# Representations of quantum superalgebras and their crystal bases

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Lecture 1. Review on crystal base - Kac-Moody case.

Lecture 2. Crystal base for $U_q(\mathfrak{gl}(m|n))$

Lecture 3 Crystal base for $U_q(\mathfrak{osp}(m|n))$

# Lecture 1. Review on crystal base

1. $q$-analogue of the universal enveloping alg. $U_q(\mathfrak{g})$

2. Representations of $U_q(\mathfrak{g})$

3. Crystal base of $V(\lambda)$

4. Combinatorial model for $B(\lambda)$ - $\mathfrak{g} = \mathfrak{gl}_n$

5. Crystal base of $U_q^-(\mathfrak{g})$

## 1. q-analogue of the universal enveloping alg.

We assume the following data

- $I$ : index set (finite)
- $P$ : free abelian group
- $\Pi = \{ \alpha_i \mid i \in I \} \subset P$ the set of simple roots (linearly independent)
- $\Pi^\vee = \{ h_i \mid i \in I \} \subset P^\vee = \mathrm{Hom}(P, \mathbb{Z})$ : the set of simple coroots.

- $(\cdot\,|\,\cdot) : P \times P \longrightarrow \mathbb{Q}$ symm bilinear form s.t.

  1) $(\alpha_i|\alpha_i) \in 2\mathbb{Z}_{>0}$ $\quad (i \in I)$

  2) $(\alpha_i|\alpha_j) \leq 0$ $\quad (i, j \in I,\ i \neq j)$

  3) $\langle h_i, \lambda \rangle = \dfrac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)}$ $\quad (i \in I,\ \lambda \in P)$

$a_{ij} := \langle h_j, \alpha_i \rangle$

$\shortparallel$

$\mathfrak{g}$ : a Kac-Moody alg assoc. to $(A, P, P^\vee, \Pi, \Pi^\vee)$

$\mathcal{U}(\mathfrak{g})$ : the universal enveloping algebra of $\mathfrak{g}$

- $q$ : formal variable.

- $\Bbbk = \mathbb{Q}(q)$

- $q_i = q^{\frac{(\alpha_i|\alpha_i)}{2}}$ $\quad (i \in I)$

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} = q_i^{n-1} + q_i^{n-3} + \cdots + q_i^{-n+1} \qquad (q\text{-integer})$$

$$[n]_i! = [n]_i [n-1]_i \cdots [2]_i [1]_i .$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i! [n-m]_i!} \qquad (q\text{-binomial coeff})$$

<u>Def.</u> $U_q(\mathfrak{g})$ = the assoc. $\Bbbk$-alg. with 1

- generators : $e_i, f_i, q^h \quad (h \in P^\vee, i \in I)$
- relations :

$$q^h q^{h'} = q^{h'} q^h = q^{h+h'}, \quad q^0 = 1$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \qquad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \qquad \text{where } t_i = q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}$$

$$\sum_{k=0}^{c_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(c_{ij}-k)} = \sum_{k=0}^{c_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(c_{ij}-k)} = 0$$

where $c_{ij} = 1 - a_{ij} \; (i \neq j)$, $\quad x_i^{(k)} = \dfrac{x_i^k}{[k]_i!}$

<u>Example</u> ① $A = (2)$ $U_q(\mathfrak{sl}_2) = \langle e, f, t^{\pm 1} \rangle$

$$t e t^{-1} = q^2 e \qquad t f t^{-1} = q^{-2} f \qquad e f - f e = \frac{t - t^{-1}}{q - q^{-1}}$$

$\langle e_i, f_i, t_i^{\pm 1} \rangle / \mathbb{Q}(q_i) \cong U_q(\mathfrak{sl}_2)$ over $\mathbb{Q}(q)$ $(i \in I)$

② $\mathfrak{g} = \mathfrak{gl}_n$ type $A_{n-1}$

$P = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \cdots \oplus \mathbb{Z}\delta_n$ $(\delta_i, \delta_j) = \delta_{ij}$

$\alpha_1 = \delta_1 - \delta_2$, $\alpha_2 = \delta_2 - \delta_3$, $\cdots$

$P^\vee = \mathbb{Z}\delta_1^\vee \oplus \mathbb{Z}\delta_2^\vee \oplus \cdots \oplus \mathbb{Z}\delta_n^\vee$ $\langle \delta_j^\vee, \delta_i \rangle = \delta_{ij}$

$U_q(\mathfrak{gl}_n) = \langle e_i, f_i \ (i = 1, \ldots, n-1) \ q^{\pm \delta_j^\vee} \ (j = 1, \ldots, n) \rangle$

- (Triangular decomposition)

$$U_q^+(\mathfrak{g}) := \langle e_i \mid i \in I \rangle , \quad U_q^-(\mathfrak{g}) := \langle f_i \mid i \in I \rangle$$

$$U_q^0(\mathfrak{g}) := \langle q^h \mid h \in P^\vee \rangle$$

$$U_q^-(\mathfrak{g}) \otimes_{\Bbbk} U_q^0(\mathfrak{g}) \otimes_{\Bbbk} U_q^+(\mathfrak{g}) \xrightarrow{\cong} U_q(\mathfrak{g}) \qquad \Bbbk\text{-linear iso}$$

$$x_- \otimes x_0 \otimes x_+ \longmapsto x_- x_0 x_+$$

- (Weight space decomposition)

$$U_q(\mathfrak{g}) = \bigoplus_{\xi \in P} U_q(\mathfrak{g})_\xi \qquad U_q(\mathfrak{g})_\xi = \{ x \mid q^h x q^{-h} = q^{\langle h, \xi \rangle} x \ (h \in P^\vee) \}$$

$$Q = \sum_{i \in I} \mathbb{Z}\alpha_i \qquad Q_+ = \sum \mathbb{Z}_+ \alpha_i \,, \qquad Q_- = -Q_+$$

$$U_q^{\pm}(\mathfrak{g}) = \bigoplus_{Q_\pm} U_q(\mathfrak{g})_\beta$$

- $U_q(\mathfrak{g})$ is a Hopf algebra with $\overset{\text{comult}}{\Delta}$, $\overset{\text{antipode}}{S}$, $\overset{\text{counit.}}{\varepsilon}$

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes t_i^{-1}$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i$$

$$\Delta(q^h) = q^h \otimes q^h$$

$$S(e_i) = -e_i t_i \qquad S(f_i) = -t_i^{-1} f_i \qquad S(q^h) = q^{-h}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(q^h) = 0 \qquad \varepsilon(1) = 1$$

## 2. Representations of $U_q(\mathfrak{g})$

- $\lambda \in P_+$ $\Big($ i.e. $\langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ $(i \in I)\Big)$

$V(\lambda)$ : a left $U_q(\mathfrak{g})$-module gen. by $v_\lambda$ subject to

$$q^h \cdot v_\lambda = q^{\langle h, \lambda \rangle} v_\lambda \qquad e_i v_\lambda = 0 \qquad f_i^{\langle h_i, \lambda \rangle + 1} v_\lambda = 0$$

$$(h \in P^\vee,\ i \in I)$$

that is, $V(\lambda) :=$

$$\frac{U_q(\mathfrak{g})}{\sum_i U_q(\mathfrak{g}) e_i + \sum_i U_q(\mathfrak{g}) f_i^{\langle h_i, \lambda \rangle + 1} + \sum_h U_q(\mathfrak{g}) (q^h - q^{\langle h, \lambda \rangle})} \ni v_\lambda = \overline{1}$$

