Weak solutions

L^1 data

Existence

Uniqueness

Renormalized solutions for elliptic equations with L^1 data

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Quasilinear elliptic equations

Let us consider the quasilinear elliptic equation in divergence form

$$(1) \qquad \begin{cases} \lambda u - \operatorname{div}\big(A(x,u)\nabla u\big) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

bounded

where Ω is an open subset of $\mathbb{R}^N$ and

- $\lambda \geq 0$
- $A(x,s) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^{N^2}$ is a Carathéodory function (measurable in x, continuous in s) such that

$$(2) \qquad A(x,s)\xi \cdot \xi \geq \alpha|\xi|^2,$$

$$(3) \qquad \exists M > 0, \quad |A(x,s)| \leq M, \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega$$

- $f \in H^{-1}(\Omega)$ (the dual space of $H_0^1(\Omega)$)

The notion of weak solution is a convenient framework

$$(4)\qquad \begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that: } \forall v \in H_0^1(\Omega) \\ \lambda \displaystyle\int_\Omega uv + \int_\Omega A(x,u)\nabla u \cdot \nabla v = < f, v >_{H^{-1},H_0^1} \end{cases}$$

Under the previous hypotheses, such a weak solution exists. It is sufficient to combine Lax-Milgram Theorem and Leray-Schauder fixed point Theorem.

However if

- $f \in L^1(\Omega)$ (and $f \notin H^{-1}(\Omega)$) : the term $\int_\Omega fv$ may not exist
- $A(x,s)$ is not uniformly bounded with respect to s : we cannot expect to have $A(x,u)\nabla u \cdot \nabla v \in L^1(\Omega)$

$\Rightarrow$ solving (1) in the sense of (4) is not possible (in general). We need an appropriate extension of the notion of weak solution.

- Mathematical question : can we have a convenient framework for elliptic (or parabolic) equations with L^1 data ?
- Some models in fluid-mechanic (Bousinessq system type, $k-\varepsilon$ model), in solid mechanic, thermistor problem, etc give PDE with L^1 expected term. For example, the Kelvin-Voigt thermo-visco-elasticity type model couples the movement equation and the energy balance equation

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left[B_1\mathcal{E}\left(\frac{\partial u}{\partial t}\right) + B_2\mathcal{E}(u)\right] + Df(\theta) = g$$

$$\frac{\partial b(\theta)}{\partial t} - \operatorname{div}(AD\theta) = \underbrace{B_1\mathcal{E}\left(\frac{\partial u}{\partial t}\right)\cdot\mathcal{E}\left(\frac{\partial u}{\partial t}\right)} - f(\theta)\operatorname{tr}(\mathcal{E}\left(\frac{\partial u}{\partial t}\right))$$

$B_1\mathcal{E}\left(\frac{\partial u}{\partial t}\right)\cdot\mathcal{E}\left(\frac{\partial u}{\partial t}\right)$ (the mechanical dissipation) is expected to belong to L^1.

- Mixing homogenization and L^1 data
- From a mathematical point of view or for some models, considering a matrix $A(x,s)$ which is not bounded with respect to s is interesting (with a control of the growth or a possible blow-up).

Solutions in the sense of distributions

Boccardo-Gallouët (1989, 1992) – in the particular quasilinear case

Assume that the matrix A is bounded. For any $f \in L^1$ (or measure) there exists $u \in W_0^{1,q}(\Omega)$ for any $1 < q < \frac{N}{N-1}$ solution to (1) in the sense of distribution:

$$\forall \varphi \in C_0^\infty(\Omega) \qquad \lambda \int_\Omega u\varphi + \int_\Omega A(x,u)\nabla u \cdot \nabla \varphi = \int_\Omega f\varphi.$$

Remarks

- u is not an admissible test function
- extension to nonlinear operator with p growth with $2 - 1/N < p$.
- can be generalized to parabolic equations with L^1 data
- f can be replaced by a Radon measure with bounded variation

