A variational formula for large deviations in FPP under tail estimates

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Joint work with C. Cosco and F. Schweiger
(Weizmann Institute of Science)

Motivation for the research on interface growing

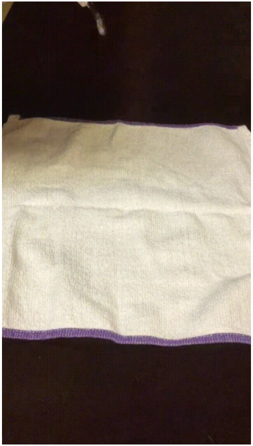

Figure: Experiment: pouring water over a towel

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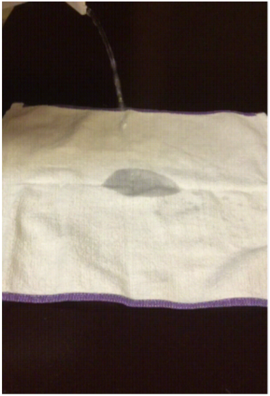


Experiment A

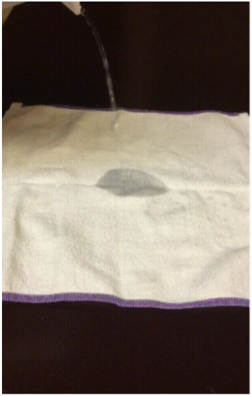
Features

Randomness

Fractal

Universality

Motivation for the research on interface growing



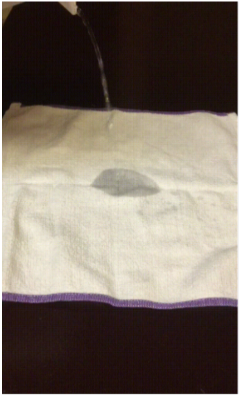
Experiment A

Features

Randomness

1st

Motivation for the research on interface growing



Experiment A

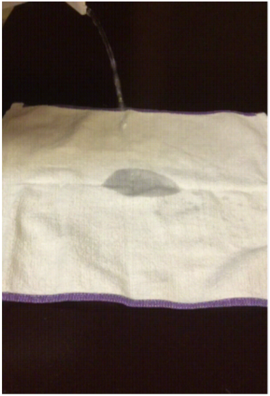
Features

Randomness

1st

2nd

Motivation for the research on interface growing



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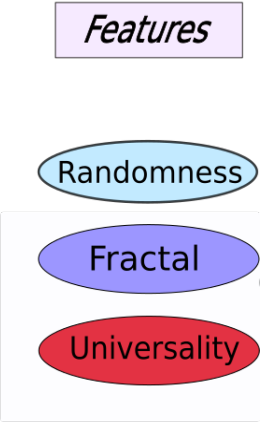


KPZ Universality Class

- In 1986, Kardar–Parisi–Zhang introduced a SPDE, called the KPZ equation, in a physics literature:

$$\partial_t h = \frac{1}{2}\nabla h + \frac{1}{2}|\nabla h|^2 + \lambda\xi.$$

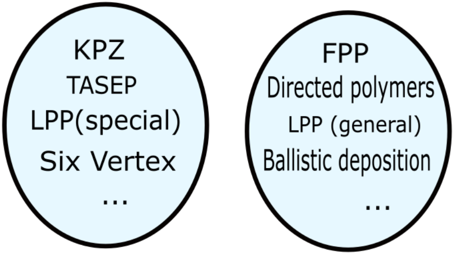
- A lot of interface growing models (e.g., spread of the infected people) and mathematical models are found to behave like a solution to the KPZ equation.

- We call the set of these kinds of models the "KPZ universality class" collectively.

KPZ Universality II

$d = 1$ (KPZ Universality Class)

Exactly Solvable | Unsolvable



$d \geq 2$

There are no solvable models so far.

Setting (FPP)

- $\mathrm{E}^d = \{\{x, y\} \mid x, y \in \mathbb{Z}^d, \ |x - y|_1 = 1|\}$.
- $\tau = \{\tau_e\}_{e \in \mathrm{E}^d}$: i.i.d. non-negative random variables.
- $\gamma : x \to y$ stands for a path from x to y.
- Given a path γ, we define $\mathrm{T}(\gamma) := \sum_{e \in \gamma} \tau_e$.