By triangular decomposition,

$$V(\lambda) = U_q^-(\mathfrak{g})\, v_\lambda = \bigoplus_{\lambda \geq \mu} V(\lambda)_\mu$$

where $V(\lambda)_\mu = \{ v \mid q^h v = q^{\langle h, \mu \rangle} v \}$

- $V(\lambda)$ is irreducible.
- $\overline{V(\lambda)}$ : the classical limit of $V(\lambda)$, $U(\mathfrak{g})$-module.

$$\mathrm{ch}_{\mathbb{k}} V(\lambda) = \mathrm{ch}_{\mathbb{Q}} \overline{V(\lambda)} = \frac{\sum_{W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\Phi^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$$

- $\mathcal{O}_{\text{int}}$ : the category of $\mathcal{U}_q(\mathfrak{g})$-modules $M$ such that

1) $M = \bigoplus_{\lambda \in P} M_\lambda$ $\quad \dim M_\lambda < \infty$

2) $\underline{\mathrm{wt}(M)} \subset (\lambda_1 - Q_+) \cup \cdots \cup (\lambda_r - Q_+)$ for some $\lambda_1, \cdots, \lambda_r$

set of weights

3) $e_i, f_i$ : locally nilpotent for $i \in I$

- $L \in \mathcal{O}_{\text{int}}$ : irreducible $\iff$ $L = V(\lambda)$ for some $\lambda \in P_+$

- $\mathcal{O}_{\text{int}}$ : semisimple & closed under $\otimes$

# Repn's of $U_q(sl_2)$

$m \in \mathbb{Z}_{\geq 0}$

$V(m)$ : irr. h.w. $U_q(sl_2)$-module with h.w. $m$

$$V(m) = \bigoplus_{k=0}^{m} f^{(k)} v_m \quad \text{where } v_m : \text{h.w. vector}$$

$$f \cdot f^{(k)} v_m = [k+1] f^{(k+1)} v_m$$

$$e \cdot f^{(k)} v_m = \left( f^{(k)} e + f^{(k-1)} \{ q^{1-k} t \} \right) v_m$$

$$= [m-k+1] f^{(k-1)} v_m$$

$$\{x\} = \frac{x - x^{-1}}{q - q^{-1}}$$

$V(m) \otimes V(n)$ : semisimple

$$= V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(|m-n|)$$

Example (Tensor product decomposition)

$V(2) \otimes V(1) \cong V(3) \oplus V(1)$ : $\Bbbk$-span of $f^{(k)} v_2 \otimes f^{(l)} v_1$

$k = 0, 1, 2 \quad l = 0, 1$

$w_3 = v_2 \otimes v_1$ : h.w. vector of wt 3

$$V(3) \cong U_q(sl_2)\, w_3 = \bigoplus_{n=0}^{3} \Bbbk f^{(n)} w_3$$

$w_1 = a v_2 \otimes f v_1 + b f v_2 \otimes v_1$ : h.w. vector of wt 1.

$$e w_1 = e(a v_2 \otimes f v_1) + e(b f_2 v_2 \otimes v_1)$$

$$= a(v_2 \otimes v_1) + b q^{-1}[2] v_2 \otimes v_1 = 0$$

$$a = -b q^{-1}[2] = -\underbrace{b}_{= -q^2}(1+q^{-2})$$

$w_1 = (1+q^2) v_2 \otimes f v_1 - q^2 f v_2 \otimes v_1$ h.w. vector of wt 1.

$$V(1) \cong U_q(sl_2) w_1 = \Bbbk w_1 \oplus \Bbbk f w_1$$

$$f^n w_3 = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{-k(n-k)} f^{n-k} t^k \otimes f^k \cdot (v_2 \otimes v_1) \qquad n = 0, 1, 2, 3$$

$$f^{(n)} w_3 = \sum_{k=0}^{n} q^{-k(n-k)} f^{(n-k)} t^k \otimes f^{(k)} . (v_2 \otimes v_1)$$

$$= \sum_{k=0}^{n} \underbrace{q^{-k(n-k)} q^{2k}}_{q^{k^2+2k-kn}} f^{(n-k)} v_2 \otimes f^{(k)} v_1$$

$$\equiv \begin{cases} f^{(n)} v_2 \otimes v_1 & (n = 0, 1, 2) \\ f^{(2)} v_2 \otimes f v_1 & (n = 3) \end{cases}$$

by taking $q = 0$ in the coeff's of $f^{(k)} v_2 \otimes f^{(l)} v_1$

$$w_1 = (1+q^2)\, v_2 \otimes f v_1 - q^2 f v_2 \otimes v_1 \equiv v_2 \otimes f v_1$$

$$f w_1 = (1+q^2)\, f v_2 \otimes f v_1 - q^2 [2] f^{(2)} v_2 \otimes v_1 - q^2 f v_2 \otimes f v_1$$

$$\equiv f v_2 \otimes f v_1$$

The $U_q(sl_2)$ - strings in $V(2) \otimes V(1)$ at $q = 0$

$$\begin{tikzcd}
 & & v_2 & f v_2 & f^{(2)} v_2 \\
\otimes & & \bullet & \bullet & \bullet \\
v_1 & \bullet & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[d] \\
f v_1 & \bullet & \bullet \arrow[r] & \bullet & \bullet
\end{tikzcd}$$

The above example can be generalized to $V(m) \otimes V(n)$

$$A_0 = \{ f(q) \in K \mid f : \text{regular at } q=0 \} \subset \mathbb{k} = \mathbb{Q}(q)$$

local ring w/ maximal ideal $qA_0$

$$V(m) = \bigoplus_{k=0}^{m} f^{(k)} v_m$$

$$L(m) = \bigoplus_{k=0}^{m} A_0 f^{(k)} v_m \qquad : A_0\text{-lattice of } V(m)$$

