Euler systems and Beilinson–Flach elements

§1 Cyclotomic units

References:

- Coates–Sujatha: cyclotomic fields and zeta values
- Lang: cyclotomic fields I and II
- Washington: Introduction to cyclotomic fields.

§1.1 Basic properties

Fix a collection of primitive m-th roots of unity ζ_m, $m \geq 1$ s.t. $(\zeta_{mn})^n = \zeta_m$

eg $\zeta_m = e^{2\pi i/m}$

The ring of integers of $\mathbb{Q}(\zeta_m)$ is $\mathbb{Z}[\zeta_m]$

We write $E_m = \mathbb{Z}[\zeta_m]^\times$

$V_m :=$ mult gp of $\mathbb{Q}(\zeta_m)^\times$ generated by $\pm\zeta_m$, $1-\zeta_m^a$, $1 \leq a \leq m-1$

The gp of cyclo. units in $\mathbb{Q}(\zeta_m)$ is $C_m = \mathbb{Z}[\zeta_m]^\times \cap V_m$

If K is an abelian number field, then $K \subset \mathbb{Q}(\zeta_m)$ (Kronecker–Weber)

We define the gp of cyclo units in K to be
$$C_K = \mathcal{O}_K^\times \cap C_m$$

Ex $\mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n) \cap \mathbb{R} = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$

$$\zeta_n^{(1-a)} \frac{1-\zeta_n^a}{1-\zeta_n} = \pm \frac{\sin \pi a/n}{\sin \pi/n}$$

From now on, we write

$$E_n^+ = (\mathbb{Z}[\zeta_{p^n}]^+)^\times$$
$$V_n^+ = V_{\mathbb{Q}(\zeta_{p^n})^+}$$
$$C_n^+ = C_{\mathbb{Q}(\zeta_{p^n})^+}$$

(p is a fixed odd prime)

Lem 1.1 (a) C_n^+ is generated by -1 and $\xi_a = \zeta_{p^n}^{\frac{(1-a)}{2}} \cdot \frac{1-\zeta_{p^n}^a}{1-\zeta_{p^n}}$ as a runs through

$1 < a < \frac{1}{2}p^n$ $(a,p)=1$.

(b) $C_{\mathbb{Q}(\zeta_{p^n})}$ is generated C_n^+ and ζ_{p^n}.

Thm 1.2 $[E_n^+ : C_n^+] < \infty$ and is equal to the class number of $\mathbb{Q}(\zeta_{p^n})^+$.

Idea: Analytic class number formula:

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|D_K|}}$$

$K = \mathbb{Q}(\zeta_{p^n})^+$, the RHS $=$

$$\frac{2^{d-1} h_K R_K}{\prod_{\substack{\chi\in\hat{G}\\ \chi\neq 1}} \tau(\chi)}, \quad \text{where } G = \mathrm{Gal}(K/\mathbb{Q}),\ d = [K:\mathbb{Q}]$$

$$R_\xi = \det(\log|\xi_a^\sigma|)_{\substack{\sigma\in G,\ \sigma\neq 1\\ 1<a<p^n/2,\ (a,p)=1}} = \prod_\chi \left(-\frac{\tau(\chi)}{2} L(1,\bar{\chi})\right)$$

We can conclude using lem 1.1.

Norm relations

Let ℓ be a prime, $m>1$ an integer.

Let $N: \mathbb{Q}(\zeta_{m\ell}) \to \mathbb{Q}(\zeta_m)$ be the norm map.

$$[\mathbb{Q}(\zeta_{m\ell}) : \mathbb{Q}(\zeta_m)] = \begin{cases} \ell & \text{if } \ell \mid m \\ \ell-1 & \text{if } \ell \nmid m\end{cases}$$

Min poly $\zeta_{m\ell}$ over $\mathbb{Q}(\zeta_m)$ is given by

$$\begin{cases} X^\ell - \zeta_m & \text{if } \ell\mid m \\ \dfrac{X^\ell - \zeta_m}{X - \zeta_m^a} & \text{if } \ell\nmid m\end{cases}$$

where $a \in \mathbb{Z}$ s.t. $a\ell \equiv 1 \mod m$

$$N(1-\zeta_{m\ell}) = \begin{cases} 1-\zeta_m & \text{if } \ell\mid m \\ (1-\zeta_m)^{1-\sigma_\ell^{-1}} & \text{if } \ell\nmid m\end{cases}$$

$\sigma_\ell \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ sending ζ_m to ζ_m^ℓ.

