Integration by Parts and the KPZ Two-Point Function.

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Kardar-Parisi-Zhang (KPZ) Equation

Let $h = (h_t(x)\,,\, x \in \mathbb{R})_{t\geq 0}$ denote the solution of

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi \\ h_0(x) = \sigma W(x)\,, \end{cases}$$

ξ space-time white noise, $W \equiv$ standard two-sided BM.

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Two-Point Function

Find $C_t(x)$ such that

$$\mathbb{E}\left[\int_{\mathbb{R}} \phi_1 \partial_x h_0 \int_{\mathbb{R}} \phi_2 \partial_x h_t\right] = \int\int_{\mathbb{R}^2} \phi_1(x)\phi_2(y) C_t(y-x)dxdy\,.$$

Integration by Parts (IP) from Malliavin Calculus

Let $W(\phi) = \int_{\mathbb{R}} \phi dW$, and $X = X(W)$ be a random variable. Then

$$\mathbb{E}\left[W(\phi)X\right] = \mathbb{E}\left[\langle DX, \phi\rangle\right],$$

where $DX = (DX(u)\,,\; u \in \mathbb{R})$ is the Malliavin derivative of X with respect to W.

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Malliavin Derivative

If $X = f(W(\phi_1), \ldots, W(\phi_d))$ then

$$DX(u) = \sum_{j=1}^{d} \partial_{x_j} f(W(\phi_1), \ldots, W(\phi_d))\phi_j(u) \in \mathbb{L}^2 (\Omega \times \mathbb{R}) .$$

In particular $D(W(\phi))(u) = \phi(u)$.

Thus, for $X = \int_{\mathbb{R}} \phi_2 \partial_x h_t$,

$$\mathbb{E}\left[\int_{\mathbb{R}} \phi_1 \partial_x h_0 \int_{\mathbb{R}} \phi_2 \partial_x h_t\right] = \sigma \mathbb{E}\left[W(\phi_1) X\right] = \sigma \mathbb{E}\left[\langle DX, \phi_1 \rangle\right].$$

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Since we can write $X = -\int_{\mathbb{R}} \phi_2'(x) h_t(x) dx$, then

$$DX(u) = -\int_{\mathbb{R}} \phi_2'(x) D(h_t(x))(u) dx \, .$$

Thus, for $X = \int_{\mathbb{R}} \phi_2 \partial_x h_t$,

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How to compute $D(h_t(x))$?

Look at the Cole-Hopf solution of the KPZ.

Stochastic Heat Equation

Let $Z_t(x, y)$ be the fundamental solution of (y fixed)

$$\begin{cases} \partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi \\ Z_0(x, y) = \delta(x - y)\,. \end{cases}$$

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Cole-Hopf Solution

Then

$$h_t(x) = \log\left(\int_{\mathbb{R}} Z_t(x,y)e^{\sigma W(y)}dy\right)\,,$$

is the free energy associated to the end-point density

$$p_{x,t}(y) = \frac{Z_t(x,y)e^{\sigma W(y)}}{\int_{\mathbb{R}} Z_t(x,z)e^{\sigma W(z)}dz}\,.$$

Thus,

$$D(h_t(x))(u) = \sigma \int_{\mathbb{R}} \zeta_y(u) p_{x,t}(y) dy = \sigma E_{x,t}\left[\zeta_Y(u)\right] ,$$

where

$$u \in \mathbb{R} \mapsto \zeta_y(u) = \begin{cases} \mathbb{1}_{(0,y]}(u) & \text{if } y > 0\,, \\ 0 & \text{if } y = 0\,, \\ -\mathbb{1}_{(y,0]}(u) & \text{if } y < 0\,. \end{cases}$$

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Therefore,

$$DX(u) = -\sigma \int_{\mathbb{R}} \phi_2'(x) E_{x,t}\left[\zeta_Y(u)\right] dx \,.$$