$$B(m) = \{ f^{(k)} v_m \pmod{qL(m)} \mid 0 \leq k \leq m \}$$

: $\mathbb{Q}$-basis of $L(m) / qL(m)$

Note that

$L(m) \underset{A_0}{\otimes} L(n)$ : $A_0$-lattice of $V(m) \otimes V(n)$

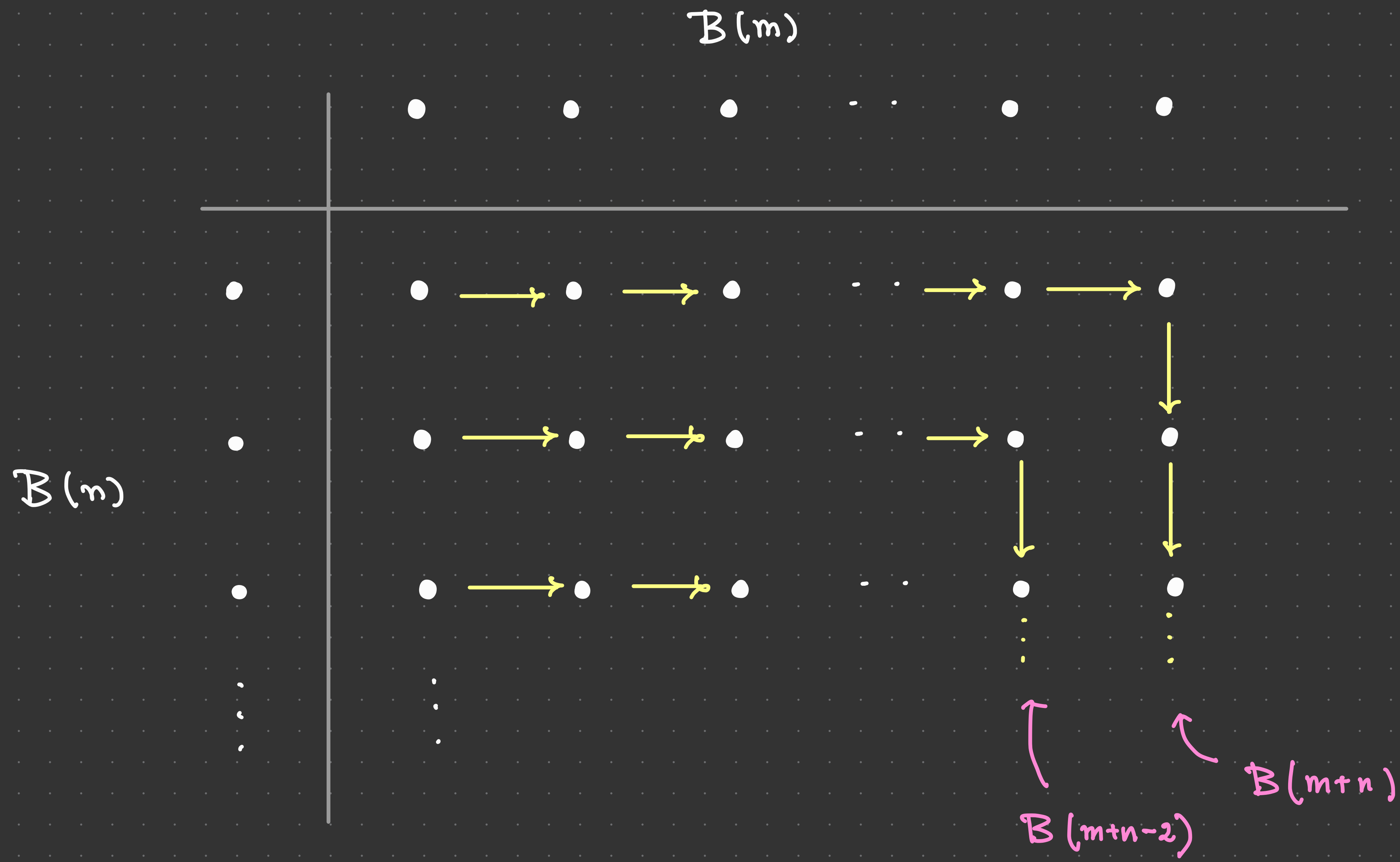
$B(m) \otimes B(n)$ : $\mathbb{Q}$-basis of $L(m)\otimes L(n) / q\,L(m)\otimes L(n) = \dfrac{L(m)}{q\,L(m)} \otimes \dfrac{L(n)}{q\,L(n)}$

Then we have the following

① $\exists\, w_\ell$ : h.w. vector of wt $\ell$ $(\ell = m+n, \ldots, |m-n|)$

such that $\underline{w_\ell \equiv v_m \otimes f^{(m+n-\ell)} v_n}$

② $\underline{f^{(k)} w_\ell \equiv f^{(s)} v_m \otimes f^{(t)} v_n}$ where $(s,t)$ are given by




$B(m) \otimes B(n)$

## 3. Crystal base of $M \in \mathcal{O}_{int}$

We first need to define operators on $M$ in general describing

$i$-string at $q = 0$ for all $i \in I$.

$M \in \mathcal{O}_{int}$ $\quad i \in I$

Define $\tilde{e}_i, \tilde{f}_i : M \longrightarrow M$ as follows:

$v \in M_\lambda$. we have

$$v = \sum_{n \geq 0} f_i^{(n)} v_n$$ for unique $v_n \in M_{\lambda - n\alpha_i}$ $e_i v_n = 0$.

Define

$$\tilde{e}_i v = \sum_{n \geq 1} f_i^{(n-1)} v_n \quad , \quad \tilde{f}_i v = \sum_{n \geq 0} f_i^{(n+1)} v_n$$

Rmk These operators coincide with the ones for $U_q(sl_2)$

moving each vertex in $B(m) \otimes B(n)$ along "$\longrightarrow$"

Using $\tilde{e}_i, \tilde{f}_i$, we define a pair $(\mathcal{L}, B)$ for $M \in \mathcal{O}_{int}$

generalizing $(L(m), B(m))$ for $V(m)$

**Def** A crystal base of $M$ is a pair $(\mathcal{L}, B)$

where $\mathcal{L}$ : $A_0$-lattice of $M$ satisfying

$B$ : $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L}$

① $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda \qquad (\mathcal{L}_\lambda = M_\lambda \cap \mathcal{L})$ wt. space decomp.

$B = \bigsqcup_{\lambda \in P} B_\lambda \qquad (B_\lambda = B \cap \mathcal{L}_\lambda / q\mathcal{L}_\lambda)$

② $\tilde{e}_i \mathcal{L} \subset \mathcal{L}, \quad \tilde{f}_i \mathcal{L} \subset \mathcal{L} \qquad (i \in I)$ invariant under crystal operators

$\tilde{e}_i B \subset B \cup \{0\}, \quad \tilde{f}_i B \subset B \cup \{0\} \qquad (i \in I)$

③ $\tilde{f}_i b = b' \iff b = \tilde{e}_i b' \qquad (b, b' \in B)$ $\tilde{e}_i, \tilde{f}_i$ : mutually inverses

**Rmk** $\mathcal{B}$ has an $I$-colored oriented graph str.

$$b \xrightarrow{i} b' \iff b' = \tilde{f}_i b \quad ( \tilde{e}_i b' = b )$$

$\mathcal{B}$ called a crystal (graph) of $M$.

can be viewed as a basis of $M$ at $q = 0$.

