BUSEMANN FUNCTIONS IN RANDOM GROWTH AND POLYMER MODELS

Timo Seppäläinen

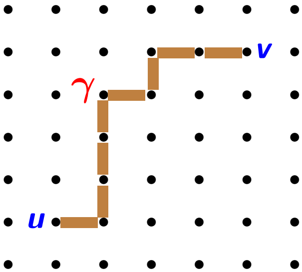
THE UNIVERSITY of WISCONSIN MADISON

Summary. Busemann processes of random growth models and directed polymer models, properties and usefulness.

- mainly in the **exponential corner growth model**
- time permitting **directed landscape**

Collaborators: Márton Balázs (Bristol), Ofer Busani (Bonn), Chris Janjigian (Purdue), Louis Fan (Indiana), Firas Rassoul-Agha (Utah), Xiao Shen (Utah), Evan Sorensen (Madison).

Corner growth model (directed last-passage percolation)



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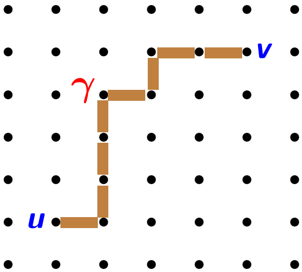

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Point-to-point last-passage percolation process:

$$L(\boldsymbol{u}, \boldsymbol{v}) \;=\; \max_{\gamma\,:\,\boldsymbol{u} \to \boldsymbol{v}} \sum_{x \in \gamma} \omega_x \qquad \text{for } \boldsymbol{u} \leqslant \boldsymbol{v} \text{ in } \mathbb{Z}^2$$

A maximizing path is a **geodesic**. Unique when ω_x assumed continuous.

Semi-infinite geodesics

A **semi-infinite geodesic** is an infinite up-right nearest-neighbor path $(x_k)_{k\geqslant 0}$ that is the geodesic between any two of its points:

$$L(x_m, x_n) = \sum_{i=m}^{n} \omega_{x_i} \quad \forall\, m < n$$

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Questions:

- Are semi-infinite geodesics directed: $\exists\, \boldsymbol{\xi} = \lim_{n\to\infty} \frac{x_n}{n}$?

- Given x and $\boldsymbol{\xi}$, existence and uniqueness of $\boldsymbol{\xi}$-directed semi-infinite geodesic from x ?

- Given x, y and $\boldsymbol{\xi}$, do the $\boldsymbol{\xi}$-directed geodesics from x and y cross ? Coalesce ?

Early work to set the stage

Given a direction $\boldsymbol{\xi}$: techniques of Newman et al. (1990s) adapted to Exp CGM (Ferrari, Pimentel, Coupier 2000s) proved almost surely:

- $\exists$ unique $\boldsymbol{\xi}$-directed semi-infinite geodesic from every $x \in \mathbb{Z}^2$.
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Question: What is the global picture and how to access it?

Busemann function and Busemann process

Busemann function $B^{\boldsymbol{\xi}}$ in direction $\boldsymbol{\xi} \in \,]\mathbf{e}_2, \mathbf{e}_1[$ is defined by

$$B^{\boldsymbol{\xi}}(x,y) = \lim_{n\to\infty} [L(x, v_n) - L(y, v_n)] \qquad x, y \in \mathbb{Z}^2$$

for a sequence $v_n \to \infty$ s.t. $v_n/n \to \boldsymbol{\xi}$.

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Busemann process combines these simultaneously in all directions:

$$\{B^{\boldsymbol{\zeta}\square}(x, y) : \boldsymbol{\zeta} \in]\mathbf{e}_2, \mathbf{e}_1[\,,\ \square \in \{\pm\},\ x, y \in \mathbb{Z}^2\}$$

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Construction. For example:

(1) Prove Busemann limits for a countable dense set of $\boldsymbol{\xi}$.

(2) Take left/right limits $\boldsymbol{\xi} \to \boldsymbol{\zeta}\pm$ to construct full process.

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(1') If almost sure limits not available, start with a weak limit on a larger probability space.

Can we understand distribution of $\{B^{\boldsymbol{\xi}\pm} : \boldsymbol{\xi} \in \,]\mathbf{e}_2, \mathbf{e}_1[\,\}$?

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$\forall\, t \in \mathbb{Z}$ let $\overline{\omega}_t = (\omega_{(k,t)})_{k\in\mathbb{Z}}$ and $\overline{B}_t^{\boldsymbol{\xi}} = (B^{\boldsymbol{\xi}}_{(k,t),(k+1,t)})_{k\in\mathbb{Z}}$.