Denote $\mathbb{E}_{x,t}[\cdot] = \mathbb{E}[E_{x,t}[\cdot]]$, $p_t \equiv p_{0,t}$ and $\mathbb{E}_t \equiv \mathbb{E}_{0,t}$. Then

$$\mathbb{E}[\langle DX, \phi_1 \rangle] = -\sigma \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_2'(x) \mathbb{E}_t [\zeta_{x+Y}(u)] \, dx \right) \phi_1(u) du$$

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\mathbb{E}[\langle DX, \phi_1 \rangle] &= -\sigma \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_2'(x) \mathbb{E}_t [\zeta_{x+Y}(u)]\, dx \right) \phi_1(u) du \\
&= -\sigma \mathbb{E}_t \left[\int_{\mathbb{R}} \phi_2'(x) \left(\int_{\mathbb{R}} \phi_1(u) \zeta_{x+Y}(u) du \right) dx \right]
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&= -\sigma \mathbb{E}_t \left[\int_{\mathbb{R}} \phi_2'(x) \left(\int_{\mathbb{R}} \phi_1(u) \zeta_{x+Y}(u) du \right) dx \right] \\
&= \sigma \mathbb{E}_t \left[\int_{\mathbb{R}} \phi_2(x) \phi_1(x+Y) dx \right].
\end{aligned}$$

Hence,

$$\mathbb{E}\left[\int_{\mathbb{R}} \phi_1 \partial_x h_0 \int_{\mathbb{R}} \phi_2 \partial_x h_t\right] = \sigma^2 \mathbb{E}_t\left[\phi_2 \star \phi_1(Y)\right] ,$$

where $\phi_2 \star \phi_1(y) = \int_{\mathbb{R}} \phi_2(x)\phi_1(x+y)dz$, and then

$$C_t(x) = \sigma^2 \mathbb{E}\left[p_t(x)\right] .$$

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$$C_t(x) = \sigma^2 \mathbb{E}\left[p_t(x)\right].$$

How to compute $\mathbb{E}\left[p_t(x)\right]$?

Denote $g_t(x) = \mathrm{Var}\left[h_t(x)\right]$. We use that

$$g_t(x) = \mathrm{Var}\left[h_t(0)\right] - \sigma^2|x| + 2\mathbb{E}\left[h_0(x)h_t(x)\right],$$

and apply IP to compute $G_t(x) = \mathbb{E}\left[h_0(x)h_t(x)\right]$.

IP implies that

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Since

$$\int_{\mathbb{R}} \zeta_Y(u)\zeta_x(u)du = \begin{cases} \min\{x, Y_+\}\,, \; x \geq 0 \\ -\max\{x, Y_-\}\,, \; x \leq 0\,, \end{cases}$$

we have that

$$G_t(x) = \sigma^2 \begin{cases} \mathbb{E}_t\left[\min\{x, (x+Y)_+\}\right] & \text{if } x \geq 0\,, \\ \mathbb{E}_t\left[-\max\{x, (x+Y)_-\}\right] & \text{if } x \leq 0\,. \end{cases}$$

For $x \geq 0$

$$\begin{aligned}\mathbb{E}_t\left[\min\{x, (x+Y)_+\}\right] &= \mathbb{E}_t\left[(x+Y)\mathbb{1}_{\{Y\in(-x,0]\}} + x\mathbb{1}_{\{Y>0]\}}\right] \\ &= \mathbb{E}_t\left[Y\mathbb{1}_{\{Y\in(-x,0]\}}\right] + x\mathbb{P}_t\left[Y > -x\right]\end{aligned}$$

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$$\implies \quad G_t'(x) = \sigma^2\mathbb{P}_t\left[Y > -x\right] = \sigma^2\mathbb{P}_t\left[Y \leq x\right]$$

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$$\implies \quad g_t'(x) = -\sigma^2 + 2\sigma^2\mathbb{P}_t\left[Y\leq x\right] = \sigma^2\left(2\mathbb{P}_t\left[Y\leq x\right]-1\right) .$$

For $x \leq 0$

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\mathbb{E}_t\left[-\max\{x,(x+Y)_-\}\right] &= \mathbb{E}_t\left[-(x+Y)\mathbb{1}_{\{Y\in(0,-x]\}} - x\mathbb{1}_{\{Y\leq 0]\}}\right] \\
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$$\Longrightarrow \quad \mathbb{E}\left[p_t(x)\right] = \frac{g_t''(x)}{2\sigma^2} \quad \text{and} \quad C_t(x) = \sigma^2\mathbb{E}\left[p_t(x)\right] = \frac{g_t''(x)}{2} .$$

Previous Work

For the stationary regime $\sigma = 1$.

