 & 7 & 4 & 2 & 5 & 3 & 6 & 4 \\
 & - & & & & & - & & & & & - \\
\end{array}$$

$$\tilde{x}_i T = T' \longleftrightarrow \tilde{x}_i w \qquad (x = e, f).$$

Applications

① (non-standard Borel)

One can consider $U_{m|n}$ w.r.t. a non-std Borel

$$\underset{1}{\circ} - \underset{2}{\circ} - \underset{3}{\otimes} - \underset{4}{\circ} - \underset{5}{\circ} \qquad 1 < 2 < 3 < 4 < 5 < 6$$

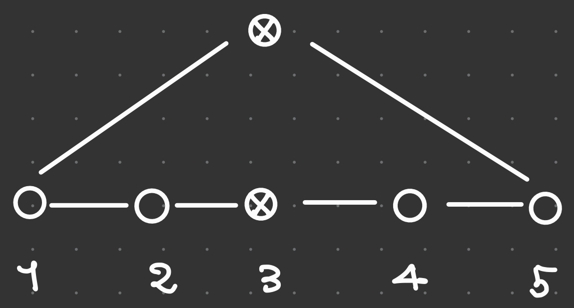
odd even standard

$$\underset{1}{\otimes} - \underset{2}{\otimes} - \underset{3}{\otimes} - \underset{4}{\otimes} - \underset{5}{\otimes} \qquad 1 < 2 < 3 < 4 < 5 < 6$$

The (non-standard) crystal structure on $B_{m|n}^{\otimes d}$ has a connection with quasi-symmetric functions. (K 09)

② (affine case)

One can define the QSA of affine type A



$\exists$ Kirillov-Reshetikhin type module $W^{r,s}$ with a crystal base

for $(r^s) = (\underbrace{r,\cdots,r}_{s}) \in \mathcal{P}_{m|n}$ (K-Okado 24)

and $B^{r,s} \cong SST_{m|n}((r^s))$ as a crystal of finite type

3. Kac modules.

- $\exists$ crystal base of a Verma module for $\mathcal{U}_{m|n}$?

- $\exists$ natural direct system on $\{ B_{m|n}(\lambda) \mid \lambda \in P_{m|n} \}$?

No presentation for $V_{m|n}(\lambda)$ is known so far.

$\rightsquigarrow$ We do not know a natural partial order on $P_{m|n}$

- How to take a limit of $B_{m|n}(\lambda)$?

- In repn theory of $\mathfrak{gl}(m|n)$,

$\exists$ an important family of fin.dim indecomp h.w. modules

$\cong$ a parabolic Verma module w.r.t $\mathfrak{gl}(m|n)_0$

- 

$$E_i := E_{\alpha_i} \qquad F_i = F_{\alpha_i}$$

$$\mathcal{U}_{m,n} = \langle E_i, F_i, K_a^{\pm 1} \mid i \neq m \rangle$$

- $P^+ = \{ \lambda = \sum \lambda_a \delta_a \in P \mid \lambda_1 \geq \cdots \geq \lambda_m , \ \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \}$

$\lambda \in P^+$ $\quad \lambda_+ = \sum_{1 \leq a \leq m} \lambda_a \delta_a \quad \lambda^- = \sum_{m < a \leq m+n} \lambda_a \delta_a$

$V_{m,n}(\lambda) := V_{m|0}(\lambda^+) \otimes V_{0|n}(\lambda^-) \curvearrowleft \underset{\langle \mathcal{U}_{m,n}, E_m \rangle}{\overset{}{P \atop \shortparallel}}$ $\quad$ (E_m : trivially)

$K(\lambda) := \mathcal{U}_{m|n} \underset{P}{\otimes} V_{m,n}(\lambda)$

: indecomposable h.w. module w/ h.w. λ

$V_{m|n}(\lambda) =$ the max. quotient of $K(\lambda)$

Theorem (K14)

① $K(\lambda)$ has a unique crystal base $(\mathcal{L}(K(\lambda)), B(K(\lambda)))$

where $B(K(\lambda))$: connected

② $\lambda \in P_{m|n}$

$$\begin{array}{ccc} \Lambda_\lambda & & \\ \| & & \\ K(\lambda) & \xrightarrow{\pi_\lambda} & V_{m|n}(\lambda) \\ \cup & & \cup \\ \mathcal{L}(K(\lambda)) & \longrightarrow & \mathcal{L}_{m|n}(\lambda) \end{array}$$

$$B(K(\lambda)) \longrightarrow B_{m|n}(\lambda) \cup \{0\} \quad q=0$$

Rmk We should define $\tilde{e}_m, \tilde{f}_m$.