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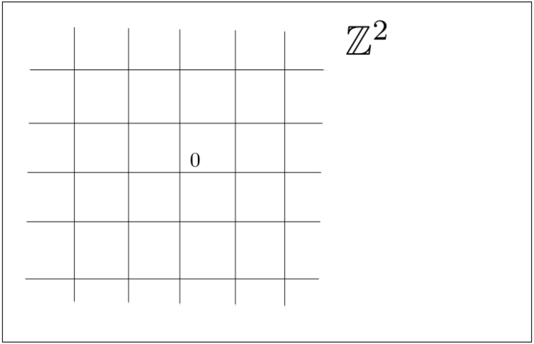

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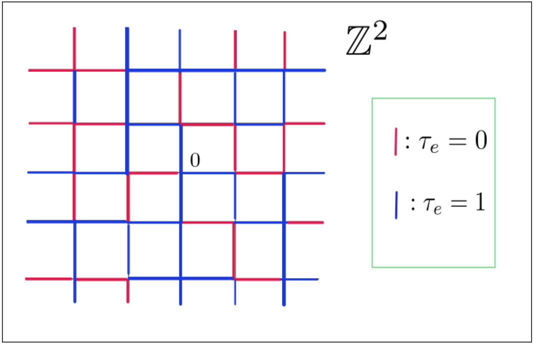

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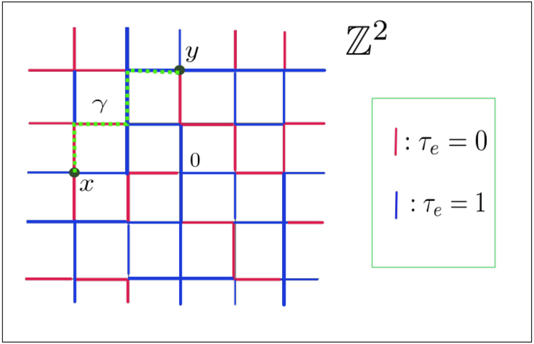

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First passage time $(x, y \in \mathbb{Z}^d)$

$$\mathrm{T}(x, y) := \inf_{\gamma : x \to y} \mathrm{T}(\gamma).$$

Wetting region $(t \geq 0)$

$$B(t) := \left\{ x \in \mathbb{Z}^d | \ \mathrm{T}(0, x) \leq t \right\}.$$

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Optimal paths Let $\mathbb{O}(x, y)$ be the set of all optimal paths:

$$\mathbb{O}(x, y) := \{\gamma : x \to y | \, \mathrm{T}(x, y) = \mathrm{T}(\gamma)\}.$$

Simulation (interface growing)

Wetting region

$$\mathrm{B}(t) := \{x \in \mathbb{Z}^d \mid \mathrm{T}(0,x) \leq t\}.$$

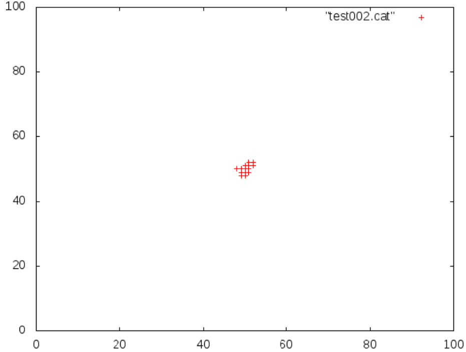

Figure: Simulation of $\mathrm{B}(t)$ when $\mathbb{P}(\tau_e = 0) = 1/4 = 1 - \mathbb{P}(\tau_e = 1)$.

Random metric space

Proposition 1 (Sub-additivity)

$\mathrm{T}(x,z) \le \mathrm{T}(x,y) + \mathrm{T}(y,z)$ *for any* $x, y, z \in \mathbb{Z}^d$

Proof.

$$\mathrm{LHS} = \inf_{\gamma : x \to z} \mathrm{T}(\gamma) \le \inf_{\gamma : x \to y \to z} \mathrm{T}(\gamma) = \mathrm{RHS},$$

where the second inf. runs over paths from x to z passing y. □

- $\mathrm{T} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$ is a pseudo-metric.
- When $\mathbb{P}(\tau_e = 0) = 0$, T is a metric a.s.