Proof

Consider an approximate problem with f_ε in place of f (f_ε regular which converges to f in L^1): let $u_\varepsilon \in H_0^1(\Omega)$ a weak solution. Then we perform

- Boccardo-Gallouët estimates
- subsequence extraction
- passing to the limit

Boccardo-Gallouët estimates

$$\begin{cases} \lambda u_\varepsilon - \operatorname{div}(A(x,u_\varepsilon)\nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \partial\Omega \end{cases}$$

Following Boccardo-Gallouët [1992], we consider the function φ_m, $m > 0$

$$\varphi_m(r) = \int_0^r \frac{1}{(1+|s|)^{1+m}} ds.$$

Since $\varphi_m(u_\varepsilon) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\|\varphi_m(u_\varepsilon)\|_{L^\infty(\Omega)} \leq \frac{1}{m}$ (the bound is independent of ε), using it as test function in the approximate problem

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(1+|u_\varepsilon|)^{1+m}} \leq \frac{C}{m} \quad \text{independently of } \varepsilon.$$

For $0 < m < 1$, with $\psi_m(u_\varepsilon) = \int_0^{u_\varepsilon} (1+|s|)^{(1-m)/2} ds$ it can be rewritten

$$\int_\Omega |\nabla \psi_m(u_\varepsilon)|^2 \leq \frac{C}{m}.$$

If $m>0$ small $\frac{1-m}{2} > 0$

Poincaré

$$\int_\Omega |\psi_m(u_\varepsilon)|^2 \leq \frac{C'}{m}$$

Since $\psi_m(u_\varepsilon) = \int_0^{u_\varepsilon} (1+|s|)^{(1-m)/2} ds \in H_0^1(\Omega)$ we obtain (Sobolev inequalities) $(N > 2)$

$$\forall 0 < m < 1, \quad \psi_m(u_\varepsilon) \text{ bounded in } L^{2N/(N-2)}(\Omega)$$

that is

$$\forall 0 < m < 1 \quad u_\varepsilon \text{ bounded in } L^q(\Omega) \quad q \leq \frac{(1-m)N}{N-2}$$

which gives the first estimate

$$u_\varepsilon \text{ bounded in } L^q(\Omega) \quad q < \frac{N}{N-2}. \tag{5}$$

As far as the gradient is concerned, using the Hölder inequality

$$\int_\Omega |\nabla u_\varepsilon|^q \le \int_\Omega \left(\frac{|\nabla u_\varepsilon|^2}{(1+|u_\varepsilon|)^{1+m}} \right)^{q/2} \left(\int_\Omega (1+|u_\varepsilon|)^{q(1+m)/(2-q)} \right)^{(2-q)/2}.$$

We obtain a bound for ∇u_ε in $(L^q(\Omega))^N$ if and only if

$$\frac{q(1+m)}{2-q} < \frac{N}{N-2},$$

N>2

that is (since $0 < m < 1$ can be chosen small enough)

$$q < \frac{N}{N-1}.$$

It follows that, for any $1 \le q < N/(N-1)$

$$u_\varepsilon \text{ is bounded in } W_0^{1,q}(\Omega).$$

By Rellich-Kondrasov Theorem, up to a subsequence (still indexed by ε), there exists a measurable function u, finite a.e. in Ω, such that $\forall\, 1 < q < \frac{N}{N-1}$

$$\begin{cases} u_\varepsilon \to u & \text{a.e. and strongly in } L^q(\Omega) \\ u_\varepsilon \rightharpoonup u & \text{weakly in } W_0^{1,q}(\Omega). \end{cases}$$

We are now in a position to pass to the limit. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$, using φ as a test function in the approximate problem

$$\lambda \int_\Omega u_\varepsilon \varphi + \int_\Omega A(x, u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi = \int_\Omega f_\varepsilon \varphi.$$