- To construct a crystal base of $K(\lambda)$

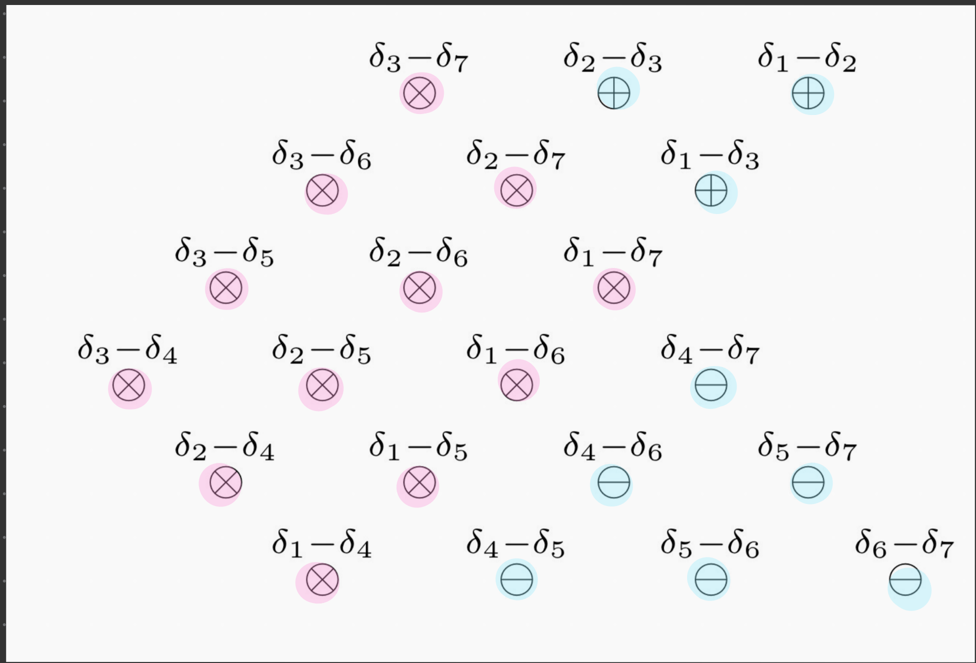
  we use a PBW type basis of $\mathcal{U}^-_{m|n}$

- $\Phi_0^+ = \{ \delta_a - \delta_b \mid a < b \leq m \,,\ m < a < b \}$ even

  $\Phi_1^+ = \{ \delta_a - \delta_b \mid a \leq m < b \}$ odd

- We take a particular convex order on Φ^+ assoc. to a reduced expression of $w_0 \in S_{m+n}$ adapted to

$$\underset{\alpha_1}{\circ} \longrightarrow \underset{\alpha_2}{\circ} \longrightarrow \cdots \longrightarrow \circ \longrightarrow \underset{\alpha_m}{\circ} \longleftarrow \circ \longleftarrow \cdots \longleftarrow \underset{\alpha_{m+n-1}}{\circ}$$

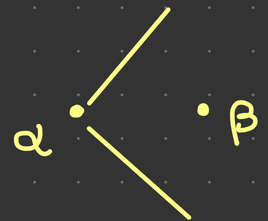


$m = 3$

$n = 4$

$\oplus : (\beta|\beta) > 0$ $\quad$ $\ominus : (\beta|\beta) < 0$ $\quad$ $\otimes : (\beta|\beta) = 0$

$\alpha < \beta \iff$



- $\beta \in \Phi^+$, define a root vector F_β by

using Lusztig's transf. T_i's $(i \neq m)$ if $\beta \in \Phi_0^+$

applying q-adjoint $ad_q(F_i)$'s $(i \neq m)$ to F_m if $\beta \in \Phi_1^+$

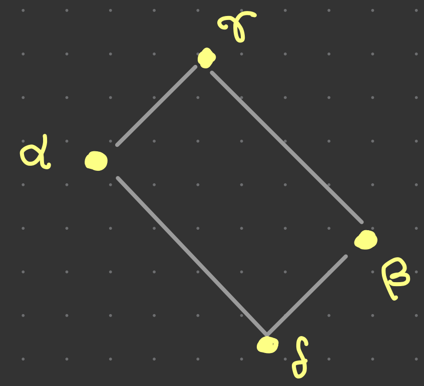
$$F_\beta = ad_q(F_j) \circ \cdots \circ ad_q(F_{m+1}) \circ ad_q(F_i) \circ \cdots \circ ad_q(F_{m-1})(F_m)$$

$$(\beta = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j)$$

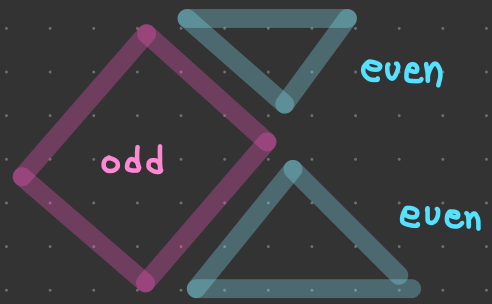
where $\quad ad_q(x)(y) = [x, y]_{\mathbb{q}} = xy - \mathbb{q}^{-1}(|x|, |y|)\, yx$