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Theorem [Fan, S; PMP 2020] Given $(\rho_1, \dots, \rho_n) \in (1,\infty)^n$, this Markov chain has a unique invariant distribution ergodic under spatial translations and with means $\mathbf{E} B^{\boldsymbol{\xi}_i}_{(k,t),(k+1,t)} = \rho_i$.

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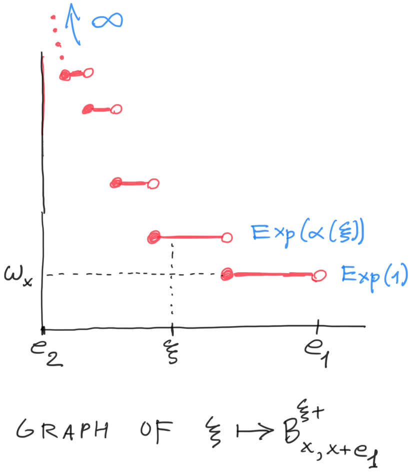
Queueing-theoretic features of $\boldsymbol{D}$ give access to explicit properties.

Exponential case, nearest-neighbor edge

Theorem. $\{B^{\boldsymbol{\xi}+}(x, x+\mathbf{e}_1) : \boldsymbol{\xi} \in]\mathbf{e}_2, \mathbf{e}_1[\,\}$ is a nonincreasing jump process.



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- $N =$ Poisson p.p. on $(0,1)$ with intensity $r^{-1}dr$, and $N\{1\} = 1$.
- To each $r \in N$ attach independent $Z_r \sim \text{Exp}(r)$.
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Corollary. $\boldsymbol{\xi} \mapsto B^{\boldsymbol{\xi}\pm}(x,y)$ is a jump process.

Busemann singularities $\iff$ geodesic nonuniqueness

Starting point: w.p.1 $\exists$ countable dense set $\mathcal{D}_0 \subset \,]\mathbf{e}_2, \mathbf{e}_1[$ of directions $\boldsymbol{\xi}$ with unique coalescing geodesics $\{\boldsymbol{\gamma}^{x,\boldsymbol{\xi}} : x \in \mathbb{Z}^2\}$.

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$$B^{\boldsymbol{\xi}\pm}(x, y) \text{ jumps at } \boldsymbol{\zeta} \implies \mathbf{z}^{\boldsymbol{\xi}\pm}(x, y) \text{ jumps at } \boldsymbol{\zeta} \implies \text{nonunique } \boldsymbol{\zeta}\text{-directed geodesics exist.}$$

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Works the other way also.

Global existence, uniqueness and coalescence in the exponential corner growth model

Theorem [Janjigian, Rassoul-Agha, S; JEMS 2022]

The following holds with probability one. $\exists$ a countable dense random set $\mathcal{V}^\omega \subset \,]\mathbf{e}_2, \mathbf{e}_1[$ of exceptional directions with these properties:

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• If $\boldsymbol{\xi} \notin \mathcal{V}^\omega$, then from each $x \in \mathbb{Z}^2$ there is a unique $\boldsymbol{\xi}$-directed geodesic and all these geodesics coalesce.

• If $\boldsymbol{\xi} \in \mathcal{V}^\omega$, then out of each $x \in \mathbb{Z}^2$ there are exactly two semi-infinite geodesics $\boldsymbol{\gamma}^{x,\boldsymbol{\xi}+}$ and $\boldsymbol{\gamma}^{x,\boldsymbol{\xi}-}$ in direction $\boldsymbol{\xi}$ that eventually separate. These form two distinct coalescing trees of geodesics.

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• $\mathcal{V}^\omega =$ the set of jumps of the Busemann functions $\boldsymbol{\xi} \mapsto B^{\boldsymbol{\xi}\pm}(x, y)$ over all $x, y \in \mathbb{Z}^2$.

• $\mathcal{V}^\omega = \{\boldsymbol{\xi}^*(x) : x \in \mathbb{Z}^2\} =$ the random set of asymptotic directions of competition interfaces.

Global existence, uniqueness and coalescence in the exponential corner growth model (cont.)

This accounts for all the semi-infinite geodesics on $\mathbb{Z}^2$, except the trivial ones $\{x + k\mathbf{e}_i : k \geqslant 0\}$ with direction $\mathbf{e}_i$. In particular:

- Every semi-infinite geodesic has a direction in $[\mathbf{e}_2, \mathbf{e}_1]$.

- There do not exist three disjoint semi-infinite geodesics with the same direction anywhere on the lattice, except the trivial ones.

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Open problem: Prove Coupier's result without relying on computations on the TASEP speed process from [Amir, Angel, Valkó; AOP 2011] !