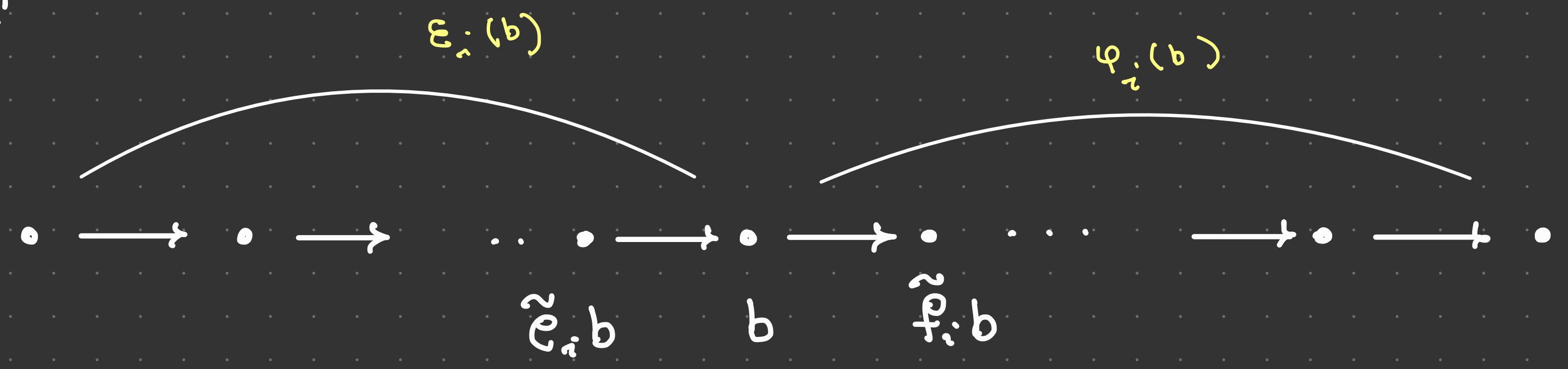
For example, $V(m) \curvearrowleft U_q(sl_2)$

$$L(m) = \bigoplus_{k=0}^{m} A_0 f^{(k)} v_m \qquad \mathcal{B}(m) = \{ f^{(k)} v_m \ (\mathrm{mod}\ qL(m)) \}$$

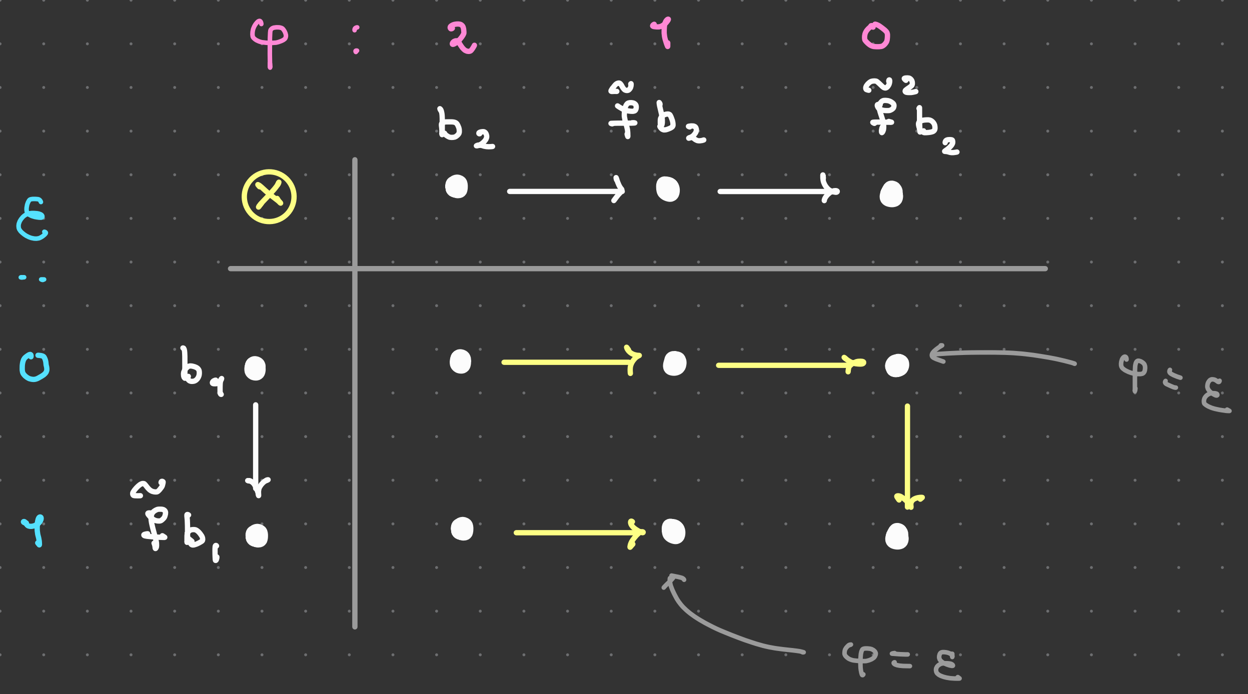
crystal base

$$\underset{v_m}{\bullet} \longrightarrow \underset{f v_m}{\bullet} \longrightarrow \cdots \longrightarrow \underset{f^{(m)} v_m}{\bullet}$$

## Recall




$\varepsilon_i(b) = \max\{ k \mid \tilde{e}_i^k b \neq 0 \}$ $\qquad$ $\varphi_i(b) = \max\{ k \mid \tilde{f}_i^k b \neq 0 \}$



The most ~~important~~ powerful property of crystal base is the following.

<u>Theorem</u> (Tensor product theorem. Kashiwara 91)

$M_1, M_2 \in \mathcal{O}_{int}$ with a crystal base $(\mathcal{L}_i, B_i)$ $(i=1,2)$

$\Rightarrow$ $(\mathcal{L}_1 \otimes \mathcal{L}_2, B_1 \otimes B_2)$ : a crystal base of $M_1 \otimes M_2$. where

$$\tilde{f}_i (b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

for $b_1 \otimes b_2 \in B_1 \otimes B_2$ and $i \in I$.

## Example

$$U_q(\mathfrak{gl}_n) = \langle e_i, f_i, q^{\pm \delta_j^\vee} \mid 1 \le i \le n-1,\ 1 \le j \le n \rangle$$

(1) $V = \bigoplus_{i=1}^{n} \Bbbk v_i$ : the natural repn

$$\begin{array}{ccc} v_i & \underset{e_i}{\overset{f_i}{\rightleftarrows}} & v_{i+1} \\ \delta_i & & \delta_{i+1} \end{array}$$

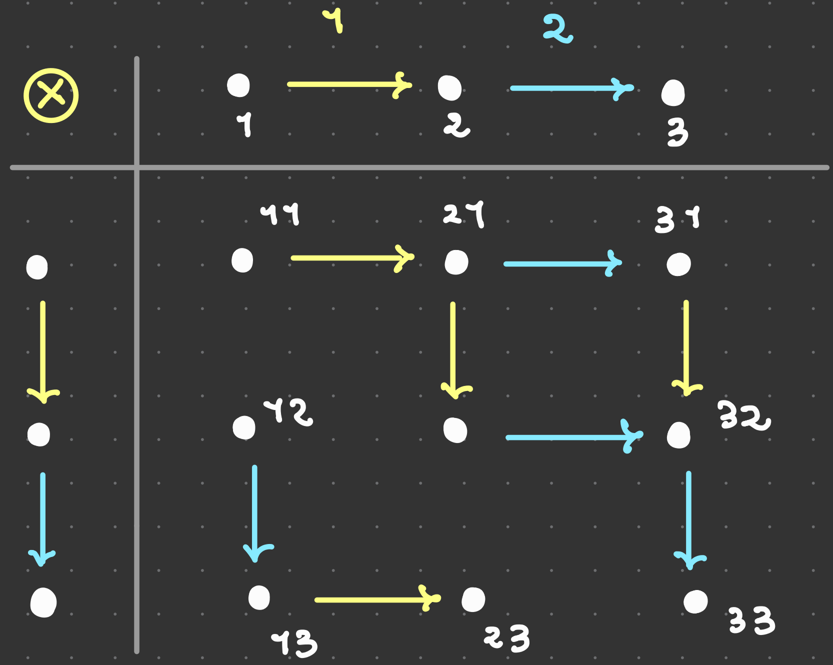
$\mathcal{L} := \bigoplus A_0 v_i \qquad B := \{ \overline{v_i} \ (\mathrm{mod}\ q\mathcal{L}) \}$ : crystal base of $V$

