Disjointness of geodesics for first passage percolation in the plane

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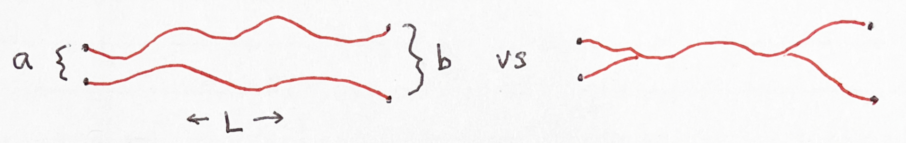
First passage percolation (FPP) on $\mathbb{Z}^2$:

We attach i.i.d. *edge passage times* τ_e to the edges e of the lattice. For a finite path γ, its passage time is $T(\gamma) = \sum_{e \in \gamma} \tau_e$. For $x, y \in \mathbb{Z}^2$, the passage time from x to y is

$$T(x, y) = \inf_{\gamma} T(\gamma),$$

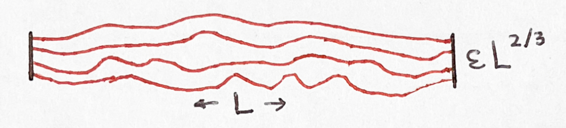
inf taken over all finite paths γ from x to y. Under mild conditions, the inf is achieved, uniquely if distribution of τ_e is continuous, which we assume. The minimizing γ is the *geodesic* Γ_{xy} from x to y.

$T(\cdot, \cdot)$ may be viewed as a random metric on the lattice; to understand this metric (especially in the context of its conjectured scaling limit, the *directed landscape*) it is important to understand the geometry of these geodesics. A basic question: when are they disjoint?



We expect: likely disjoint if $a, b \gg L^{2/3}$, not if $a, b \ll L^{2/3}$.

A closely related question was examined by Hammond (2020) for Brownian last passage percolation (LPP), an integrable model: for two short transverse intervals, what is the probability that k disjoint geodesics connect them? Answer: approximately $\epsilon^{(k^2-1)/2}$, obtained via the RSK correspondence which provides no real probabilistic/geometric intuition. (Also "Brownian Gibbs property.")



Central to a sequence of papers by Hammond examining how passage times vary as one moves tranversally to the direction of the geodesics in a rescaled system, and related questions.

Our context is like $k = 2$, so for $a = b$ of order $L^{2/3}$ it suggests the answer

$$P(\text{disjoint}) \asymp \frac{a^{3/2}}{L}.$$

What can we say for FPP, necessarily by purely probabilistic/geometric methods? Why 3/2? And what if $a \ll b$ or $a, b \ll L^{2/3}$?

"Fluctuations of $T(0,x)$ are of order $|x|^\chi$." Two ways to formalize this:

Lower fluctuation exponent:

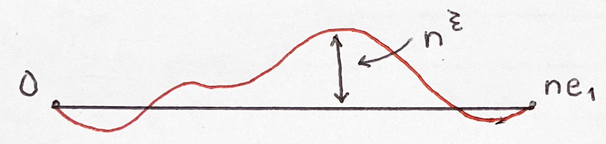
$$\chi^- = \liminf_{n\to\infty} \frac{\log \mathrm{var}(T(0,ne_1))}{2\log n}$$

Upper fluctuation exponent: (assuming τ_e has an exponential moment)

$$\chi^+ = \inf\left\{\chi > 0 : Ee^{|T(0,ne_1)-ET(0,ne_1)|/n^\chi} \text{ is bounded in } n\right\}$$

Our assumption throughout is a *unique fluctuation exponent*, $\chi^- = \chi^+ = \chi > 0$; this is not proved for any FPP model!

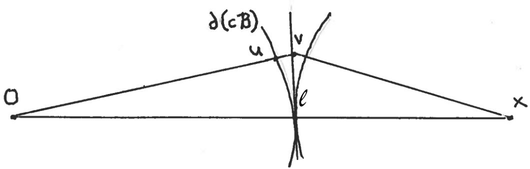
Transverse fluctuation exponent: ξ such that the transverse wandering of a geodesic Γ_{0,ne_1} from the straight path are typically of order n^ξ.