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What about general continuous i.i.d. weights?

What are the precise assumptions used for the previous global description?

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In all cases of i.i.d. LPP where these issues have been resolved, both assumptions are valid.

Distribution of Busemann jumps on the x-axis

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Abbreviate $B^{\boldsymbol{\xi}\pm}_{\mathbf{k}} = B^{\boldsymbol{\xi}\pm}_{(\mathbf{k},0),(\mathbf{k}+1,0)}$. Let $\cdots < \tau^{\boldsymbol{\xi}}_{-1} < 0 \leqslant \tau^{\boldsymbol{\xi}}_{0} < \tau^{\boldsymbol{\xi}}_{1} < \cdots$ be the ordered set of indices $\mathbf{k}$ such that $B^{\boldsymbol{\xi}-}_{\mathbf{k}} \neq B^{\boldsymbol{\xi}+}_{\mathbf{k}}$.

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Theorem. [Janjigian, Rassoul-Agha, S; JEMS 2022]
Condition on $B_0^{\boldsymbol{\xi}-} > B_0^{\boldsymbol{\xi}+}$ in the Palm sense. Then
$\{\tau_{i+1}^{\boldsymbol{\xi}} - \tau_i^{\boldsymbol{\xi}},\ B_{\tau^{\boldsymbol{\xi}}(i)}^{\boldsymbol{\xi}-} - B_{\tau^{\boldsymbol{\xi}}(i)}^{\boldsymbol{\xi}+} : i \in \mathbb{Z}\}$ is **i.i.d.**

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$$\mathbb{P}\big\{\tau^{\boldsymbol{\xi}}_{i+1} - \tau^{\boldsymbol{\xi}}_i = n,\ B^{\boldsymbol{\xi}-}_{\tau^{\boldsymbol{\xi}}(i)} - B^{\boldsymbol{\xi}+}_{\tau^{\boldsymbol{\xi}}(i)} > r \,\big\|\, B^{\boldsymbol{\xi}-}_0 > B^{\boldsymbol{\xi}+}_0\big\} = C_{n-1}\,\frac{1}{2^{2n-1}}\,e^{-\alpha(\boldsymbol{\xi})r} \qquad \forall\, i \in \mathbb{Z},\ n \in \mathbb{N},\ r \in \mathbb{R}_+.$$

$\|$ is Palm conditioning. Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$.

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Jump locations come like the zeroes of simple random walk!

For example, $[\![-N, N]\!]$ contains order $\sqrt{N}$ occurrences of each $\boldsymbol{\xi} \in \mathcal{V}^{\omega}$.

Applications: coalescence estimates

Start semi-infinite $\boldsymbol{\xi}$-geodesics from $k^{2/3}\mathbf{e}_1$ and $k^{2/3}\mathbf{e}_2$.

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For large $\boldsymbol{R} > 0$:

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Proof uses joint Busemann distribution (among other inputs).

The wrong correction $(\log \boldsymbol{R})^{2/3}$ is the price we pay for not using integrable probability. Sharp result by [Basu, Sarkar, Sly 2019].

Non-existence of bi-infinite geodesics

Theorem. With probability one, there are no nontrivial bi-infinite geodesics in the Exp CGM.

[Basu, Hoffman, Sly; CMP 2022 (FIRST). Balázs, Busani, S; Forum Sigma 2020. Geometric CGM by Groathouse, Janjigian, Rassoul-Agha.]

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Approach generalizes to positive temperature.

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From zero temperature LPP to positive temperature polymers: finite geodesics are replaced by quenched point-to-point polymer distributions: the probability of a path γ between points $\boldsymbol{u}$ and $\boldsymbol{v}$ is

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Theorem. [Busani, S; EJP 2022] With probability one, there are no nontrivial bi-infinite Gibbs measures in the inverse-gamma directed polymer model.

Directed landscape

Directed landscape $\mathcal{L}(x, s; y, t)$ is the 4-parameter scaling limit of last-passage percolation models in the KPZ class in the KPZ scaling window: time increments of order N, spatial increments of order $N^{2/3}$, fluctuations of order $N^{1/3}$. [Dauvergne, Ortmann, Virág; Acta Math]