- We also have the Levendorskii - Soibelman type relation

$$[F_\beta, F_\alpha]_q = \begin{cases} (q^{-1}-q)\, F_\gamma F_\delta & \left(\alpha < \begin{matrix}\gamma \\ \delta\end{matrix} < \beta, \quad \alpha+\beta = \gamma+\delta\right) \\ F_\gamma & (\gamma = \alpha+\beta) \\ 0 & \end{cases}$$



+ $F_\alpha^2 = 0 \qquad (\alpha \in \Phi_1^+)$



- $\left\{ \prod_{\Phi^+}^{<} F_\alpha^{(c_\alpha)} \;\middle|\; c_\alpha \in \mathbb{Z}_{\geq 0},\ c_\beta = 0, 1.\ (\beta \in \Phi_1^+) \right\}$: $\Bbbk$-basis of $\mathcal{U}_{m|n}^-$

- $K = \langle F_\beta \mid \beta \in \Phi_1^+ \rangle$ = span of $\prod_{\text{odd}}^{<} F_\gamma^{(c_\gamma)}$

  : the subalg. generated by odd root vectors.

$$\mathcal{U}_{m|n}^- \cong K \otimes \mathcal{U}_{m,n}^- = K \otimes \mathcal{U}_{m,0}^- \otimes \mathcal{U}_{0,n}^- \quad \text{as } \Bbbk\text{-spaces}$$

- $K(\lambda) \cong \mathcal{U}_{m|n}^- \otimes V_{m,n}(\lambda)$ as $\Bbbk$-spaces

$$= K \underset{\Bbbk}{\otimes} V_{m|0}(\lambda^+) \underset{\Bbbk}{\otimes} V_{0|n}(\lambda^-)$$

- $K(\lambda)$: fin-dimensional &

$$\mathrm{ch}\, K(\lambda) = \mathrm{ch}\, K \cdot \mathrm{ch}\, V_{m,n}(\lambda) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (1 + x_i^{-1} y_j)\, s_{\lambda^+}(x)\, s_{(\lambda^-)^t}(y)$$

which can be viewed as a q-deformed Kac-module

- Define $\tilde{e}_i, \tilde{f}_i$ on $K(\lambda)$ by

$$\begin{cases} \text{lower crystal operator for } 1 \le i < m & ((\alpha_i|\alpha_i) > 0) \\ \text{upper crystal operator for } m < i \le m+n-1 & ((\alpha_i|\alpha_i) < 0) \\ f_m \text{ (multiplication)} \quad \text{where } \tilde{e}_m = e_m' \text{ (left derivation)} & \end{cases}$$

Sketch of proof) (Existence)

$$\begin{array}{ccc} K & \xrightarrow[\cong]{\psi} & \Lambda^q(\Bbbk^m \otimes \Bbbk^n) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ F_\beta & \longmapsto & v_i \otimes w_j \end{array} \quad (\beta = \delta_i - \delta_j)$$

: q-deformed exterior alg.
gen. by $v_i \otimes w_j$
(Uglov.)

$\exists$ an action of $\underline{\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_p(\mathfrak{gl}_n)}$ on $\Lambda^q(\Bbbk^m \otimes \Bbbk^n)$ $\quad p = -q^{-1}$

$$\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_p(\mathfrak{gl}_n) \cong \mathcal{U}_{m,n}$$

$\mathcal{U}_{m,n}$-module K induced by ψ $\cong$ $K(0) = K \otimes V_{m|n}(0)$

$$\Rightarrow\ K(\lambda) \cong \Lambda^q(\Bbbk^m \otimes \Bbbk^n) \otimes V_{m|n}(\lambda)$$ as $\mathcal{U}_{m,n}$-module

$$\mathcal{L}(K) := \bigoplus_{(c_\beta)} A_0 \prod_{\Phi_1^+} F_\beta^{c_\beta} \qquad B(K) := \left\{ \pm \prod_{\Phi_1^+} F_\beta^{c_\beta} \pmod{q\mathcal{L}(K)} \right\}$$

$$\Rightarrow\ \mathcal{L}(K(\lambda)) := \mathcal{L}(K) \otimes \mathcal{L}(V_{m|n}(\lambda))$$

$$B(K(\lambda)) := B(K) \times B(V_{m|n}(\lambda))$$

forms a crystal base of $K(\lambda)$ as a $\mathcal{U}_{m,n}$-module

Finally, one can check $\underline{\tilde{x}_m \mathcal{L}(K(\lambda)) \subset \mathcal{L}(K(\lambda))}$ $(x = e, f)$