Since the matrix field $A(x, u_\varepsilon)$ is bounded, with the previous convergences, we can pass to the limit and we obtain that

$$\begin{cases} & u \in \bigcap_{q<N/(N-1)} W_0^{1,q}(\Omega), \\ \forall \varphi \in \mathcal{C}_0^\infty(\Omega) & \lambda \int_\Omega u\varphi + \int_\Omega A(x,u)\nabla u \cdot \nabla \varphi = \int_\Omega f\varphi. \end{cases}$$

Some limitations

We have an existence theorem but

- for general operator with p growth we have restriction on p, since we cannot expect to have u_ε in a Sobolev space (even with a small exponent) (could be relaxed)
- u is not an admissible test function
- no stability result
- uniqueness may fail (see counterexample in Serrin (1964)) even in the linear case
- the matrix $A(x, s)$ should be bounded

Convenient frameworks

During the 90's different notions of solutions have been developed to deal with L^1 data, that is to have existence, stability and uniqueness results:

- Entropy solutions, introduced by Bénilan et al (1995)
- Renormalized solutions, introduced by DiPerna and Lions for first order equations and adapted to elliptic, parabolic equations
- SOLA or Solution Obtained as a Limit of Approximation, introduced by Dall'Aglio (1996)

Moreover the common point is that these three a posteriori definitions are obtained by considering an approximate problem (data approximation, operator approximation) and by passing to the limit.

Remark

- For L^1 data, these three notions are (in general) equivalent.
- In the linear (and quasi-linear) case it is also possible to use the notion of "duality" solution introduced by Stampacchia (1965) (see also Murat (1994), Droniou (2000))

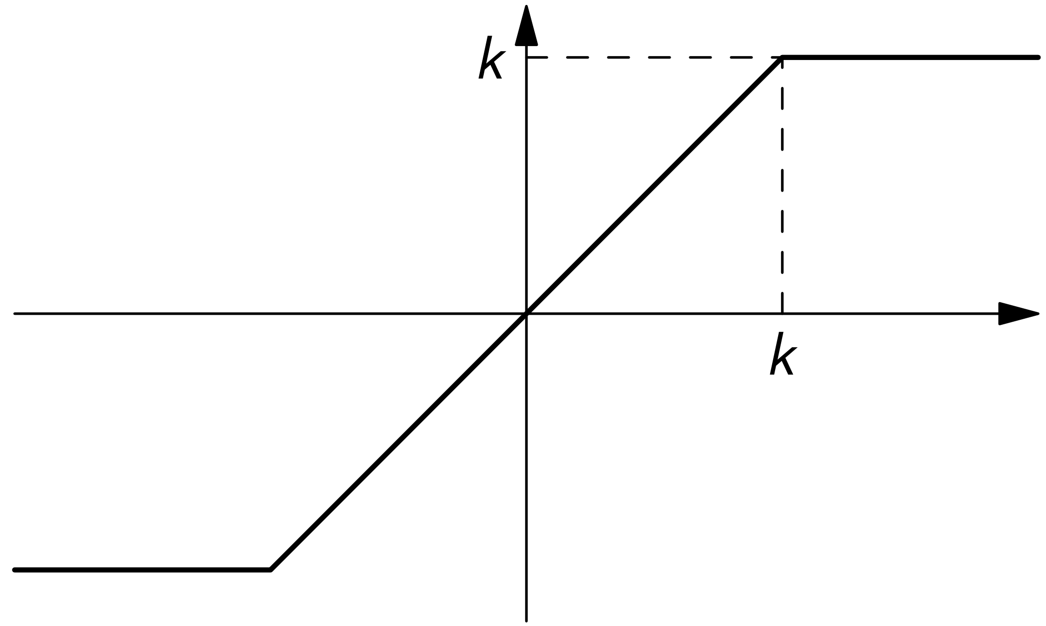
A convenient framework: renormalized solutions

- First order and Boltzmann equations: DiPerna and Lions (1989)
- Elliptic equations: Murat (1993-1994, L^1 data), Boccardo-Diaz-Giachetti-Murat (1993), Lions-Murat, Dal Maso-Murat-Orsina-Prignet (1999, bounded Radon measure as data), etc.
- Parabolic equations: Blanchard (1994), Blanchard-Murat (1997), Blanchard-Redwane (1998), Carrillo-Wittbold (1999), Porretta (1999), Blanchard-Porretta (2005), Andreu et al (2009), etc.
- with extension to anisotropic equations, to Orlicz-Sobolev spaces, etc.