If $(a, 2m\ell)=1$, then

$$N\left(\zeta_{m\ell}^{\frac{1-a}{2}} \frac{1-\zeta_{m\ell}^a}{1-\zeta_{m\ell}}\right) = \begin{cases} \zeta_m^{\frac{1-a}{2}} \cdot \dfrac{1-\zeta_m^a}{1-\zeta_m} & \text{if } \ell\mid m \\ (\cdots)^{1-\sigma_\ell^{-1}} & \text{if } \ell\nmid m\end{cases}$$

$$(\cdots)^{1-\sigma_\ell^{-1}} = \frac{(\cdots)}{(\cdots)^{\sigma_\ell^{-1}}}$$

§1.2 Iwasawa main conjecture

p odd prime

$F_n = \mathbb{Q}(\zeta_{p^n})^+$, $G_n = \mathrm{Gal}(F_n/\mathbb{Q})$

$F_\infty = \cup F_n$

$$G = \mathrm{Gal}(F_\infty/\mathbb{Q}) \cong \underbrace{\Delta}_{\text{cyclic } \frac{p-1}{2}} \times \mathbb{Z}_p$$

$U_n^1 =$ local units of $\mathbb{Q}_p(\zeta_{p^n})^+$ cong to 1 modulo the max ideal.

$E_n^1 =$ completion of $E_n^+ \cap U_n^1$

$C_n^1 =$ — " — $C_n^+ \cap U_n^1$

$U_\infty^1 = \varprojlim_n U_n^1$, $E_\infty^1 = \varprojlim_n E_n^1$,

$C_\infty^1 = \varprojlim_n C_n^1$ where the connecting maps are the norm maps.

$$\Lambda = \mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G_n] \cong \mathbb{Z}_p[\Delta][[X]]$$

power series in X with coeffs in $\mathbb{Z}_p[\Delta]$ and $X \leftrightarrow \gamma - 1$ where γ is a top gen of the $\mathbb{Z}_p$ part of G.

U_∞^1, E_∞^1 and C_∞^1 are Λ-mods.

Lem 1.3 $C_\infty^1 = \Lambda b$ where $b = (b_n)$

where $b_n = \dfrac{\zeta_{p^n}^{-a/2} - \zeta_{p^n}^{a/2}}{\zeta_{p^n}^{-1/2} - \zeta_{p^n}^{1/2}} \cdot u$

where $(a,p)=1$, $u \in \mathbb{Z}_p^\times$ is the $(p-1)$-st root of unity s.t. $ua \equiv 1 \mod p$.

Let M_n be the max'l ab p-extn of F_n unramified outside p.

Let L_n is the max'l ab p-extn of F_n unramified everywhere. (p-part of the Hilbert class field)

$M_\infty = \cup M_n$, $L_\infty = \cup L_n$

$$X_\infty\left\{\begin{array}{c} M_\infty \\ | \\ \left.\begin{array}{c} L_\infty \\ | \\ F_\infty \end{array}\right\} Y_\infty \end{array}\right.$$

Thm 1.2 $E_n^1/C_n^1 \leftrightsquigarrow \mathrm{Gal}(L_n/F_n)$

This suggests that $Y_\infty \leftrightsquigarrow E_\infty^1/C_\infty^1$

Both X_∞ and Y_∞ are Λ-mods

There is an exact sequence of Λ-mods:

$$0 \to E_\infty^1/C_\infty^1 \to U_\infty^1/C_\infty^1 \to X_\infty \to Y_\infty \to 0 \quad (*)$$

Thm 1.4 $\exists$ Λ-morphism

$$\mathcal{L}': U_\infty^1 \xrightarrow{\sim} \Lambda \quad \text{(Coleman map)}$$

with $\mathcal{L}'(C_\infty^1) = I(G)\,\zeta_p$ where $I(G)$ is the augmentation ideal and ζ_p is the Kubota–Leopoldt p-adic L-function.