$\tilde{f}_m \prod_{\Phi_1^+} F_\beta^{c_\beta}$ is given by $\begin{cases} c_{\alpha_m} \longmapsto c_{\alpha_m} + 1 & (c_{\alpha_m} = 0) \\ 0 & (c_{\alpha_m} = 1) \end{cases}$

$$B(K) \overset{1-1}{\longleftrightarrow} \mathcal{P}(\Phi_1^+)$$: power set of Φ_1^+

We may regard

$$B(K(\lambda)) = \mathcal{P}(\Phi_1^+) \times B(V_{m|0}(\lambda^+)) \times B(V_{0|n}(\lambda^-))$$

$B(K(\lambda))$ can be described explicitly since

$U_{m,n}$-crystal structure is well-known + tensor product rule

This implies

① the connectedness of $B(K(\lambda))$

② $\mathcal{L}(K(\lambda)) = \sum A_0 \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \quad (i_1, \cdots, i_r,\ x = e, f)$

$B(K(\lambda)) = \{ \pm \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} v_\lambda \ (\mathrm{mod}\ q\mathcal{L}(K(\lambda)) \} \setminus \{0\}$

③ Uniqueness of a crystal base of $K(\lambda)$

(Compatibility with $V_{m|n}(\lambda)$ for $\lambda \in \mathcal{P}_{m|n}$)

- $K(\lambda)$: irreducible $\iff$ λ : typical

  i.e. $(\lambda + \rho_{m|n} \,|\, \beta) = 0$ for all $\beta \in \Phi_1^+$

  where $\rho_{m|n} = \frac{1}{2}\sum_{\Phi_0^+} \alpha - \frac{1}{2}\sum_{\Phi_1^+} \beta$

(It follows from the fact at $q=1$ due to Kac)

In particular,

$\lambda \in \mathcal{P}_{m|n}$ $\quad \Lambda_\lambda$: typical $\iff (n^m) \subset \lambda$

$$\iff K(\lambda) = V_{m|n}(\lambda)$$

$$K(\lambda) = K(\Lambda_\lambda)$$

- $\lambda \in P_{m|n}$ (identifying w/ Λ_λ)

  $\delta_+ = \delta_1 + \cdots + \delta_m$ $\quad \lambda + \ell\delta_+$: typical for $\ell \gg 0$.

Consider the following comm. diagram :

$$\begin{tikzcd}
K(\lambda+\ell\delta_+) \arrow[r, "\pi_{\lambda+\ell\delta_+}"] \arrow[d, "\theta_\ell"] & V_{m|n}(\lambda+\ell\delta_+) \arrow[r, "\Phi_{\lambda,\ell}"] \arrow[d, "\theta_\ell"] & V_{m|n}(\lambda)\otimes V_{m|n}(\ell\delta_+) \arrow[dl, "S_{\lambda,\ell}"] \\
K(\lambda) \arrow[r, "\pi_\lambda"] & V_{m|n}(\lambda) &
\end{tikzcd}$$

$$1\otimes v_{\lambda+\ell\delta_+} \mapsto 1\otimes v_\lambda \qquad\qquad v\otimes v_{\ell\delta_+} \mapsto v$$

horizontal : $U_{m|n}$-linear

vertical : $U^-_{m|n}$-linear

$$\begin{tikzcd}
K(\lambda + \ell\delta_+) \arrow[r, "\pi_{\lambda+\ell\delta_+}", "\cong"'] \arrow[d, "\wr"', "\theta_\ell"] & V_{min}(\lambda + \ell\delta_+) \arrow[d, "\theta_\ell"] \\
K(\lambda) \arrow[r, "\pi_\lambda"'] & V_{min}(\lambda)
\end{tikzcd}$$

each map sends crystal base to crystal base

$\Rightarrow$ π_λ sends $\mathcal{L}(K(\lambda)) \longrightarrow \mathcal{L}_{min}(\lambda)$