The main tool

The main tool is the truncation function at height $\pm k$

$$T_k(r) = \max(-k, \min(r, k))$$



$f \in L^1(\Omega)$ – renormalized solution

Definition 1

A renormalized solution u of

(6) $$\begin{cases} \lambda u - \operatorname{div}\big(A(x,u)\nabla u)\big) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

is measurable function defined on Ω, finite a.e. in Ω, such that

(7) $$T_K(u) \in H_0^1(\Omega) \quad \forall K \geq 0,$$

$\forall h \in W^{1,\infty}(\mathbb{R})$ with h having compact support

(8) $$\lambda u h(u) - \operatorname{div}(h(u)A(x,u)\nabla u) + h'(u)A(x,u)\nabla u \cdot \nabla u = fh(u)$$

(formally (6) x h(u))

in $\mathcal{D}'(\Omega)$.

(9) $$\frac{1}{n}\int_{\{|u|<n\}} A(x,u)\nabla u \cdot \nabla u \longrightarrow 0 \text{ as } n \to +\infty.$$

Remarks

- $T_K(u) \in H_0^1(\Omega)$ and u finite a.e. allow one to define the gradient in the truncate sense (see Bénilan et al (1995)). There exists a unique vector field, denoted by ∇u, such that

$$\nabla T_K(u) = \mathbb{1}_{\{|u|<K\}} \nabla u \text{ a.e. in } \Omega.$$

$T_{k'}(T_k(u)) = T_{k'}(u)$

$0 < k' < k$

- the equation

$$\lambda u h(u) - \operatorname{div}(h(u)A(x,u)\nabla u) + h'(u)A(x,u)\nabla u \cdot \nabla u = fh(u)$$

has a sense in $\mathcal{D}'(\Omega)$: assuming that $\operatorname{supp}(h) \subset [-K, K]$,

$$\lambda u h(u) \in L^\infty(\Omega);$$

$$h(u)A(x,u)\nabla u = h(u)A(x, T_K(u))\nabla T_K(u) \in (L^2(\Omega))^N;$$

$$h'(u)A(x,u)\nabla u \cdot \nabla u = h(u)A(x, T_K(u))\nabla T_K(u) \cdot \nabla T_K(u) \in L^1(\Omega);$$

$$fh(u) \in L^1(\Omega).$$

test function $\in L^\infty(\Omega) \cap H_0^1(\Omega)$

Remarks

- In (8) we can take any test function belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$
- The condition (9) namely

$$\frac{1}{n}\int_{\{|u|<n\}} A(x,u)\nabla u \cdot \nabla u \longrightarrow 0 \text{ as } n \to +\infty$$

  is in some sense the decay of the truncated energy. Since the equation stands for bounded value of u (whatever the bound is), the decay of the energy is a crucial information on the behavior of u near $\pm\infty$. It allows
  - to prove stability results
  - to use (formally) $T_k(u)$ as a test function
  - to prove uniqueness results.
- Even if the linear case, i.e. $A(x,s) = A(x)$, the formulation (8) is is nonlinear.