“Law of large numbers”

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, $\mathrm{T}(\boldsymbol{x}, \boldsymbol{y}) \equiv \mathrm{T}(\lfloor \boldsymbol{x} \rfloor, \lfloor \boldsymbol{y} \rfloor)$ where $\lfloor \cdot \rfloor$ is a floor function.

Proposition 2 (Kingman ’68)

Suppose $\mathbb{E}\tau_e < \infty$. *For any* $\boldsymbol{x} \in \mathbb{R}^d$,

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{T}(0, n\boldsymbol{x}) = \mu(\boldsymbol{x}) \quad \text{a.s.},$$

where $\mu(\boldsymbol{x}) := \liminf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}[\mathrm{T}(0, n\boldsymbol{x})]$ (time constant).

Proof.

Apply Kingman’s sub-additive ergodic theorem. □

What is large deviations?

For simplicity, we write $\mathrm{T}_n = \mathrm{T}(0, n\mathbf{e}_1)$ and $\mu = \mu(\mathbf{e}_1)$.

Large deviations?

Large deviations concerns the asymptotic behaviour of remote tails of sequences of probability distributions. (ref. Wiki)

By law of large numbers, for any $\epsilon > 0$,

$$\mathbb{P}\left(|\mathrm{T}_n - \mu n| < \epsilon n\right) \to 1.$$

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$$\mathbb{P}\left(|\mathrm{T}_n - \mu n| < \epsilon n\right) \to 1.$$

We consider the following two events: Given $\xi > 0$,

$$\mathcal{L}_\xi^-(n) = \{\mathrm{T}_n < (\mu - \xi)n\}, \text{ (lower tail)}$$
$$\mathcal{U}_\xi^+(n) = \{\mathrm{T}_n > (\mu + \xi)n\}. \text{ (upper tail)}$$

Kesten's work

Theorem 3 (Lower tail LDP: Kesten 1986)

Suppose $\mathbb{P}(\tau_e = 0) < \mathrm{p}_c(d)$ *and* $\mathbb{E}\tau_e < \infty$. *Then for* ϵ *small enough,*

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}(\mathcal{L}_\epsilon^-(n)) = \exists I^-(\epsilon).$$

The limit I^- *is called the rate function.*

Kesten's work

Theorem 4 (Lower tail LDP: Kesten 1986)

Suppose $\mathbb{P}(\tau_e = 0) < \mathrm{p}_c(d)$ *and* $\mathbb{E}\tau_e < \infty$. *Then for* ϵ *small enough,*

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}(\mathcal{L}_\epsilon^-(n)) = \exists I^-(\epsilon).$$

Sketch of the proof.

By FKG inequality,

$$\begin{aligned}
&\mathbb{P}(\mathrm{T}_{n+m} < (\mu - \xi)(n+m)) \\
&= \mathbb{P}(\mathrm{T}_n < (\mu - \xi)n)\mathbb{P}(\mathrm{T}_{n+m} < (\mu - \xi)(n+m) |\ \mathrm{T}_n < (\mu - \xi)n) \\
&\geq \mathbb{P}(\mathrm{T}_n < (\mu - \xi)n)\mathbb{P}(\mathrm{T}_m < (\mu - \xi)m).
\end{aligned}$$

By Fekete's subadditive lemma, the limit $I^-(\epsilon)$ exists. □

Kesten's work

Theorem 5 (Lower tail LDP: Kesten 1986)

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$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}(\mathcal{L}_\epsilon^-(n)) = \exists I^-(\epsilon) < 0.$$

Theorem 6 (Upper tail LDP: Kesten 1986)

If τ_e *is a bounded non-degenerate random variable, then for* $\epsilon > 0$ *small enough,*

$$-\infty < \varliminf_{n\to\infty} n^{-d} \log \mathbb{P}(\mathcal{U}_\epsilon^+(n)) \leq \varlimsup_{n\to\infty} n^{-d} \log \mathbb{P}(\mathcal{U}_\epsilon^+(n)) < 0.$$

Related work II

Existence of rate function for bounded τ_e

Theorem 7 (Basu, Ganguly, and Sly, 2021, CPAM.)