$B(K(\lambda)) \longrightarrow B_{min}(\lambda) \cup \{0\}$

$\blacksquare$

Rmk

① We have a combinatorial description of crystal embedding

$$\begin{array}{ccc} B_{m|n}(\lambda) & \longrightarrow & \mathcal{P}(\Phi_1^+) \times B_{m|0}(\lambda^+) \times B_{0|n}(\lambda^-) \\ \| & & \qquad\qquad \| \qquad\qquad\quad \| \\ SST_{m|n}(\lambda) & & \qquad SST_{m|0}(\lambda_{\leq m}) \quad SST_{0|n}(\lambda_{>m}) \end{array}$$

$$T = (T^{\leq m}, T^{>m}) \longmapsto (S, T', T^{>m})$$

$$T^{\leq m} = (T_0^{\leq m}, T_1^{\leq m})$$

Sagan-Stanley's skew RSK.

pair of skew SST's in $\{\check{m}, \cdots, \check{1}\}$, $\{m+1, \cdots, m+n-1\}$

of same inner shape

② Crystal structure of $B(K(\lambda))$

$$B(K(\lambda)) = \mathcal{P}(\Phi_1^+) \times B_{m|0}(\lambda^+) \times B_{0|n}(\lambda^-)$$

$$\cong \overrightarrow{\mathcal{P}(\Phi_1^+) \otimes B_{m|0}(\lambda^+)} \times B_{0|n}(\lambda^-) \quad \text{as } \mathcal{U}_{m|0}\text{-crystal}$$

$$\cong \mathcal{P}(\Phi_1^+) \times \overleftarrow{B_{m|0}(\lambda^+) \otimes B_{0|n}(\lambda^-)} \quad \text{as } \mathcal{U}_{0|n}\text{-crystal}$$

$$\cong \mathcal{P}(\Phi_1^+) \times B_{m|0}(\lambda^+) \times B_{0|n}(\lambda^-) \quad \text{for } \tilde{e}_m, \tilde{f}_m$$

So it suffices to consider $B(K(0)) \cong \mathcal{P}(\Phi_1^+)$

③ (Jang-K-Uruno 23)

We construct a crystal base $(\mathcal{L}(\infty), B(\infty))$ of $U_q^-(\mathfrak{gl}(m|n))$

$$B(\infty) := \mathcal{P}(\Phi_1^+) \times B_{m|0}(\infty) \times B_{0|n}(\infty)$$

$$\uparrow$$

$$B(K(\lambda)) = \mathcal{P}(\Phi_1^+) \times B_{m|0}(\lambda^+) \times B_{0|n}(\lambda^-)$$

where $B_{m|0}(\infty)$: crystal of $U_q^-(\mathfrak{gl}(m|0))$

$B_{0|n}(\infty)$: crystal of $U_q^-(\mathfrak{gl}(0|n))$

Example of $B(K(0))$ or $P(\Phi_1^+)$ $m=3$ $n=4$ $\delta_a - \delta_b$

$$
\begin{array}{cccccc}
 & & & \underset{\otimes}{\delta_3-\delta_7} & & \\
 & & \underset{\otimes}{\delta_3-\delta_6} & & \underset{\otimes}{\delta_2-\delta_7} & \\
 & \underset{\otimes}{\delta_3-\delta_5} & & \underset{\otimes}{\delta_2-\delta_6} & & \underset{\otimes}{\delta_1-\delta_7} \\
\underset{\otimes}{\delta_3-\delta_4} & & \underset{\otimes}{\delta_2-\delta_5} & & \underset{\otimes}{\delta_1-\delta_6} & \\
 & \underset{\otimes}{\delta_2-\delta_4} & & \underset{\otimes}{\delta_1-\delta_5} & & \\
 & & \underset{\otimes}{\delta_1-\delta_4} & & &
\end{array}
\qquad
\begin{array}{cccccc}
 & & & c_{37} & & \\
 & & c_{36} & & c_{27} & \\
 & c_{35} & & c_{26} & & c_{17} \\
c_{34} & & c_{25} & & c_{16} & \\
 & c_{24} & & c_{15} & & \\
 & & c_{14} & & &
\end{array}
$$