- Using stationary weakly ASEP approximations (due to Bertini-Giacomin) Balázs-Quastel-Seppäläinen showed that $g_t(0) \sim t^{2/3}$, $C_t(x) = \frac{g_t''(x)}{2}$, and that $C_t(x)$ can be written in terms of a symmetric probability measure on $\mathbb{R}$.

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- Imamura-Sasamoto derived exact formula for $g_t(x)$.
- Maes-Thierry indicated the connection between g_t'' and the polymer end-point distribution (nonrigorous argument).
- Gu-Komorowski proved that $g_t(0) = \mathbb{E}_{0,t}[Y]$ to show that $g_t(0) \sim t^{2/3}$ (Malliavin calculus).

Previous Work

KPZ universality.

- 1:2:3 scaling $\epsilon h_{\epsilon^{-3}t}(x\epsilon^{-2}) + C_\epsilon t \to \mathfrak{h}_t(x)$ (KPZ Fixed Point) .

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- References: Prähofer-Spohn, Ferrari-Spohn, Baik-Ferrari-Péché.

KPZ Fixed Point $(\mathfrak{h}_t(x)\,,\, x \in \mathbb{R})_{t\geq 0}$ and $\mathfrak{h}_0 = \sigma W$

Let $g(x) = \mathrm{Var}\left[\mathfrak{h}_1(x)\right]$ and

$$Z = \arg\max_{z\in\mathbb{R}} \left\{\sigma W(z) + \mathcal{A}_2(z) - z^2\right\} \,.$$

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Then

- $\mathbb{E}\,\left[\int_{\mathbb{R}} \phi_1 \partial_x \mathfrak{h}_0 \int_{\mathbb{R}} \phi_2 \partial_x \mathfrak{h}_t\right] = \sigma^2 \mathbb{E}\,\left[\phi_2 \star \phi_1(t^{2/3} Z)\right]$.
- $g'(x) = \sigma^2\,(2\mathbb{P}\,[Z \leq x] - 1)$.
- $C_t(x) = \frac{g''(xt^{-2/3})}{2t^{2/3}}$.

The Flat Case $\sigma = 0$

In both models (KPZ - Eq & FP) one obtains

$$g_t'(x,\sigma) = \sigma^2 \left(2\mathbb{F}_t(x,\sigma) - 1\right) \ .$$

For $\sigma = 0$ one gets $\mathbb{F}_t(x,0)$, the corresponding end-point distribution for the flat initial data. Therefore

$$\mathbb{F}_t(x,0) = \frac{1}{2}\left(\lim_{\sigma\downarrow 0} \frac{g_t'(x,\sigma)}{\sigma^2} + 1\right) \ .$$

Wasserstein Distance from Independence

Let (X_1, X_2) be a random vector with probability law $\theta = \mathbb{P}_{X_1,X_2}$, and let $\eta = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$ be the product measure induced by the marginals of θ.

$$\mathrm{Wass}(\theta, \eta) = \sup_{\ell \in \mathrm{Lip}_1} \left\{ \left| \int_{\mathbb{R}^2} \ell \, d\theta - \int_{\mathbb{R}^2} \ell \, d\eta \right| \right\} .$$

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Malliavin-Stein method for Asymptotic Independence

Denote $\theta = \mathbb{P}_{\sigma W(\phi_1), X}$ and $\eta = \mathbb{P}_{\sigma W(\phi_1)} \otimes \mathbb{P}_X$. Then

$$\mathrm{Wass}(\theta, \eta) \leq \frac{1}{\|\phi_1\|_2} \sqrt{\frac{\pi}{2}} \Big| \mathbb{E} \left[\langle DX \, , \, \phi_1 \rangle \right] \Big| .$$