Let L_n is the max'l ab p-extn of F_n unramified everywhere. (p-part of the Hilbert class field)

F_∞

Both X_∞ and Y_∞ are Λ-mods

Iwasawa main conj predicts that there is a link between Y_∞ and E'_∞ / C'_∞ as Λ-mods.

$I(G)$ is the augmentation ideal and $\mathcal{S}_p$ is the Kubota-Leopoldt p-adic L-function.

Cor 1.5 $U'_\infty / C'_\infty \cong \Lambda(G) / I(G)\,\mathcal{S}_p$.

In particular, it's a torsion Λ-mod.

Y_∞ satisfies a control thm

$(Y_\infty)_{\Gamma_n} \doteq Gal(L_n/F_n)$ where

$\Gamma_n = Gal(F_\infty/F_n)$

$\Rightarrow Y_\infty$ is a torsion Λ-mod.

So every module in $(*)$ is Λ-torsion

Thm 1.6 If M is a f.g. torsion Λ-mod, then $\exists\, f_1, \dots, f_r \in \Lambda$ and $\exists$ a Λ-mod Q with $|Q| < \infty$ s.t.

$$0 \to \bigoplus_{i=1}^{r} \Lambda/f_i \to M \to Q \to 0$$

is an exact seq of Λ-mod.

We define the characteristic ideal of M to be

$$ch_\Lambda(M) = (f_1 \cdots f_r) \subset \Lambda$$

Iwasawa main conj

$$ch_\Lambda(E'_\infty / C'_\infty) = ch_\Lambda(Y_\infty)$$

This is equivalent to

$$ch_\Lambda(X_\infty) = ch_\Lambda(U'_\infty / C'_\infty) = I(G)\,\mathcal{S}_p$$

Goal: Explain how to prove

$ch_\Lambda(Y_\infty) \mid ch_\Lambda\, E'_\infty / C'_\infty$.

$$0 \to \bigoplus_{i=1}^{r} \frac{\Lambda}{f_i} \to Y_\infty \to Q \to 0, \quad |Q| < \infty$$

Fact $E'_\infty / C'_\infty \doteq \Lambda(G)/(\beta)$ for some $\beta \in \Lambda(G)$.

We want to prove $\prod f_i \mid \beta$

We will project to finite levels

$F = F_m$

Let $A = Gal(L_m/F_m)$ (related to $(Y_\infty)_{\Gamma_m}$ by the control thm)

Let $\Pi = Gal(F/\mathbb{Q})$

$R = \mathbb{Z}_p[\Pi]$

$pr : \Lambda(G) \to R$

$R/pr(\beta)$ is finite

Fix δ an annihilator of Q

Fix s a power of p that annihilates $R/pr(\beta)$ and $R/pr(\delta)$

Let $t = \#A\,\#Q\,p^m s^{r+1}$

is a p-power $\geq p^m$

We will look at projection modulo t.

Write $\mathcal{R} = \mathbb{Z}/t\,[\Pi]$.

We will aim to prove divisibility over $\mathcal{R}$, then let $m \to \infty$

Define $\Lambda(G) \to \mathcal{R}$

$x \mapsto x^*$

Thm 1.7 For $i = 1, \dots, r$, we have $(f_1 \cdots f_i)^* \mid ((\gamma - 1)\beta\,\delta^{i+1})^*$ in $\mathcal{R}$

as $m \to \infty$, we get

$f_1 \cdots f_r \mid (\gamma-1)\beta\,\delta^{r+1}$ in Λ.

Since Q is a pseudo-null Λ-mod, we can find two δ_1 and δ_2 that are coprime that are both annihilators of Q

$f_1 \cdots f_r \mid (\gamma-1)\beta\,\delta_1^{r+1}$

and $(\gamma-1)\beta\,\delta_2^{r+1}$

$\Rightarrow f_1 \cdots f_r \mid (\gamma-1)\beta$

Using control thm

$(Y_\infty)_\Gamma$ is finite $\Rightarrow$

$ch_\Lambda(Y_\infty)$ is coprime to $\gamma - 1$

$\Rightarrow f_1 \cdots f_r \mid \beta$.