Existence of a renormalized solution

Theorem 2
Assume that $\lambda \geq 0$ *and* $A(x,s) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^{N^2}$ *is a Carathéodory function (measurable in x, continuous in s) such that*

$$A(x,s)\xi \cdot \xi \geq \alpha|\xi|^2, \tag{10}$$

$$\exists\forall K > 0, M_K > 0, \quad |A(x,s)| \leq M_K, \quad \forall s \in [-K,K], \ \textit{a.e. in}\ \Omega. \tag{11}$$

Then for any $f \in L^1(\Omega)$ *there exists at least a renormalized solution.*

Step 1 : construction of the approximate problem

Since the matrix A is not supposed to be bounded, for $\varepsilon > 0$ let us define

$$A_\varepsilon(x,s) = A(x, T_{1/\varepsilon}(s))$$

and let $f_\varepsilon \in L^2(\Omega)$ such that

$$f_\varepsilon \to f \text{ strongly in } L^1(\Omega).$$

We now consider $u_\varepsilon \in H_0^1(\Omega)$ a weak solution of the approximated problem : $\forall v \in H_0^1(\Omega)$

$$\lambda \int_\Omega u_\varepsilon v + \int_\Omega A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v = \int_\Omega f_\varepsilon v \tag{12}$$

" $\varepsilon \to 0$ "

Step 2 : a priori estimates

$T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ uniformly with respect to ε

$$\lim_{M\to+\infty} \limsup_{\varepsilon\to 0} \operatorname{meas}\{|u_\varepsilon| \geq M\} = 0$$

$T_k(u_\varepsilon)$ as a test function

$$\lambda \int_\Omega u_\varepsilon T_k(u_\varepsilon) + \int_\Omega A_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon) = \int_\Omega f_\varepsilon T_k(u_\varepsilon).$$

ellipticity condition on A, $u_\varepsilon T_k(u_\varepsilon) \geq 0$, $f_\varepsilon \to f$ strongly in L^1

$$\alpha \int_\Omega |\nabla T_k(u_\varepsilon)|^2 \leq k M$$

(M depends on $\left(\|f_\varepsilon\|_{L^1}\right)$) $\|f\|$

$\Rightarrow$ $T_k(u_\varepsilon)$ is bounded $H_0^1(\Omega)$

+ (Poincaré inequality)

Poincaré inequality $\int_\Omega |T_k(u_\varepsilon)|^2 \leq k M'$ (uniformly/ε)

$$k^2 \operatorname{mes}\{|u_\varepsilon| > k\} \leq k M'$$

Uniform estimate $\operatorname{mes}\{|u_\varepsilon| > k\} \leq \dfrac{M'}{k}$

$$\Rightarrow \quad \lim_{k \to +\infty} \limsup_{\varepsilon \to 0} \operatorname{mes}\{|u_\varepsilon| > k\} = 0 .$$

Step 3 : extraction of subsequences

There exists a measurable function u, finite a.e. such that, up to a subsequence

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

$$\forall k > 0, \quad T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } H_0^1(\Omega)$$

$k \in \mathbb{N}$ + $T_k(u_\varepsilon)$ bounded in $H_0^1(\Omega)$. Diagonal process, subsequence

$\forall k \in \mathbb{N}$ $\exists u_k \in H_0^1(\Omega) \cap L^\infty(\Omega)$ / $T_k(u_\varepsilon) \longrightarrow u_k$ a.e. in Ω strongly in $L^2(\Omega)$

$T_k(u_\varepsilon) \longrightarrow u_k$ weakly in $H_0^1(\Omega)$.

$(u_\varepsilon)_\varepsilon$ is a Cauchy sequence in measure.

$$\{|u_\varepsilon - u_{\varepsilon'}| > \eta\} \subset \{|T_k(u_\varepsilon) - T_k(u_{\varepsilon'})| > \eta\} \cup \{|u_\varepsilon| > k\} \cup \{|u_{\varepsilon'}| > k\}$$

. First we choose k / meas $\{|u_\varepsilon| > k\}$ small enough

. $T_k(u_\varepsilon) \longrightarrow T_k(u_{\varepsilon'})$ $\quad \varepsilon_0$ / $\forall \varepsilon < \varepsilon_0$ meas $\{|T_k(u_\varepsilon) - T_k(u_{\varepsilon'})| > \eta\}$ small enough.