In this case $\tilde{e}_i = e_i$, $\tilde{f}_i = f_i$

$$v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3 \longrightarrow \cdots \xrightarrow{n-1} v_n \qquad \xrightarrow{i} = \tilde{f}_i$$

$V^{\otimes 2}$ has a crystal base $(\mathcal{L}^{\otimes 2}, B^{\otimes 2})$

$n = 3$




$(ij = v_i \otimes v_j)$

Recall $\quad V^{\otimes 2} \cong \underset{\substack{\| \\ 2\varpi_1}}{V(2\delta_1)} \oplus \underset{\substack{\| \\ \varpi_2}}{V(\delta_1+\delta_2)} \quad$ (e.g. by Pieri rule.)

How are these two decompositions are related?

② $W^{\epsilon} = \bigoplus_{m \in X_\epsilon} K\, m$ $\quad \epsilon = 0, 1 \quad X_\epsilon = \begin{cases} \mathbb{Z}^n_{\geq 0} & (\epsilon = 0) \\ \mathbb{Z}_2^n & (\epsilon = 1). \end{cases}$

$W^\epsilon$ : a $U_q(gl_n)$ - module

$$e_i\, m = [m_{i+1}](\cdots, m_i + 1, m_{i+1} - 1, \cdots)$$

$$f_i\, m = [m_i](\cdots m_i - 1, m_{i+1} + 1, \cdots)$$

$$q^{\delta_i^\vee} m = q^{m_i} m$$

$W^\epsilon_k := \bigoplus_{\sum m_i = k} \Bbbk\, m$ submodule $\cong \begin{cases} V(k\varpi_1) & \epsilon = 0 \\ V(\varpi_k) & \epsilon = 1.\ k \leq n \end{cases}$

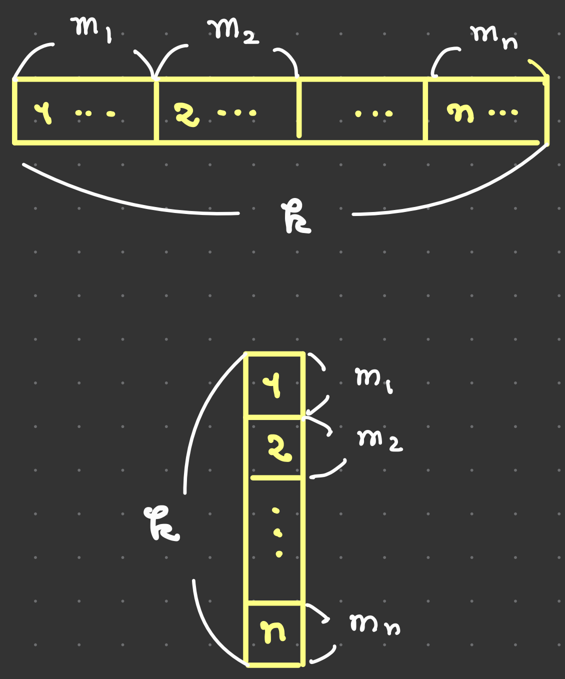
$$\mathcal{L}_k^{\epsilon} := \bigoplus A_0 \frac{1}{\prod [m_i]!} \mathbf{m} \qquad B_k^{\epsilon} = \left\{ \frac{1}{\prod [m_i]!} \overline{\mathbf{m}} \pmod{q \mathcal{L}_k^{\epsilon}} \right\}$$

: crystal base of $W_k^{\epsilon}$

$$(\cdots, m_i, 0, \cdots) \xrightarrow{f_i^{(k)}} (\cdots, m_i - k, k, \cdots)$$

$U_q(gl_2)$-h.w. vector

Identify $\mathbf{m}$ with




$\epsilon = 0$

$\epsilon = 1$

Young tableaux

$n = 3$

$B(2\varpi_1)$

$$\begin{tikzcd} 11 \arrow[r] & 12 \arrow[r] \arrow[d] & 13 \arrow[d] \\ & 22 \arrow[r] & 23 \arrow[d] \\ & & 33 \end{tikzcd}$$

$B(\varpi_2)$

$$\begin{tikzcd} \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[d] & \\ \begin{smallmatrix}1\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\end{smallmatrix} \end{tikzcd}$$

$$\underline{B^{\otimes 2} \cong B(2\varpi_1) \sqcup B(\varpi_2)}$$

In the previous example, we have seen that

decomposition of a crystal = decomposition of a module

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**Theorem** (Existence of a crystal base. Kashiwara 91)

$\lambda \in P_+$

$V(\lambda)$ has a unique crystal base $(\mathcal{L}(\lambda), B(\lambda))$ where

$$\mathcal{L}(\lambda) = \sum_{i_1, \cdots, i_r} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$$

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \ (\mathrm{mod}\ q\mathcal{L}(\lambda)) \} \setminus \{0\}$$

Theorem (Uniqueness of a crystal, Kashiwara 91)

$M \in \mathcal{O}_{\mathrm{int.}}$ with a crystal base $(\mathcal{L}, B)$

Then $\exists\, \varphi : M \xrightarrow{\cong} \bigoplus_{P_+} V(\lambda)^{\oplus m_\lambda}$ such that

$$\varphi|_{\mathcal{L}} : \mathcal{L} \longrightarrow \bigoplus \mathcal{L}(\lambda)^{\oplus m_\lambda}$$

$$\bar{\varphi}|_{B} : B \longrightarrow \bigsqcup B(\lambda)^{\oplus m_\lambda}$$

where $\bar{\varphi} : \mathcal{L}/q\mathcal{L} \longrightarrow \bigoplus \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)^{\oplus m_\lambda}$

Rmk

① By Existence thm + Complete reducibility of $\mathcal{O}_{int}$

any $M \in \mathcal{O}_{int}$ has a crystal base

② $M \in \mathcal{O}_{int}$ with a crystal base $(\mathcal{L}, B)$

By Uniqueness of crystals,

$$B \cong \coprod B(\lambda)^{\oplus m_\lambda} \qquad m_\lambda = \dim_{\mathbb{k}} \mathrm{Hom}_{U_q(\mathfrak{g})}(V(\lambda), M)$$

decomp of $M$ into $V(\lambda)$'s

$\iff$ decomp. of $B$ into $B(\lambda)$'s.