The pf is by induction on i.

We will make use of a filtration on A:

$Fil^0(A) \subset Fil^1(A) \subset \cdots \subset Fil^r(A) = A$

s.t.

$$0 \to \underset{\substack{\uparrow \\ \text{killed by } \delta}}{Q_i} \to \mathcal{R}/pr(f_i) \to \underset{\substack{\uparrow \\ \text{cyclic } R\text{-mod.}}}{\frac{Fil^i(A)}{Fil^{i-1}(A)}} \to 0 \quad (**)$$

§ 1.3 An Euler system argument

Fix integers $k \geq 2$, $a_1, \dots, a_k \neq 0$

$n_1, \dots, n_k$ s.t. $\sum_{i=1}^{k} n_i = 0$

Let $\alpha(T) = \prod_{i=1}^{k} (T^{a_i/2} - T^{-a_i/2})^{n_i}$

$S = \{2, \text{ primes} \mid \prod a_i\}$

$\alpha(\zeta_d)$ is a cyclotomic unit.

Lem 1.8 If $(d, S) = 1$, q a prime s.t. $(q, d \cup S)$. For $\zeta \in \mu_d$ and $\rho \in \mu_q \setminus \{1\}$, we have

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$$N_{\mathbb{Q}(\zeta_{dq})^+/\mathbb{Q}(\zeta_d)^+}(\alpha(\rho\zeta)) = \alpha(\zeta)^{\sigma_q - 1}$$

Let $Z_S = \{$square-free integers prime to S and $p\}$

$Z'_S = \{\prod q_i \in Z_S \mid q_i \equiv 1 \bmod t\}$.

If $q \in Z_S$, Δ_q is cyclic of order $q - 1$.

Fix τ_q a generator of Δ_q.

Define the Kolyvagin derivative

$$D(q) = \sum_{i=0}^{q-2} i\,\tau_q^i \in \mathbb{Z}[\Delta_q]$$

Lem 1.9 (a) $F^\times / F^{\times t} = (J_n^\times / J_n^{\times t})^{\Delta_n}$

(b) Let $n \in Z'_S$, ζ a primitive n-th root of unity, ρ a primitive p^m-th root of unity. Then the image of $\alpha(\rho\zeta)^{D(n)}$ in $J_n^\times / (J_n^\times)^t$ is invariant under Δ_n

... to S and p }

$Z_S' = \{ \prod q_i \in Z_S \mid q_i \equiv 1 \bmod t \}$.

For a positive integer n, we write

$\mathcal{J}_n = F(\zeta_n)^+$

$\Delta_n = \mathrm{Gal}(\mathcal{J}_n / F)$.

$D(q) = \sum_{i=0} i \tau_q^i \in \mathbb{Z}[\Delta_q]$

If $n \in Z_S$, $n = \prod q_i$, define

$D(n) = \prod D(q_i) \in \mathbb{Z}[\Delta_n]$.

$$\begin{array}{c} \mathcal{J}_n \\ \big| \; \Delta_n \\ F \end{array}$$

$\alpha(\rho\zeta)$ in $\mathcal{J}_n/(\mathcal{J}_n)$ is invariant under Δ_n

In other words, $\alpha(\rho\zeta)^{D(n)}$ gives an elt in $F^\times / F^{\times t}$. We will denote this by $R_\alpha(\zeta)$ [Fix ρ]

Lem 1.8 tells us that if

$\zeta \in \mu_n$ a primitive n-th root of unity

$\zeta = \prod u_{q_i}$ $\quad n = \prod q_i$, with $u_{q_i} \in \mu_{q_i}$,

$R_\alpha(\zeta)$ compatible with

$R_\alpha(\zeta / u_{q_i})$ in certain sense.