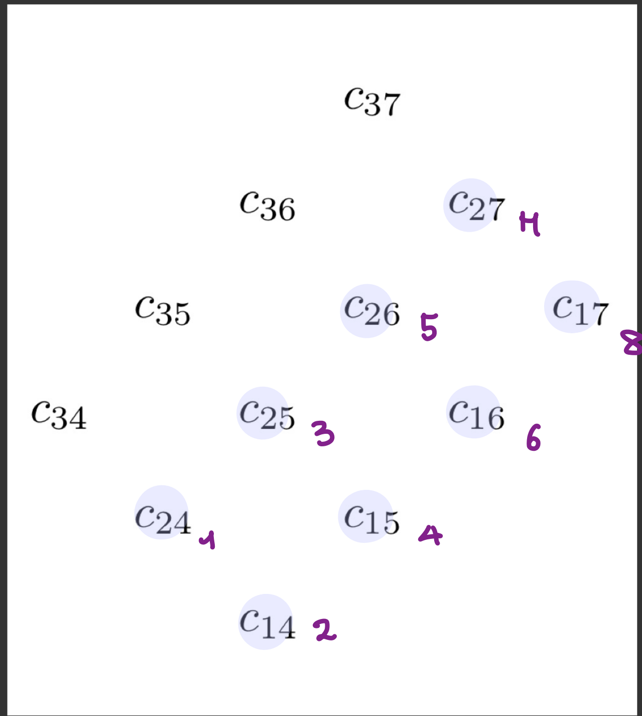
Φ_1^+ in convex order

$\mathbb{C} = (c_{ij}) \in P(\Phi_1^+)$

$(c_{ij} = 0, 1)$

(4) $(\alpha_1 | \alpha_1) > 0$

$\tilde{f}_1$



$= \mathbb{C} = (c_{ij})$

$\mathbb{C} = (c_{24}, c_{14}, c_{25}, c_{15}, \cdots) \longleftrightarrow$ a seq. of $+, -$'s.

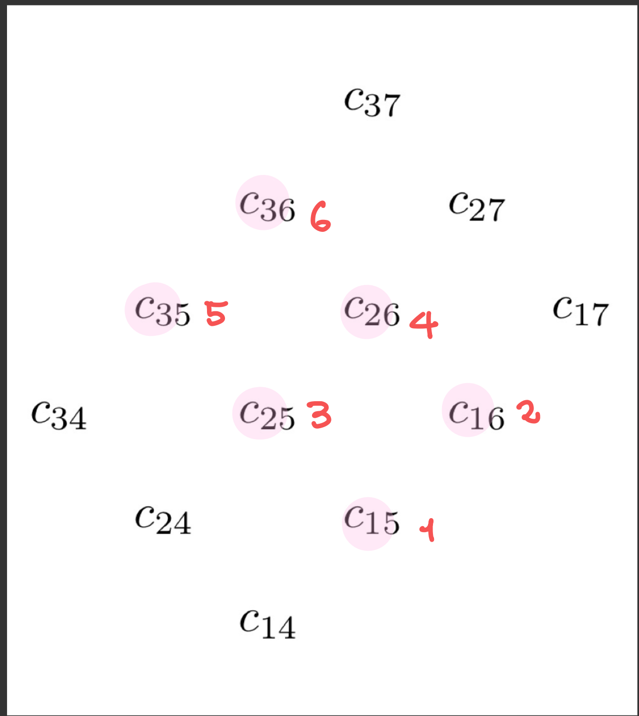
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow +$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow -$

$\tilde{f}_1 \mathbb{C}$ obtained by applying "<u>signature rule</u>" $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\tilde{f}_i} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

② $(\alpha_5|\alpha_5) < 0$

$\tilde{f}_5$



$= \mathbb{C} = (c_{ij})$

$\mathbb{C} = (c_{15}, c_{16}, c_{25}, c_{26}, \cdots) \longleftrightarrow$ a seq. of $+, -$'s.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow +$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow -$

$\tilde{f}_5 \mathbb{C}$ obtained by applying "signature rule"

Example of embedding $\quad B_{min}(\lambda) \longrightarrow B(K(\lambda))$

$$m = 3, \quad n = 4, \quad \lambda = (5,3,2,2) = \underbrace{5\delta_1 + 3\delta_2 + 2\delta_3}_{\lambda^+} + \underbrace{\delta_4 + \delta_5}_{\lambda^-}$$

$$T = \begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 3 & 6 \\
\hline
2 & 5 & 7 \\
\cline{1-3}
3 & 5 \\
\cline{1-2}
4 & 6 \\
\cline{1-2}
\end{array} \in SST_{3|4}(5,3,2,2)$$

$$\longrightarrow \quad (S, T', T^{>3}) \in \mathcal{P}(\Phi_1^+) \times B_{3|0}(\lambda^+) \times B_{0|4}(\lambda^-)$$