③ By construction,

$$B(\lambda) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda \pmod{q\mathcal{L}(\lambda)} \} \setminus \{0\}$$ : connected.

$b \in B(\lambda)$ $\quad \tilde{e}_i b = 0$ for all $i$ $\iff$ $b = v_\lambda$

decomp. of $B$ into $B(\lambda)$'s.

$\iff$ finding all $b \in B$ such that $\tilde{e}_i b = 0$ for all $i$.

In fact. the connected component of $b$ under $\tilde{f}_i$'s

$\cong B(\lambda)$ where $\lambda = wt(b)$.

## 4. Combinatorial model for $B(\lambda)$

Tableaux realization $\mathfrak{g} = \mathfrak{gl}_n$

Recall that we have constructed a crystal base of $V(\ell\varpi_1)$, $V(\varpi_k)$

where the crystal can be identified as a set

$$B(\ell\varpi_1) = \left\{ \boxed{a_1}\boxed{a_2}\boxed{\quad\cdots\quad}\boxed{a_\ell} \;\middle|\; 1 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell \leq n \right\}$$

$$B(\varpi_k) = \left\{ \begin{array}{|c|} \hline a_1 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \;\middle|\; 1 \leq a_1 < \cdots < a_k \leq n \right\}$$

$B = B(l\varpi_1)$ or $B(\varpi_k)$

$T \in B \qquad wt(T) = \sum_{i \geq 1} \delta_{a_i} = \delta_{a_1} + \delta_{a_2} + \cdots$

$$\tilde{f}_i T = \begin{cases} T' \in B & \text{obtained from } T \text{ by replacing } i \text{ w/ } i+1 \\ 0 & \end{cases}$$

e.g. $$\boxed{\overbrace{i \cdots i}^{a}} \boxed{\overbrace{i+1 \cdots i+1}^{b}} \xrightarrow{\tilde{f}_i} \boxed{\overbrace{i \cdots i}^{a-1}} \boxed{\overbrace{i+1 \cdots i+1}^{b+1}}$$

- One can realize $B(\lambda)$ for $\lambda \in P_+$ using $B$ as building block.

( like fundamental representations in repn's of semisimple Lie alg's )

- The basic strategy is to describe the connected component

$$C(b) \subset B(\varpi_{k_1}) \otimes \cdots \otimes B(\varpi_{k_r}) \quad \text{or} \quad B(l_1\varpi_1) \otimes \cdots \otimes B(l_s\varpi_1)$$

where $\tilde{e}_i b = 0$ for all $i$ $\quad wt(b) = \lambda.$ $\quad \Rightarrow \quad C(b) \cong B(\lambda)$

* This can be applied to any $\mathfrak{g}$

In particular,

$$B(\varpi_k) \subset B(\varpi_1)^{\otimes k} \qquad\qquad B(l\varpi_1) \subset B(\varpi_1)^{\otimes l}$$

$$\begin{array}{|c|}\hline a_1 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \longmapsto \boxed{a_1} \otimes \cdots \otimes \boxed{a_k} \qquad\qquad \begin{array}{|c|c|c|}\hline a_1 & \cdots & a_l \\ \hline \end{array} \longmapsto \boxed{a_l} \otimes \cdots \otimes \boxed{a_1}$$

$\lambda \in P_+$ $\quad \lambda = \sum_{a=1}^{n} \lambda_a \delta_a \quad (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$

$\lambda$ : polynomial if $\lambda_i \geq 0 \ \forall i$.

$\lambda$ : polynomial $\longleftrightarrow$ $\lambda = (\lambda_1, \cdots, \lambda_n)$ : partition

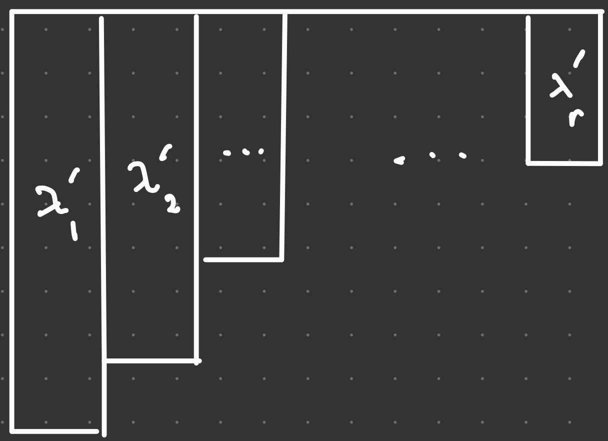
$SST_n(\lambda)$ = the set of semi-standard tableaux of shape $\lambda$ with the entries in $\{1, \cdots, n\}$

e.g.

$B(\varpi_k) = SST_n(1^k) \qquad B(l\varpi_1) = SST_n(l)$

$\lambda$ : a partition $\qquad \mu = \lambda'$ : the conjugate of $\lambda$

$\lambda =$




$\lambda'_j$ : the length of $j$ th column

$\shortparallel$

$\mu_j$

Want to describe $B(\lambda)$ in

$$B(\varpi_{\mu_r}) \otimes \cdots \otimes B(\varpi_{\mu_1}) = SST_n(1^{\mu_r}) \otimes \cdots \otimes SST_n(1^{\mu_1})$$

$\cup$ as a set

$SST_n(\lambda)$

regarding $j$ th column

as an elt in $SST_n(1^{\mu_j})$

Highest weight vectors :

$$B(\varpi_{\mu_r}) \otimes \cdots \otimes B(\varpi_{\mu_1})$$

$$\Cup$$

$v_{\varpi_\mu} := v_{\varpi_{\mu_r}} \otimes \cdots \otimes v_{\varpi_{\mu_1}}$ the cnn. comp of $v_{\varpi_\mu} \cong B(\lambda)$

$$SST_n(1^{\mu_r}) \otimes \cdots \otimes SST_n(1^{\mu_1})$$

$$v_{\varpi_\mu} = v_{\varpi_{\mu_r}} \otimes \cdots \otimes v_{\varpi_{\mu_1}} \longmapsto H_{(1^{\mu_r})} \otimes \cdots \otimes H_{(1^{\mu_1})} =: H_\lambda$$

$$H_{(1^\ell)} = \boxed{\begin{matrix} 1 \\ 2 \\ \vdots \\ \ell \end{matrix}}$$

So. it is enough to consider cnn. comp. of $H_\lambda$. $\subset B(\varpi_1)^{\otimes|\lambda|}$

The following formula is very useful ("signature rule")

$B_1, B_2$ : crystals. $b_1 \otimes b_2 \in B_1 \otimes B_2$ $\quad i \in I$

$$\sigma_i = (\underbrace{- \cdots -}_{\varepsilon_i(b_1)} \underbrace{+ \cdots +}_{\varphi_i(b_1)}) \cdot (\underbrace{- \cdots -}_{\varepsilon_i(b_2)} \underbrace{+ \cdots +}_{\varphi_i(b_2)})$$

i) replace any two neighboring $(+, -)$ w/ $(\cdot, \cdot)$

ii) repeat the process i) (ignoring $\cdot$) as far as possible

to have a seq. of the form

$$\overline{\sigma_i} = (- \cdots - + \cdots +) \quad (\text{ignoring } \cdot)$$

## Example

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \exists \text{ right most } - \text{ in } \overline{\sigma_i} \in b_1 \\ b_1 \otimes \tilde{e}_i b_2 & \text{otherwise.} \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \exists \text{ leftmost } + \text{ in } \overline{\sigma_i} \in b_1 \\ b_1 \otimes \tilde{f}_i b_2 & \text{otherwise.} \end{cases}$$