By a Chebotarev argument, we can find appropriate $q_1, \dots, q_r$ and

$v_j = R_\alpha(u_{q_1} \cdots u_{q_j}) \in F^\times / F^{\times t}$

we get $\mathcal{R}$-morphism

$\eta_j : \mathcal{R} v_j \longrightarrow \mathcal{R}$

$\quad \cap$

$F^\times / F^{\times t}$

$v_j \mapsto y_j^*$ for some $y_j \in R$

y_j annihilates $F_1 p^{j+1} / F_1 p^j$ in (**)

and $\delta^* y_{j-1}^* = f_j^* y_j^*$ up to a unit and $y_0 = pr((\gamma - 1)\beta\delta)$

$\Rightarrow ((\gamma-1)\beta\delta^{j+1})^* \in (f_1 \cdots f_j y_j)^* \mathcal{R}$

Further, from (**)

$pr(\delta)\, y_j \in f_{j+1} R$.

$\Rightarrow ((\gamma-1)\beta\delta^{j+2})^* \in (f_1 \cdots f_{j+1})^* \mathcal{R}$.

$\Rightarrow$ Thm 1.7.

For the other inclusion of the main conj, we can compare leading terms via control thms.

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§2 General defn of an Euler system

Ref Rubin: Euler systems.

Consider the short exact seq:

$0 \to \mu_{p^n} \to \bar{\mathbb{Q}}^\times \xrightarrow{p^n} \bar{\mathbb{Q}}^\times \to 0$

Take $G_{\mathbb{Q}(\zeta_m)}$-cohomology:

$\mathbb{Q}(\zeta_m)^\times / \mathbb{Q}(\zeta_m)^{\times p^n} \to H^1(\mathbb{Q}(\zeta_m), \mu_{p^n})$

Take $\varprojlim_n$,

$\mathbb{Q}(\zeta_m)^\times \hookrightarrow H^1(\mathbb{Q}(\zeta_m), \mathbb{Z}_p(1))$

So cyclotomic units can be regarded as elts in $H^1(\mathbb{Q}(\zeta_m), \mathbb{Z}_p(1))$.

Defn Let T be a f.g. free $\mathbb{Z}_p$-mod, equipped with a cont $G_{\mathbb{Q}}$-action.

$V = T \otimes \mathbb{Q}_p$, $\quad W = V/T$

$T^* = \mathrm{Hom}(T, \mathbb{Z}_p)$, $\quad V^* = \mathrm{Hom}(V, \mathbb{Q}_p)$

$W^* = V^* / T^*$.

$P_q(X \mid V) = \det(1 - X\sigma_q^{-1} \mid V^{I_q})$

q a prime

$L(V, s) = \prod_q P_q(q^{-s} \mid V)$.

$\Sigma \supseteq \{ q : V^{I_q} \neq V \} \cup \{p\}$.

We assume Σ is finite.

An Euler system for T is a collection of classes

$\{ c_m \in H^1(\mathbb{Q}(\zeta_m), T) : m = np^n$, n is a square-free product of primes $\notin \Sigma \}$

s.t. $\mathrm{cor}^{\mathbb{Q}(\zeta_{m\ell})}_{\mathbb{Q}(\zeta_m)}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \mid m \\ P_\ell(\sigma_\ell^{-1} \mid V^*(1))\, c_m & \text{if } \ell \nmid m \end{cases}$

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eg When $T = \mathbb{Z}_p(1)$, $\Sigma = \{p\}$

$V = \mathbb{Q}_p(1)$, $\quad V^*(1) = \mathbb{Q}_p$

$P_\ell(X \mid V^*(1)) = 1 - X$

$P_\ell(\sigma_\ell \mid V^*(1)) = 1 - \sigma_\ell^{-1}$.

If K is a number field let $S_p(K, W) =$

$\ker\left(H^1(K, W) \to \bigoplus_{v \nmid p} \frac{H^1(K_v, W)}{H^1_{ur}(K_v, W)} \oplus \bigoplus_{v \mid p} H^1(K_v, W)\right)$

$\mathbb{Q}_{cyc}$ = cyclotomic $\mathbb{Z}_p$-extn of $\mathbb{Q}$

$S_p(\mathbb{Q}_{cyc}, W) = \varinjlim_{\mathbb{Q} \subset K \subset \mathbb{Q}_{cyc}} S_p(K, W)$.

$Y_\infty = S_p(\mathbb{Q}_{cyc}, W)^\vee$, this is a Λ-mod, where $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q})]]$.