If τ_e is bouonded and has a continuous density, then

$$\exists I^+(\epsilon) = \lim_{n\to\infty} n^{-d} \log \mathbb{P}(\mathcal{U}_\epsilon^+(n)).$$

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Q. What if τ_e is unbounded?

Related work III

Upper tail LDP for unbounded τ_e

Suppose there exist $c_1, c_2, \alpha_1, \alpha_2 > 0$ and $r \in (0, \infty)$:

$$c_1 \exp\left(-\alpha_1 t^r\right) \leq \mathbb{P}(\tau_e > t) \leq c_2 \exp\left(-\alpha_2 t^r\right),$$

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Theorem 8 (Cranston, Gauthier, and Mountford, 2009, AAP.)

When $r > d$, then

$$-\infty < \varliminf_{n\to\infty} n^{-d} \log \mathbb{P}(\mathcal{U}^+_\epsilon(n)) \le \varlimsup_{n\to\infty} n^{-d} \log \mathbb{P}(\mathcal{U}^+_\epsilon(n)) < 0.$$

When $r = d = 2$, then

$$-\infty < \varliminf_{n\to\infty} \frac{\log n}{n^2} \log \mathbb{P}(\mathcal{U}^+_\epsilon(n)) \le \varlimsup_{n\to\infty} \frac{\log n}{n^2} \log \mathbb{P}(\mathcal{U}^+_\epsilon(n)) < 0.$$

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Theorem 9 (Cranston, Gauthier, and Mountford, 2009, AAP.)

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Theorem 10 (N 2016, unpublished)

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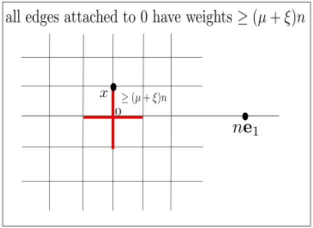
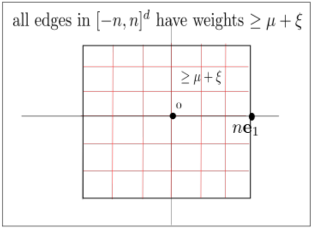
Why are the scalings diffrent?

Let's consider two scenarios:

Senario A



Senario B



When $r > d$,

$$\mathbb{P}(\text{Scenario A}) = e^{-O(n^r)} \ll \mathbb{P}(\text{Scenario B}) = e^{-O(n^d)}.$$

When $r < d$,

$$\mathbb{P}(\text{Scenario A}) = e^{-O(n^r)} \gg \mathbb{P}(\text{Scenario B}) = e^{-O(n^d)}.$$

Main result I

Hereafter, we suppose there exist $c_1, c_2, \alpha > 0$ and $r \in (0, \infty)$:

$$c_1 \exp\left(-\alpha t^r\right) \leq \mathbb{P}(\tau_e > t) \leq c_2 \exp\left(-\alpha t^r\right),$$

Theorem 11 (Cosco-N +21)

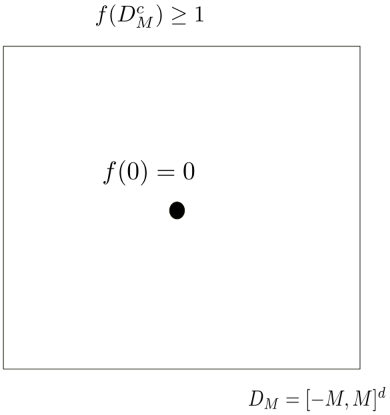
Suppose $r \leq 1$. Then for all $\xi > 0$,

$$\lim_{n\to\infty} \frac{1}{n^r} \log \mathbb{P}(\mathcal{U}_\xi^+(n)) = -2d\alpha\xi^r.$$

Notation (p-capacity)