The above combinatorial rule can be applied to

$$B_1 \otimes \cdots \otimes B_r \quad (r \geq 2)$$

**Example** $\quad B = V(\varpi_1) = SST_n(1)$

$$B^{\otimes k} \ni b_1 \otimes \cdots \otimes b_k = b \quad ( b_i \in SST_n(1) )$$

$$\parallel$$

$b_1 \cdots b_k$ : a word of length $k$ w/ letters in $\{1, \cdots, n\}$

$$\begin{array}{cccccc} 1 & 2 & 1 & 3 & 3 & 1 \\ + & - & + & \cdot & \cdot & + \\ \cancel{+} & \cancel{-} & + & \cdot & \cdot & + \end{array} \xrightarrow{\tilde{f}_1} \begin{array}{cccccc} 1 & 2 & 2 & 3 & 3 & 1 \\ \cancel{+} & \cancel{-} & - & \cdot & \cdot & + \end{array} \xrightarrow{\tilde{f}_1} 1\ 2\ 2\ 3\ 3\ 2.$$

$$\begin{array}{cccccc} 1 & 2 & 1 & 3 & 3 & 1 \\ \cdot & + & \cdot & - & - & \cdot \\ \cdot & \cancel{+} & \cdot & \cancel{-} & - & \cdot \end{array} \xleftarrow{\tilde{f}_2} 1\ 2\ 1\ 3\ 2\ 1$$

Then

① $H_\lambda \in SST_n(\lambda)$

② $SST_n(\lambda) \cup \{0\}$ stable under $\tilde{e}_i, \tilde{f}_i$ $(i \in I)$

③ $SST_n(\lambda)$ is connected as an $I$-colored oriented graph.

( i.e. any $T \in SST_n(\lambda)$ is connected to $H_\lambda$ )

$\therefore$ $SST_n(\lambda) \cong B(\lambda)$

We may obtain the same result by considering

$$B(\lambda) \subset B(\lambda_1 \varpi_1) \otimes \cdots \otimes B(\lambda_n \varpi_1) = SST_n(\lambda_1) \otimes \cdots \otimes SST_n(\lambda_n)$$

## 5. Crystal base of $U_q^-(\mathfrak{g})$

Recall that for $\lambda \in P_+$

$$V(\lambda) = \frac{U_q(\mathfrak{g})}{\sum_i U_q(\mathfrak{g}) e_i + \sum_i U_q(\mathfrak{g}) f_i^{\langle h_i, \lambda \rangle + 1} + \sum_h U_q(\mathfrak{g}) (q^h - q^{\langle h, \lambda \rangle})} \quad *$$

Define a partial order on $P_+$ by $\lambda > \mu \iff \lambda - \mu \in P_+$

Then $*$ implies

$$\exists \quad V(\lambda) \xrightarrow{\varphi_{\lambda\mu}} V(\mu) \quad : \quad U_q^-(\mathfrak{g})\text{-linear}$$

$$v_\lambda \longmapsto v_\mu$$

We will show that $(\varphi_{\lambda\mu})_{\lambda,\mu\in P_+}$ induces

a natural direct system on $\{ B(\lambda) \mid \lambda \in P_+ \}$

Then we construct $(\mathcal{L}(\infty), B(\infty))$ such that

$\mathcal{L}(\infty)$ : an $A_0$-lattice of $U_q^-(\mathfrak{g})$

local base of $U_q^-(\mathfrak{g})$ at $q=0$

$B(\infty)$ : a $k$-basis of $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$

with an $I$-colored oriented graph str. crystal of $U_q^-(\mathfrak{g})$

where $B(\infty)$ is iso. to the limit of $B(\lambda)$ $(\lambda \to \infty)$

In order to define a limit of $\{ B(\lambda) \mid \lambda \in P_+ \}$, we consider

## Category of abstract crystals

- **object** : a set $B$ with

$$wt : B \longrightarrow P$$

$$\tilde{e}_i, \tilde{f}_i : B \cup \{0\} \longrightarrow B \cup \{0\} \quad i \in I. \quad (0 : \text{formal symbol})$$

$$\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \cup \{-\infty\}$$

satisfying

① $\tilde{e}_i 0 = \tilde{f}_i 0 = 0 \quad (i \in I)$

② $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, wt(b) \rangle$

③ $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1 \quad$ if $\tilde{e}_i b \neq 0$

$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1 \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1 \quad$ if $\tilde{f}_i b \neq 0$

④ $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'$

⑤ $\varphi_i(b) = -\infty \Rightarrow \tilde{e}_i b = \tilde{f}_i b = 0$

We call $B$ a ($q$-) crystal.

**Morphism** A morphism $B_1 \xrightarrow{\psi} B_2$ is a map $B_1 \xrightarrow{\psi} B_2$

such that 1) $\psi$ preserves wt. $\varepsilon_i$. $\varphi_i$

2) $b \xrightarrow{i} b' \Rightarrow \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$

**Example**

① $B(\lambda)$

② $T_\lambda = \{t_\lambda\}$ $(\lambda \in P)$

$wt(t_\lambda) = \lambda$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$

crystals which do not come from $B(\lambda)$.

# Tensor product of crystals

$B_1 \otimes B_2 = B_1 \times B_2$ as a set

$$\mathrm{wt}(b_1 \otimes b_2) := \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$$

$$\varepsilon_i(b_1 \otimes b_2) := \max\{\varepsilon_i(b_1),\ \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle\}$$

$$\varphi_i(b_1 \otimes b_2) := \max\{\varphi_i(b_1) + \mathrm{wt}\langle h_i, \mathrm{wt}(b_2)\rangle,\ \varphi_i(b_2)\}$$

$$\tilde{e}_i(b_1 \otimes b_2) := \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) := \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

## Example

① $B(\lambda)\otimes T_{-\lambda} \longleftarrow \cong B(\lambda)$ as an $I$-colored oriented graph

$$\tilde{e}_i\,(b\otimes t_\lambda) = \tilde{e}_i b\otimes t_\lambda$$

$$\tilde{f}_i\,(b\otimes t_{-\lambda}) = \tilde{f}_i b\otimes t_\lambda$$

$$\mathrm{wt}\,(b\otimes t_{-\lambda}) = \mathrm{wt}(b) - \lambda.$$

② 
$$\begin{tikzcd}
B(m)\otimes T_{-m} \arrow[d, phantom, ""] & = & \bullet \arrow[r] \arrow[d, "u_0"'] & \bullet \arrow[r] \arrow[d, "u_1"'] & \cdots \arrow[r] & \bullet \arrow[d, "u_m"'] & & \\
\mathcal{B} & = & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r, dash] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots
\end{tikzcd}$$