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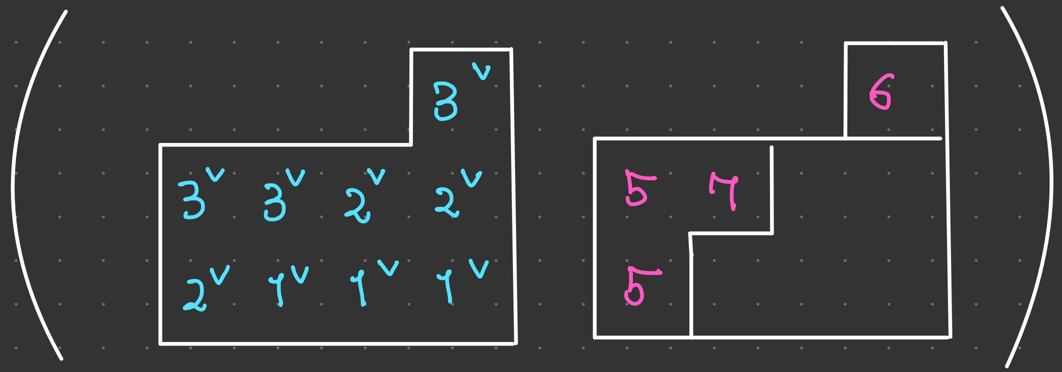
$$T^{\leqslant 3} = \begin{array}{|ccccc|} \hline 1 & 1 & 2 & 3 & 6 \\ 2 & 5 & 7 & & \\ 3 & 5 & & & \\ \hline \end{array} \qquad T^{>3} = \boxed{4 \quad 6}$$

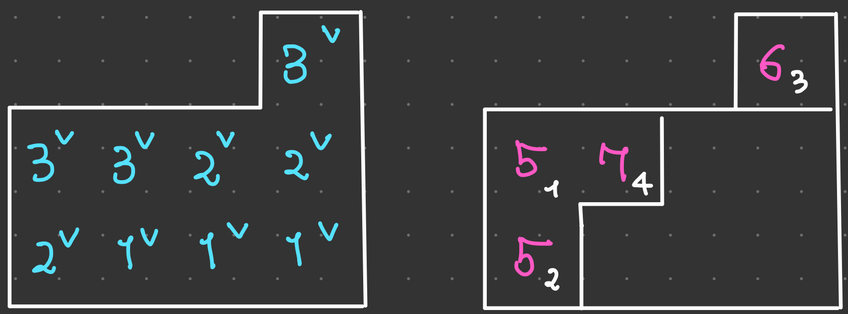
$$T_0^{\leqslant 3} = \begin{array}{|cccc|} \hline 1 & 1 & 2 & 3 \\ 2 & & & \\ 3 & & & \\ \hline \end{array} \longrightarrow \begin{array}{ccccc} 1 & 1 & 2 & 3 & 3^\vee \\ 2 & 3^\vee & 3^\vee & 2^\vee & 2^\vee \\ 3 & 2^\vee & 1^\vee & 1^\vee & 1^\vee \end{array} = \left(T_0^{\leqslant 3}\right)^*$$

tensor product

$$B_{m10}\left(\det{}^\vee\right)^{\otimes 5} \quad * \quad B_{310}(1)^\vee$$

$$3^\vee \to 2^\vee \to 1^\vee$$



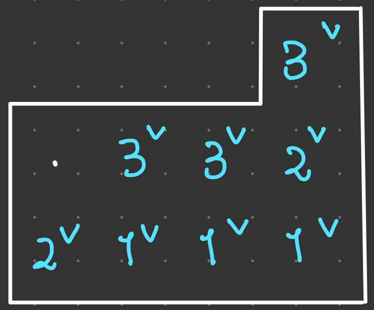


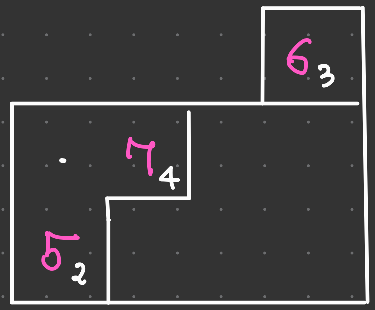
applying the inverse RSK.