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Limit of Exp CGM, along the diagonal from time level $\boldsymbol{s}N$ to $\boldsymbol{t}N$, with lateral spatial deviations $\boldsymbol{x}N^{2/3}$ and $\boldsymbol{y}N^{2/3}$:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{s}; \boldsymbol{y}, \boldsymbol{t}) = \lim_{N\to\infty} \frac{1}{2^{4/3}N^{1/3}}\Big\{L\big((\boldsymbol{s}N + 2^{5/3}\boldsymbol{x}N^{2/3}, \boldsymbol{s}N), (\boldsymbol{t}N + 2^{5/3}\boldsymbol{y}N^{2/3}, \boldsymbol{t}N)\big) - 4N(\boldsymbol{t} - \boldsymbol{s}) - 2^{8/3}N^{2/3}(\boldsymbol{y} - \boldsymbol{x})\Big\}.$$

[Dauvergne, Virág]

Directed landscape and the KPZ fixed point

KPZ fixed point is a height function that evolves in the DL environment: for $s < t$,

$$h_t(y) = \sup_{\mathbf{x} \in \mathbb{R}} \{h_s(\mathbf{x}) + \mathcal{L}(\mathbf{x}, s; y, t)\}$$

[Matetski, Quastel, Remenik; Acta Math 2021. Nica, Quastel, Remenik; Forum Sigma 2020]

Busemann process in DL

Busemann function in a fixed direction $\boldsymbol{\xi}$ constructed by Rahman-Virág:

$$W_{\boldsymbol{\xi}}(x, s; y, t) = \lim_{\mathbf{u} \to \infty} \left[\mathcal{L}(x, s; \mathbf{u}\boldsymbol{\xi}, \mathbf{u}) - \mathcal{L}(y, t; \mathbf{u}\boldsymbol{\xi}, \mathbf{u}) \right]$$

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Extend to full Busemann process as before:

$$\big\{W_{\boldsymbol{\xi}\square}(x,s;y,t) : \boldsymbol{\xi}\in\mathbb{R},\ \square\in\{\pm\},\ (x,s;y,t)\in\mathbb{R}^4\big\}$$

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Busemann process evolves as a backward KPZ FP: for $s<t$,

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Question: Can we find the joint Busemann distribution as the invariant distribution of the KPZ FP?

Stationary horizon

Stationary horizon (SH) is a cadlag process $\{G_{\boldsymbol{\xi}}\}_{\boldsymbol{\xi}\in\mathbb{R}}$ with paths $\boldsymbol{\xi} \mapsto G_{\boldsymbol{\xi}}$ in $D(\mathbb{R}, C(\mathbb{R}))$ whose finite-dimensional distributions $(G_{\boldsymbol{\xi}_1}, \ldots, G_{\boldsymbol{\xi}_n})$ for $\boldsymbol{\xi}_1 < \cdots < \boldsymbol{\xi}_n$ can be constructed as follows.

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Then

$$(G_{\xi_1}, G_{\xi_2}, \dots, G_{\xi_n}) \stackrel{d}{=} \big(f_1, \Psi^2(f_1, f_2), \dots, \Psi^n(f_1, \dots, f_n)\big)$$

where $\Psi^k : C(\mathbb{R})^k \to C(\mathbb{R})$ is the centered k-level Brownian last-passage value:

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$$- \sup_{-\infty < x_{k-1} \leqslant \cdots \leqslant x_1 \leqslant 0} \left\{ \sum_{i=1}^{k-1} (f_i(x_{i-1}) - f_i(x_i)) + f_k(x_{k-1}) \right\}$$

Stationary horizon: immediate properties

$\{G_{\xi}\}_{\xi \in \mathbb{R}}$ is a process of monotonically ordered coupled Brownian motions with drift: marginally

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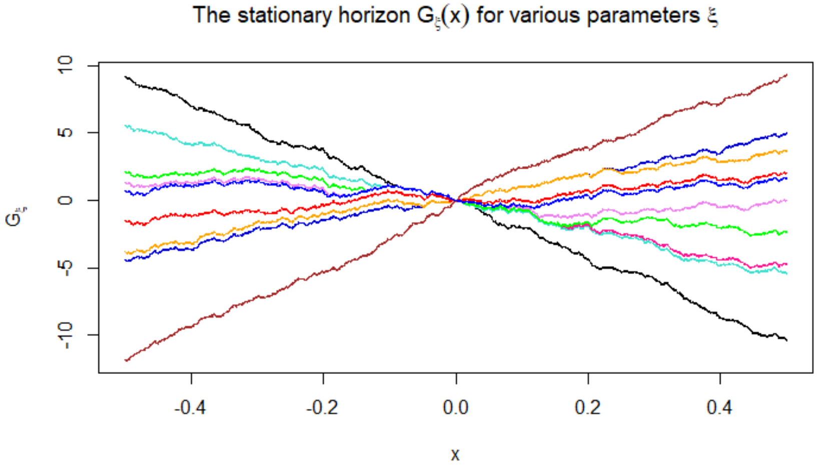
Ofer Busani derived SH as a limit from the Busemann process of Exp CGM and baptized it. SH arose concurrently and independently as the Busemann process of Brownian LPP [S, Sorensen].