Let us denote $D_M = [-M, M]^d$. We define the set of functions

$$\mathcal{C}(M) = \{f : \mathbb{Z}^d \to \mathbb{R} : \ \forall x \in D_M^c,\ f(x) \geq 1,\ f(0) = 0\}.$$



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We define the discrete p-Capacity:

$$\kappa_{d,r}(M) = \inf_{f \in \mathcal{C}(M)} \sum_{\langle x,y \rangle \in \mathrm{E}^d} |f(x) - f(y)|^r.$$

Remark

When $r = 2$,

$$\kappa_{d,2}(M) = \mathbb{P}(\tau_0 < \tau_M),$$

where τ_0 and τ_M are the hitting times of the simple random walk (starting at 0) at 0 and in D_M^c, respectively.

Main result II

We begin with the case $1 < r < d$ and define

$$\kappa_{d,r} = \lim_{M\to\infty} \kappa_{d,r}(M),$$

(the limit exists since $M \to \kappa_{d,r}(M)$ is non-increasing).

Theorem 12 (Cosco-N +21)

Assume that $1 < r < d$*. For all* $\xi > 0$,

$$\lim_{n\to\infty} \frac{1}{n^r} \log \mathbb{P}(\mathcal{U}_\xi^+(n)) = -\alpha 2^{1-r} \xi^r \kappa_{d,r} < 0.$$

Main result III

Theorem 13 (Cosco-N +21)

Suppose that $r = d$. *For all* $\xi > 0$,

$$\lim_{n\to\infty} \frac{1}{n^d\, \kappa_{d,d}(n)} \log \mathbb{P}(\mathcal{U}_\xi^+(n)) = -\alpha 2^{1-d}\, \xi^d.$$

Theorem 14 (Cosco-Schweiger-N 21+)

For $d \geq 2$,

$$\lim_{n\to\infty} (\log n)^{d-1} \kappa_{d,d}(n) = \mathrm{Vol_{d-1}}\left(\left\{x \in \mathbb{R}^n : \; \|x\|_{\frac{d}{d-1}} = 1\right\}\right).$$

Sketch of the proof

For simplicity, we only consider the case that the weight follows an exponential distribution of mean 1 ($\mathbb{P}(\tau_e \geq t) = e^{-t}$).

The goal is to prove

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\mathrm{T}_n > n(\mu + \xi)) = -2d\xi.$$

Outline of the lower bound

Let $\mathbb{E}_0 = \{e \in \mathrm{E}^d : \ 0 \in e\}$.

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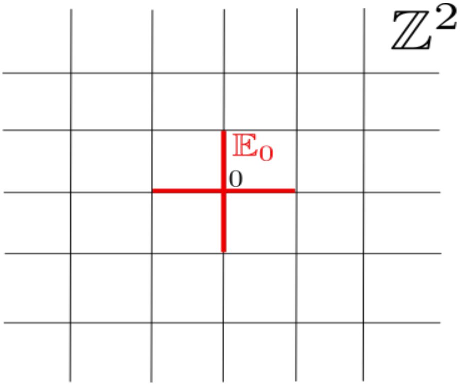

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$$\forall e \in \mathbb{E}_0,\ \tau_e \geq \xi n \Rightarrow \mathrm{T}_n > (\mu + \xi) n.$$

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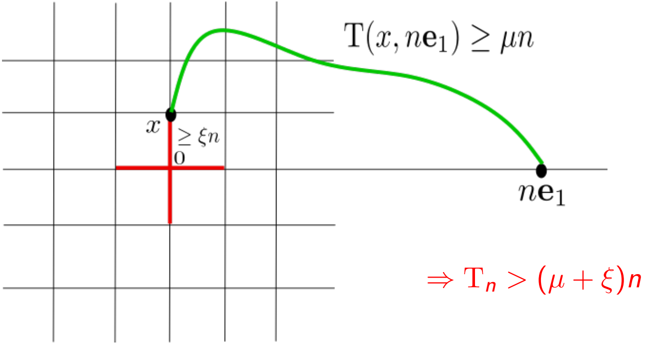