Asymptotic Independence for KPZ

$$(Eq)\ \mathrm{Wass}(\theta_t,\eta_t) \leq \frac{1}{\sigma\|\phi_1\|_2}\sqrt{\frac{\pi}{2}}\Big|\int_{\mathbb{R}} \phi_2 \star \phi_1(x)\frac{g_t''(x)}{2}dx\Big|.$$

$$(FP)\ \mathrm{Wass}(\theta_t,\eta_t) \leq \frac{1}{\sigma\|\phi_1\|_2}\sqrt{\frac{\pi}{2}}\Big|\int_{\mathbb{R}} \phi_2 \star \phi_1(x)\frac{g''(xt^{-2/3})}{2t^{2/3}}dx\Big|.$$

Stein Method for Asymp. Independence

For $f : \mathbb{R}^2 \to \mathbb{R}$ (nice) and $X_1 \sim N(0, \sigma_1^2)$ let

$$\mathcal{N} f(x_1, x_2) = \sigma_1^2 \partial_{x_1} f(x_1, x_2) - x_1 f(x_1, x_2) \,.$$

There exists a unique bounded solution $f_\ell : \mathbb{R}^2 \to \mathbb{R}$ of the equation

$$\mathcal{N} f(x_1, x_2) = \ell(x_1, x_2) - \mathbb{E}\left[\ell(X_1, x_2)\right] .$$

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Furthermore,

- $\|f_\ell\|_\infty \leq \|\partial_{x_1} \ell\|_\infty\,;$
- $\|\partial_{x_1} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{2}{\pi}}\|\partial_{x_1} \ell\|_\infty\,;$
- $\|\partial_{x_2} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{\pi}{2}}\|\partial_{x_2} \ell\|_\infty\,.$

Malliavin-Stein Method for Asymp. Independence

We want to bound

$$\int_{\mathbb{R}^2} \ell \, d\theta - \int_{\mathbb{R}^2} \ell \, d\eta = \mathbb{E}\left[\mathcal{N} f_\ell(X_1, X_2)\right]$$

for $X_1 = \sigma W(\phi_1)$ with $\sigma_1^2 = \sigma^2 \|\phi_1\|_2^2$ and $X_2 = X$.

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combined with IP:

$$\begin{aligned}\mathbb{E}\left[X_1 f(X_1, X_2)\right] &= \sigma \mathbb{E}\left[\langle Df(X_1, X_2), \phi_1 \rangle\right] \\ &= \sigma_1^2 \mathbb{E}\left[\partial_{x_1} f(X_1, X_2)\right] + \sigma \mathbb{E}\left[\partial_{x_2} f(X_1, X_2) \langle DX_2, \phi_1 \rangle\right].\end{aligned}$$

Malliavin-Stein Method for Asymp. Independence

Since $\mathcal{N}f(x_1, x_2) = \sigma_1^2 \partial_{x_1} f(x_1, x_2) - x_1 f(x_1, x_2)$, we have

$$\mathbb{E}\left[\mathcal{N}f(X_1, X_2)\right] = -\sigma \mathbb{E}\left[\partial_{x_2} f(X_1, X_2)\langle DX_2, \phi_1\rangle\right].$$

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Recall the upper bound $\|\partial_{x_2} f_\ell\|_\infty \leq \frac{1}{\sigma_1}\sqrt{\frac{\pi}{2}}$, to conclude that

$$\mathrm{Wass}(\theta, \eta) \leq \sup_\ell \left| \mathbb{E}\left[\mathcal{N}f_\ell(X_1, X_2)\right] \right| \leq \frac{1}{\|\phi_1\|_2}\sqrt{\frac{\pi}{2}} \left| \mathbb{E}\left[\langle DX_2, \phi_1 \rangle\right] \right|.$$

References for this work

- (P) Integration by parts and the KPZ two-point function. To appear in the AOP.
- (Lopez-P) On the two-point function of the one-dimensional KPZ equation.