insertion tableau

recording tableau

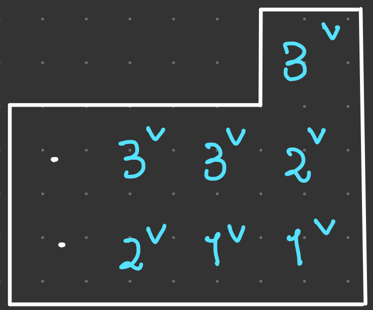
#52

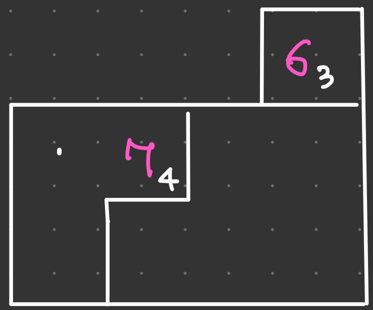




$2^\vee$ 5_1

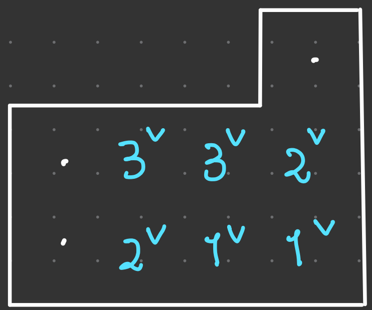
$\delta_2 - \delta_5$

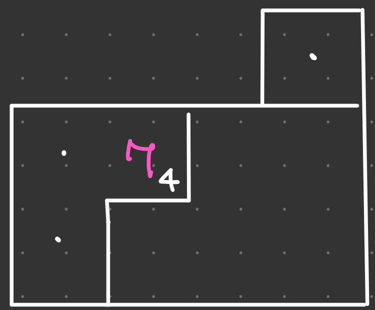




$1^\vee$ 5_2

$\delta_1 - \delta_5$

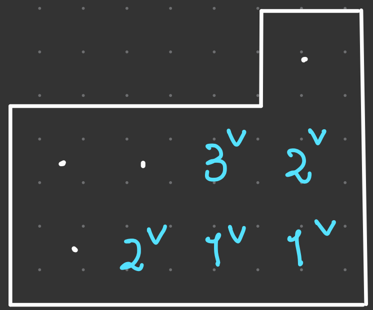


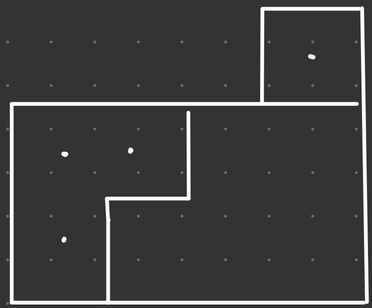


$3^\vee$ 6_3

$\delta_3 - \delta_6$.

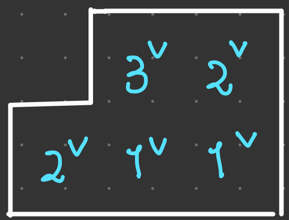
#53



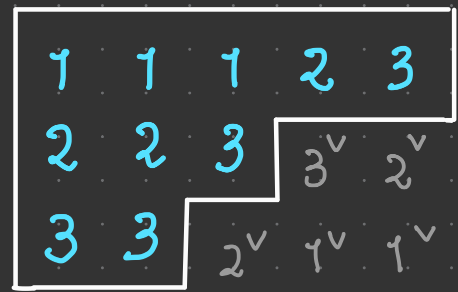


$3^\vee \quad \tau_4$

$\delta_3 - \delta_\tau$



$\longrightarrow$



$= T'$

tensor product

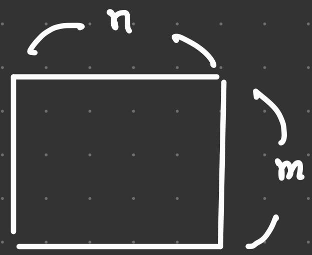
$B_{m10}(\det)^{\otimes 5}$

$$
\left(\begin{array}{ccc}
S & T' & T^{>3} \\
\{\delta_2-\delta_5 \;\; \delta_1-\delta_5 \;\; \delta_3-\delta_6. \;\; \delta_3-\delta_7\}, &
\boxed{\begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & & \\ 3 & 3 & & & \end{array}}, &
\boxed{\begin{array}{cc} 4 & 6 \end{array}}
\end{array} \right)
$$

$$
\in \mathcal{P}(\Phi_1^+) \times B_{310}(\lambda^+) \times B_{014}(\lambda^-)
$$

Rmk

$$\lambda = (m^n)$$



$\Lambda_\lambda = n\delta_+$: typical $\qquad V_{m|n}(\lambda) = K(\lambda)$

$$\begin{array}{ccc} B_{m|n}(\lambda) & \xrightarrow{\quad\kappa\quad} & B(K(\lambda)) \\ \rotatebox[origin=c]{90}{$\in$} & & \\ T = \left((T_{\leq m})^*, T_{>m} \right) & \longmapsto & (S, H_{n\delta_+}, \phi) \end{array}$$

κ is nothing but RSK (binary)

& morphism of $\mathcal{U}_{m|n}$-crystals.