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Hence,

$$\begin{aligned}\mathbb{P}(\mathrm{T}_n > (\mu + \xi)n) &\geq (1 - o(1))\mathbb{P}(\forall e \in \mathbb{E}_0, \ \tau_e \geq \xi n) \\ &= (1 - o(1))e^{-2d\xi n}.\end{aligned}$$

This implies

$$\liminf_{n \to \infty} n^{-1} \log \mathbb{P}(\mathrm{T}_n > (\mu + \xi)n) \geq -2d\xi.$$

Outline of the upper bound

Lemma 1 (Large deviations on a slab)

For any $\epsilon > 0$*, there exist* $K \in \mathbb{N}$ *and* $c > 0$ *such that*

$$\mathbb{P}\left(\mathrm{T}_{\mathbb{R}\times[-K,K]^{d-1}}(0, n\mathbf{e}_1) \geq (\mu+\epsilon)n\right) \leq \exp(-cn),$$

where

$$\mathrm{T}_A(x,y) = \inf_{\substack{\gamma: x\to y \\ \gamma\subset A}} \mathrm{T}(\gamma).$$

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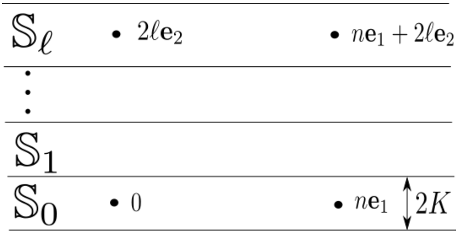
Let $L \in \mathbb{N}$. We consider L slabs ($\mathbb{S}_\ell = 2\ell\mathbf{e}_2 + \mathbb{R} \times [-K, K]$):



Outline of the upper bound II

Let $\epsilon > 0$ arbitrary. Then,

$$\mathbb{P}(\forall \ell \in \{0, \cdots, L-1\}, \mathrm{T}_{\mathbb{S}_\ell}(2\ell \mathbf{e}_2, n\mathbf{e}_1 + 2\ell \mathbf{e}_2) \geq (\mu + \epsilon)n)$$
$$\overset{\text{(indep.)}}{=} \mathbb{P}\left(\mathrm{T}_{\mathbb{R}\times[-K,K]^{d-1}}(0, n\mathbf{e}_1) \geq (\mu+\epsilon)n\right)^L$$
$$\overset{\text{(lemma)}}{\leq} e^{-cLn},$$

which is negligible for L large enough.
Hence, we can suppose that there exists a slab $\mathbb{S}_\ell$ such that

$$\mathrm{T}_{\mathbb{S}_\ell}(2\ell \mathbf{e}_2, n\mathbf{e}_1 + 2\ell \mathbf{e}_2) < (\mu + \epsilon)n.$$

Outline of the upper bound III

Suppose $\mathrm{T}_{\mathbb{S}_\ell}(2\ell\mathbf{e}_2, n\mathbf{e}_1 + 2\ell\mathbf{e}_2) < (\mu+\epsilon)n$. If $\mathrm{T}_n > (\mu+\xi)n$, then

$$
\begin{aligned}
&(\mu+\xi)n < \mathrm{T}_n \\
&\leq \mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}_{\mathbb{S}_\ell}(2\ell\mathbf{e}_2, n\mathbf{e}_1 + 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) \\
&\leq (\mu+\epsilon)n + \mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1).
\end{aligned}
$$

Therefore, $\mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) > (\xi-\epsilon)n$. Hence,

$$
\mathbb{P}(\mathrm{T}_n > (\mu+\xi)n) \leq \mathbb{P}\left(\mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) > (\xi-\epsilon)n\right)
$$

Outline of the upper bound III

Suppose $\mathrm{T}_{\mathbb{S}_\ell}(2\ell\mathbf{e}_2, n\mathbf{e}_1 + 2\ell\mathbf{e}_2) < (\mu+\epsilon)n$. If $\mathrm{T}_n > (\mu+\xi)n$, then

$$\begin{aligned}
&(\mu+\xi)n < \mathrm{T}_n \\
&\le \mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}_{\mathbb{S}_\ell}(2\ell\mathbf{e}_2, n\mathbf{e}_1 + 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) \\
&\le (\mu+\epsilon)n + \mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1).
\end{aligned}$$