Thm 2.1 (Kato, Rubin, Perrin-Riou) Under certain hypothese, if $\{c_m\}$ is an Euler system for m, let c be the image of $\{c_m\}$ in

$H^1_{Iw}(\mathbb{Q}_{cyc}, T) = \varprojlim H^1(K_n, T)$

and assume c is not in the torsion part of $H^1_{Iw}(\mathbb{Q}_{cyc}, T)$,

$$\mathbb{Z}_p \left(\begin{array}{c} \mathbb{Q}_{cyc} \\ | \\ K_n \\ | \; \mathbb{Z}/p^n \\ \mathbb{Q} \end{array} \right.$$

Then Y_∞ is a Λ-torsion mod, and $ch_\Lambda(Y_\infty)$ divides $ind_\Lambda(c)$, where

$ind_\Lambda(c)$ is the ideal generated by $\phi(c)$, as ϕ runs through elts in $\mathrm{Hom}_\Lambda(H^1_{Iw}(K_\infty, T), \Lambda)$.

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§3. Euler systems for elliptic curves and modular forms

Ref Kato: p-adic Hodge theory and values of zeta functions of modular forms.

Let $E/\mathbb{Q}$ be an elliptic curve

We have an exact seq:

$0 \to H^1_{Iw}(\mathbb{Q}_{cyc}, T) \to \mathrm{Im}\,\mathrm{Col}_{E,p} \to \mathrm{Sel}_{p^\infty}(E/\mathbb{Q}_{cyc})^\vee \to Y_\infty \to 0$

$\ni c_{BK}$ (Beilinson-Kato elt) (ie forms part of an E.S.)

an E.S. argument $\Rightarrow$

$L_{E,p} \in ch_\Lambda \mathrm{Sel}^\vee$

$N \geq 3$ an integer, $Y(N)$ the modular curve parametrizing (E, e_1, e_2)

e_1, e_2 is a basis of $E[N]$

values of zeta functions of modular forms.

Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at p.

$T = T_p E$ Tate module.

Perrin-Riou map: (cyclo $\mathbb{Z}_p$-extn of $\mathbb{Q}_p$.)

$\mathrm{Col}_{E,p} : H^1_{iw}(\mathbb{Q}_{cyc,p}, T) \to \Lambda$

(generalizes thm 1.4)

c_{BK} (Beilinson-Kato elt) (ie forms part of an E.S.)

$\mathrm{Col}_{E,p}(\mathrm{loc}_p(c_{BK})) = \underset{\text{p-adic L-fn. of E at p.}}{L_{E,p}}$

get

$0 \to \dfrac{H^1_{iw}(\mathbb{Q}_{cyc}, T)}{(c_{BK})} \to \dfrac{\Lambda}{(L_{E,p})} \to \mathrm{Sel}^\vee \to Y_\infty \to 0$

$N \geq 3$ an integer, $Y(N)$ the modular curve parametrizing (E, e_1, e_2) e_1, e_2 is a basis of $E[N]$

Fix c an integer s.t. $(c, 6N) = 1$.

For all $(\alpha, \beta) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2 \setminus \{0\}$

$\exists$ a Siegel unit ${}_c g_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times$

satisfying ${}_c g_{\alpha,\beta} = \prod {}_c g_{\alpha',\beta'}$

where $\alpha', \beta' \in \mathbb{Q}/\mathbb{Z}$ s.t. $m\alpha' = \alpha$, $m\beta' = \beta$

Let $Y(M,N)$ be the modular curve parametrizing (E, e_1, e_2) $e_1 \in E[M]$, $e_2 \in E[N]$

For L, M, N s.t. $M | L$ and $N | L$ we have

$Y(L) \to Y(M,N)$

For $(c, 6M) = 1$, $(c, 6N) = 1$, can define

${}_{c,d}Z_{M,N} = \{ {}_c g_{\frac{1}{M},0}, {}_d g_{0,\frac{1}{N}} \} \in K_2(Y(M,N))$

$\left(K_2(X) = \mathcal{O}(X)^\times \otimes \mathcal{O}(X)^\times / \langle u \otimes 1-u \rangle \right)$