$e$-strict embedding

## Limit of $B(\lambda)$ $\quad \lambda, \mu \in P_+$

$$B(\lambda+\mu) \longrightarrow B(\lambda) \otimes B(\mu) \quad \text{strict embedding}$$

$$v_{\lambda+\mu} \longmapsto v_\lambda \otimes v_\mu$$

$$B(\lambda) \otimes T_\mu \xrightarrow{\psi} B(\lambda) \otimes B(\mu) \quad e\text{-strict embedding}$$

$$b \otimes t_\mu \longmapsto b \otimes v_\mu$$

$\Rightarrow$ $\underline{\operatorname{Im} \psi \subset B(\lambda+\mu)}$ & we have

using $\otimes$ rule.

$$B(\lambda) \otimes T_\mu \longrightarrow B(\lambda+\mu) \quad e\text{-strict embedding}$$

$$F v_\lambda \otimes t_\mu \longmapsto F(v_\lambda \otimes v_\mu) = F v_\lambda \otimes v_\mu$$

$$\Rightarrow \quad B(\lambda)\otimes T_{-\lambda} \longrightarrow B(\lambda+\mu)\otimes T_{-\lambda-\mu}$$

We have a direct system of crystals

$$\{B(\lambda)\otimes T_{-\lambda}\}_{\lambda\in P_+} \qquad \{I_{\mu,\nu}\}_{\mu\geq\nu}$$

$$B(\nu)\otimes T_{-\nu} \xrightarrow{I_{\mu,\nu}} B(\mu)\otimes T_{-\mu}$$ : e-strict embedding

Define $B(\infty) = \varinjlim B(\lambda)\otimes T_{-\lambda}$ : a crystal

$$u_\infty = \varinjlim v_\lambda \otimes t_{-\lambda}$$

$$
\begin{tikzcd}
B(m)\otimes T_{-m} \arrow[r, phantom, "="] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[d] & & & \text{e-strict embedding} \\
B(n)\otimes T_{-n} \arrow[r, phantom, "="] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[d] & \\
B(\infty) \arrow[r, phantom, "="] & \underset{u_\infty}{\bullet} \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \cdots
\end{tikzcd}
$$

In general,

① $B(\lambda)\otimes T_{-\lambda} \hookrightarrow B(\infty)$ : e-strict embedding

$$\underset{F v_\lambda}{\overset{\shortparallel}{b}} \otimes t_{-\lambda} \longmapsto F u_\infty$$

② $B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\infty \mid i_1, \cdots, i_l \in I,\ l \geq 0 \}$

In particular, it is connected.

Theorem (Kashiwara 91)

$\exists$ a local base $(\mathcal{L}, B)$ of $U_q^-(\mathfrak{g})$ at $q=0$ s.t.

① $B \overset{1-1}{\longleftrightarrow} B(\infty)$ ($\Rightarrow B$ : a crystal)

② $\pi_\lambda : U_q^-(\mathfrak{g}) \longrightarrow V(\lambda)$ induces a map

$$\mathcal{L} \xrightarrow{\pi_\lambda} \mathcal{L}(\lambda)$$

$$B \xrightarrow{\overline{\pi_\lambda}} B(\lambda) \cup \{0\}$$

and $\{ b \in B \mid \overline{\pi_\lambda}(b) \neq 0 \} \cong B(\lambda)$ as a crystal

We call $(\mathcal{L}(\infty), B(\infty)) := (\mathcal{L}, B)$ : a crystal base of $U_q^-(\mathfrak{g})$

Sketch of proof) $\lambda, \mu \in P_+$

Consider the following commutative diagram

$$
\begin{tikzcd}
U_q^-(\mathfrak{g}) \arrow[r, "\pi_{\lambda+\mu}"] \arrow[dr, "\pi_\lambda"'] & V(\lambda+\mu) \arrow[r, hook, "S_{\lambda,\mu}"] \arrow[d, "p_{\lambda,\lambda+\mu}"] & V(\lambda)\otimes V(\mu) \arrow[d, "\mathrm{id}_{V(\lambda)}\otimes p_\mu"] \\
 & V(\lambda) & V(\lambda)\otimes K \arrow[l]
\end{tikzcd}
$$

vertical map, $\pi_*$ : $U_q^-(\mathfrak{g})$ - linear

the others : $U_q(\mathfrak{g})$ - linear

Then ① $\mathcal{L}(\lambda+\mu) \xrightarrow{p_{\lambda,\lambda+\mu}} \mathcal{L}(\lambda)$ (surjective)

$$B(\lambda+\mu) \xrightarrow{\overline{p_{\lambda,\lambda+\mu}}} B(\lambda) \cup \{0\}$$ (surjective)

Moreover, $\{ b \in B(\lambda+\mu) \mid \overline{p_{\lambda,\lambda+\mu}}(b) \neq 0 \} \cong B(\lambda) \otimes \{v_\mu\}$

② $\xi \in Q_+$ , $\lambda \gg 0$ $\quad U_q^-(\mathfrak{g})_{-\xi} \cong V(\lambda)_{\lambda-\xi}$

Define $\quad \mathcal{L}_{-\xi} := \pi_\lambda^{-1}(\mathcal{L}(\lambda)_{\lambda-\xi}) \quad B_{-\xi} := \overline{\pi}_\lambda^{-1}(B(\lambda)_{\lambda-\xi})$

$$\mathcal{L} := \bigoplus_{\xi \in Q_+} \mathcal{L}_{-\xi} \qquad B = \coprod_{\xi \in Q_+} B_{-\xi}$$

$\square$

**Rmk**

4 $\quad \mu > \nu$

$$\begin{tikzcd}
 & U_q^-(\mathfrak{g}) \arrow[dl, two heads, "\pi_\mu"'] \arrow[dr, two heads, "\pi_\nu"] & \\
V(\mu) \arrow[rr, two heads, "p_{\nu,\mu}"] & & V(\nu)
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
 & \mathcal{L}(\infty) \arrow[dl, two heads, "\pi_\mu"'] \arrow[dr, two heads, "\pi_\nu"] & \\
\mathcal{L}(\mu) \arrow[rr, two heads, "p_{\nu,\mu}"] & & \mathcal{L}(\nu)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B(\infty) & \\
B(\mu)\otimes T_{-\mu} \arrow[ur, hook] & & B(\nu)\otimes T_{-\nu} \arrow[ul, hook'] \arrow[ll, hook', "\iota_{\nu,\mu}"]
\end{tikzcd}$$

② One can define another crystal operators $\tilde{e}_i, \tilde{f}_i$ directly on $U_q^-(\mathfrak{g})$ as a module of the $q$-boson algebra assoc. to $\mathfrak{g}$

$( \because U_q^-(\mathfrak{g}) \notin \mathcal{O}_{int} )$

Verma module of $U_q(\mathfrak{g})$

<u>Theorem</u> (Kashiwara)

$$\mathcal{L}(\infty) = \sum_{i_1, \cdots, i_r} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1$$

$$B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \cdot 1 \pmod{q\mathcal{L}(\infty)} \}$$