Therefore, $\mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) > (\xi-\epsilon)n$. Hence,

$$\mathbb{P}(\mathrm{T}_n > (\mu+\xi)n) \le \mathbb{P}\left(\mathrm{T}(0, 2\ell\mathbf{e}_2) + \mathrm{T}(n\mathbf{e}_1 + 2\ell\mathbf{e}_2, n\mathbf{e}_1) > (\xi-\epsilon)n\right)$$

Let's take $2d$ (disjoint) paths $(\gamma_i)_{i=1}^{2d}$ from 0 to $2\ell\mathbf{e}_2$ and $(\gamma'_i)_{i=1}^{2d}$ from $n\mathbf{e}_1$ to $n\mathbf{e}_1 + 2\ell\mathbf{e}_2$. Then, the last probability is

$$\begin{aligned}
&\le \mathbb{P}(\forall i \in \{1\cdots, 2d\},\ \mathrm{T}(\gamma_i) + \mathrm{T}(\gamma'_i) > (\xi-\epsilon)n) \\
&= \prod_i \mathbb{P}(\mathrm{T}(\gamma_i) + \mathrm{T}(\gamma'_i) > (\xi-\epsilon)n) \le e^{-2d(\xi-\epsilon-o(1))n}.
\end{aligned}$$

A remark on $r > 1$

We are able to rewrite the p-Capacity as

$$\kappa_{d,r}(M) = \inf_{(t_e)\in\mathbb{R}^{\mathrm{E}^d}} \left\{ \sum_{e\in\mathrm{E}^d} |t_e|^r \mid \forall\gamma : 0 \to D_M^c, \sum_{e\in\gamma} t_e \geq 1 \right\}.$$

The argument roughly goes as

$$\begin{aligned}
&\mathbb{P}(\mathrm{T}_n > (\mu+\xi)n) \approx \mathbb{P}(\mathrm{T}(0, D_M^c) > \xi n) \\
&\approx \exp\left(-\alpha \inf_{(t_e)\in\mathbb{R}^{\mathrm{E}^d}} \left\{ \sum_{e\in\mathrm{E}^d} |t_e|^r \mid \forall\gamma : 0 \to D_M^c, \sum_{e\in\gamma} t_e \geq \xi n \right\} \right) \\
&= \exp\left(-\alpha(\xi n)^r \inf_{(t_e)\in\mathbb{R}^{\mathrm{E}^d}} \left\{ \sum_{e\in\mathrm{E}^d} |t_e|^r \mid \forall\gamma : 0 \to D_M^c, \sum_{e\in\gamma} t_e \geq 1 \right\} \right) \\
&= \exp\left(-\alpha\kappa_{d,r}(M)\xi^r n^r \right).
\end{aligned}$$

Related models

In the models below, we may confirm a similar phenomenon that the rate function is given by a power function.

- LPP, Directed polymers with Weibull distributions.
- Frog models (Ongoing work with CV. Hao and N. Kubota)

I believe this phonemonon holds for a wide range of random environment models with "heavy tail" distributions.

Some problems

Suppose τ_e obayes exponential distribution.

- Moderate deviations, i.e.

$$\mathbb{P}(\mathrm{T}_n > \mu n + n^{\alpha}),\ \alpha \in (0,1).$$

- Upper tail large deviations for Box-to-Box First-passage time:

$$\mathbb{P}(\mathrm{T}(D_{n^{\alpha}}, n\mathbf{e}_1 + D_{n^{\alpha}}) > (\mu + \xi)n),\ \alpha \in (0,1).$$

My curious problem

Theorem 15 (Cosco-N 21+)

Let

$$I^+(\xi) = \limsup_{n\to\infty} n^{-d} \log \mathbb{P}(\mathrm{T}_n > (\mu + \xi)n).$$

If τ_e is unbounded and satisfies $\mathbb{P}(\tau_e \geq t) \leq e^{-bt^r}$ with $r > d$, then

$$I^+(\xi) = O(\xi^d) \text{ as } \xi \to 0.$$

Question

$$\limsup_{\xi\to 0+} \xi^d I^+(\xi) \text{ is 0 or positive?}$$