Propn 3.1 $K_2(Y(M\ell, N\ell)) \to K_2(Y(M,N))$

sends ${}_{c,d}Z_{M\ell,N\ell}$ to

$$\begin{cases} {}_{c,d}Z_{M,N} & \text{if } \ell | M, \ell | N \\ \begin{pmatrix}\text{Euler factor} \\ \text{in Hecke} \\ \text{operator}\end{pmatrix} {}_{c,d}Z_{M,N} & \text{o/wise.} \end{cases}$$

Let $Y = Y(M,N)$

We have $h : \mathcal{O}(Y)^\times \to H^1(Y, \mathbb{Z}/p^n(1))$

This gives

$K_2(Y) \to H^2(Y, \mathbb{Z}/p^n(2))$

$\{f, g\} \mapsto h(f) \cup h(g)$

We have thus a system of elts

$({}_{c,d}Z_{Mp^n, Np^n}) \mapsto H^2(Y(Mp^n, Np^n), \mathbb{Z}/p^n(2))$

$Y(M,N) \to Y_1(N) \otimes \mathbb{Q}(\zeta_M)$

$\rightsquigarrow z_{Mp^n} \in H^1(\mathbb{Q}(\zeta_{Mp^n}), H^1(Y_1(N)\otimes\bar{\mathbb{Q}}, \mathbb{Z}_p)(1))$

If f is a wt 2 modular form of level N (eigen-cuspform) then T_f (the repn attached to f) is a quotient of $H^1(Y_1(N)\otimes\bar{\mathbb{Q}}, \mathbb{Z}_p)(1)$

$H^1(\mathbb{Q}(\zeta_{Mp^n}), H^1(Y_1(N)\otimes\bar{\mathbb{Q}}, \mathbb{Z}_p)(1))$

$\downarrow$

$H^1(\mathbb{Q}(\zeta_{Mp^n}), T_f) \ni$

gives $z^f_{Mp^n}$

These elts are related to L-values of f twisted by Dirichlet charaters

§4. Beilinson-Flach elts

Ref Flach: A finiteness theorem for the symmetric square of an elliptic curve.

L.-Loeffler-Zerbes: Euler systems for Rankin-Selberg convolutions of modular forms.

Flach was interested in the Selmer gp of the symmetric square rep of $T_p E$ (E an elliptic curve with good redn at p).

$T_p E \otimes T_p E = \underbrace{T}_{\text{rank 3}} \oplus \underbrace{T'}_{\text{rank 1.}}$

He constructed classes

$c(\ell) \in H^1(\mathbb{Q}, T)$, ℓ is indexed by a set of primes (not p, ∞, bad primes)

He used the Gersten complex:

For X a smooth variety of finite type over k, we have

$G_2(X)$:

$\coprod_{x \in X^0} K_2(k(x)) \to \coprod_{x \in X^1} k(x)^\times \xrightarrow{\text{divisor}} \coprod_{x \in X^2} \mathbb{Z}$

where X^i is the set of points of codim i.

Thm 4.1 $H^1(G_2(X)) \cong CH^2(X, 1)$

Elts in $CH^2(X,1)$ can be sent to $H^3(X, \mathbb{Z}_p(2))$ via some regulator map.

If $X = E \times E$, we get

$H^3(X, \mathbb{Z}_p(2)) \to H^1(\mathbb{Q}, H^2(\bar{E}\times\bar{E}, \mathbb{Z}_p(2)))$

$\to H^1(\mathbb{Q}, T)$

To construct $c(\ell)$, we construct an elt in $H^1(G_2(X))$

We construct some

(C, ϕ) where $C \in X^1$

$\phi \in k(C)^\times$ s.t. $\mathrm{div}(\phi) = 0$

Consider

$\pi \times \pi \circ w_\ell : X_0(N\ell) \to X_0(N) \times X_0(N)$

π is the projection map

w_ℓ is Atkin-Lehner.

Write $\mathcal{Z}_\ell(N)$ for the image

We can define

$u_g = \ell^{-k/2} \dfrac{w_\ell g}{g} \in \mathcal{O}(X_0(N\ell))^\times$

where g is a modular form of wt k and level N

Flach constructed an elt in $H^1(G_2(X))$ using $(\mathcal{Z}_\ell(N), u_\Delta)$

Δ = discriminant fn.