Each color marks the graph of $G_\xi(x) \overset{d}{=} \sqrt{2}B(x) + 2\xi x$ for a particular value ξ. Any two graphs coincide on a nondegenerate interval around 0.

Invariance of SH under KPZ fixed point

Theorem. [Busani, S, Sorensen]

Run KPZ fixed point with coupled initial data $\{h_0^{\boldsymbol{\xi}}(\bullet)\} \stackrel{d}{=} \{G_{\boldsymbol{\xi}}(\bullet)\}$:

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Case $n = 1$ improves on Pimentel's earlier uniqueness result.

Busemann process in DL

Corollary. Joint distribution of Busemann process on a time level:

$$\{W_{\xi+}(\bullet, t; 0, t)\}_{\xi\in\mathbb{R}} \overset{d}{=} \{G_{\xi}(\bullet)\}_{\xi\in\mathbb{R}}$$

as random elements of $D(\mathbb{R}, C(\mathbb{R}))$.

Discontinuities of the Busemann process

$$\text{For} \quad p, q \in \mathbb{R}^2 : \; \Xi(p;q) = \{\xi \in \mathbb{R} : W_{\xi-}(p;q) \neq W_{\xi+}(p;q)\}$$

Set of all discontinuities: $\Xi = \bigcup_{p,q \in \mathbb{R}^2} \Xi(p;q).$

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$Ɉ$ is countably infinite and dense in $\mathbb{R}$.

Picture analogous to CGM arises.

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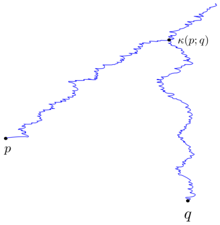
When $\xi \notin \Xi$, all semi-infinite geodesics in direction ξ coalesce. **BUT** $\forall\,\xi$ and on each time level $\exists$ countably infinitely many points from which the ξ-directed geodesic is not unique before the point of coalescence.

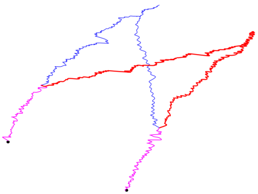
Uniqueness and coalescence

Ŧ = dense countable set of directions where Busemann process jumps.

When $\boldsymbol{\xi} \notin$ Ŧ, all semi-infinite geodesics in direction $\boldsymbol{\xi}$ coalesce. **BUT** $\forall \boldsymbol{\xi}$ and on each time level $\exists$ countably infinitely many points from which the $\boldsymbol{\xi}$-directed geodesic is not unique before the point of coalescence.

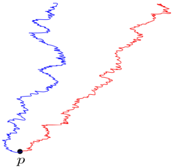
When $\boldsymbol{\xi} \in$ Ŧ, there exist two coalescing families of semi-infinite geodesics in direction $\boldsymbol{\xi}$. **BUT** presently we cannot rule out additional $\boldsymbol{\xi}$-directed semi-infinite geodesics.





Geodesic separation points

$$\mathfrak{S} = \{p \in \mathbb{R}^2 : \exists \boldsymbol{\xi} \in \mathbb{R} \text{ s.t. } \exists \text{disjoint geodesics in direction } \boldsymbol{\xi} \text{ from } p\}$$



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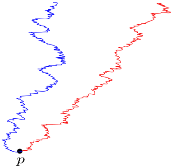

Theorem. [Busani, S, Sorensen]

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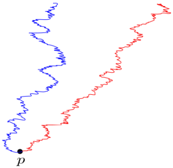

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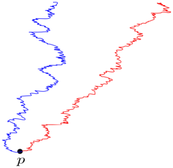

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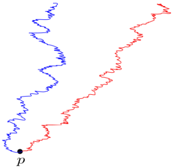

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CGM: **every** lattice point x has disjoint geodesics in direction $\boldsymbol{\xi}^*(x)$.

Busemann difference profile

$$f_{\boldsymbol{\xi}}(x) \;=\; W_{\boldsymbol{\xi}+}(x,0;0,0) - W_{\boldsymbol{\xi}-}(x,0;0,0).$$

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Theorem. [Busani, S, Sorensen]
Conditionally on $\boldsymbol{\xi} \in Ɫ$ in the Palm sense, the restarted function

$$x \;\mapsto\; f_{\boldsymbol{\xi}}(\tau_{\boldsymbol{\xi}} + x), \quad x \geqslant 0,$$

is equal in distribution to Brownian local time.