Beilinson-Flach elts combine Siegel units with the method of Flach.

Consider

$$\phi_{m,N} : Y(m, mN) \to Y(m, mN)^2$$
$$z \mapsto (z, z + \tfrac{1}{m})$$

Write $C_{m,N} = \mathrm{im}(\phi_{m,N})$.

$(C_{m,N}, g_{0,\frac{1}{mN}})$ gives an elt

$$z_{m,N} \in H^1(G_2(Y(m, mN)^2))$$

via certain regulator maps, we then get an elt in

$$H^1(\mathbb{Q}(\zeta_m), H^1(\overline{Y_1(N)}, \mathbb{Z}_p) \times H^1(\overline{Y(N)}, \mathbb{Z}_p)(2))$$
$$\downarrow$$
$$H^1(\mathbb{Q}(\zeta_m), T_f \otimes T_g) \ni z_m^{f,g}$$

where f and g are modular forms of wt 2 and level N.

<u>Thm 4.2</u> Let $\ell \mid N$ and $\ell \mid m$

Write $\tau_\ell : Y(m\ell, m\ell N) \to Y(m, mN)$ for the degeneracy map given by $z \mapsto z/\ell$ on $\mathcal{H}$.

Then

$$(\tau_\ell \times \tau_\ell)_* (z_{\ell m,N}) \mapsto (U_\ell' \times U_\ell')(z_{m,N})$$

where U_ℓ' is given by $(\pi_2)_* (\pi_1)^*$ where

$$\begin{tikzcd} Y(M(\ell), N) \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ Y(M,N) & Y(M,N) \end{tikzcd}$$

$\pi_1 : z \mapsto z$, $\pi_2 : z \mapsto z/\ell$ and $Y(M(\ell), N)$ is the modular curve parametrizing (E, e_1, e_2, C) $e_1 \in E[M]$, $e_2 \in E[N]$, $C \subseteq E[M\ell]$.

If $\ell \mid N$, $U_\ell' f = \alpha_\ell f$

$U_\ell' g = \alpha_\ell g$, then

$$\mathrm{cor}^{m\ell}_{m}(z^{f,g}_{m\ell}) = \alpha_f \alpha_g z_m^{f,g}$$

We define $BF^{f,g}_{mp^n} = \dfrac{1}{(\alpha_f \alpha_g)^n} z^{f,g}_{mp^n}$

- If f and g are ordinary at p then we do get an E.S.
- Bertolini-Darmon-Rotger: these elts are related to L-fn of the Rankin-Selberg product of f and g: $L(f \times g, s) = \sum \frac{a_n(f) a_n(g)}{n^s}$

- Can prove one inclusion of the Iwasawa main conjecture which relates the Selmer gp defined using Greenberg's condition and Hida's p-adic L-fn.
- Can also define these elts for higher weight modular forms and even deform them in Hida families (Kings-Loeffler-Zerbes)
- Buyukboduk-Ochiai proved one inclusion of the 3-variable Iwasawa main conjecture.

- When $f = g$, we can study the Iwasawa main conjecture for $\mathrm{Sym}^2 T_f$ (Loeffler-Zerbes)
- If f and g are of good red at p, then we get 4 sets of elts

$$BF^{\lambda_f \mu_g}_{mp^n}, \quad \lambda, \mu \in \{\alpha, \beta\}.$$

When f is ord at p, but g is not ord at p, then we get bounded elts

$$BF^{\alpha_f \sharp}_{mp^n}, \quad BF^{\alpha_f \flat}_{mp^n}$$

using +/- or $\sharp/\flat$ theory of Kobayashi and Sprung.

We get an E.S. and can prove one inclusion of some signed main conjectures. (Buyukbod-L.)

- We can also deform f in a CM Hida family, we get an E.S. for the $\mathbb{Z}_p^2$-extn of an imag quad field where p splits
- Can study IMC for $\mathbb{Z}_p^2$-extn.
- More works can be done for other groups ($GSp(4)$, $GU(2,1)$, etc.) by Loeffler-Zerbes-Skinner, et al.