Connection between curvature in $\partial\mathcal{B}$, and geodesic transverse wandering:

Asymptotic approximation $g(x) = \lim_n ET(0, nx)/n$ is a norm, unit ball $\mathcal{B}$ is the limit shape. Curvature in $\partial\mathcal{B}$ controls strictness of the triangle inequality for norm g: for v at distance ℓ from line $0x$, we have $g(v-u) \asymp \ell^2/|x|$, so

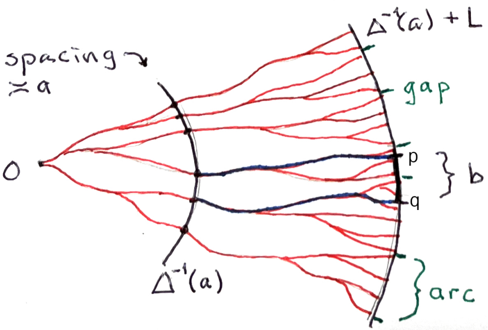
$$g(v) + g(x-v) \approx g(x) + c\frac{\ell^2}{|x|}.$$



Idea: For $|x|$ of order r, if $\mathrm{var}(T(0,x))^{1/2}$ is of some order $\sigma(r)$ then transverse fluctuations should be order $\Delta(r) = (r\sigma(r))^{1/2}$, which grows like r^ξ for $\xi = (1+\chi)/2$. Chatterjee (2011) proved this exponent relation assuming a unique χ and ξ. Here we take it as the *definition* of ξ, and use results that show $\Delta(\cdot)$ bounds the order of fluctuations.

A hint of the answer:

Consider a ball of radius $\Delta^{-1}(a) + L$, and tree of geodesics from root 0 to the boundary. How many cross a circle of radius r? *Crossing points* should be spaced by $\asymp \Delta(r)$ (the natural spacing at radius r.) Crossing points divide ball boundary into arcs sharing a crossing point; separated by gaps. Gap spacing for $r = \Delta^{-1}(a)$ is $\asymp aL/\Delta^{-1}(a)$. Arc of length b has prob. $\asymp \Delta^{-1}(a)b/aL$ to contain a gap.



Then we have two length–L geodesics w/ spacing a, b at left, right, which are disjoint.

$$\text{Is it true that } P(\text{disjoint}) \asymp \frac{\Delta^{-1}(a)b}{aL} \asymp \frac{a^{\frac{1}{\xi}-1}b}{L}??$$

a, b get different exponents! Agrees with Hammond if $\xi = 2/3$ and $a = b$...

The isotropic random graph

Our proof will exploit *radial symmetry in distribution*...which doesn't exist for the lattice. So instead of the lattice we construct FPP on an isotropic random graph $\mathbb{G}$ in $\mathbb{R}^2$ which is "as lattice–like as possible." Desired properties of this graph:

(i) planar, stationary, ergodic, isotropic

(ii) bounded hole size: every unit ball contains ≥ 1 vertex

(iii) finite range of dependence: $\exists \rho : d(A, B) \geq \rho \implies$ restrictions of graph to A, B are independent

(iv) bounded dilation: (Euclidean length of shortest graph path $x \leftrightarrow y$) $\leq C|y - x|$ for all x, y.

(v) controlled local density: $P(n$ vertices in a unit ball$) \leq c_1 e^{-c_2 n}$.

Note (iv) is true $\forall$ Delaunay triangulation in $\mathbb{R}^2$, with $C \leq 1.998$ (Xia, 2013).

Lemma 1

(A. 2021) There exists a point process in $\mathbb{R}^2$ for which the Delaunay triangulation has properties (i)–(v).

We may add: (vi) the point process is built from a space–time Poisson process.

To make our FPP even more lattice–like, we define passage times as $\tau_e = \omega_e |e|$, where the *speeds* ω_e are iid, given $\mathbb{G}$, and $|e|$ is the Euclidean length of e.

Since $\mathbb{G}$ is isotropic, the limit shape is a Euclidean ball—no need to prove or assume boundary curvature!

CAUTION: FPP on $\mathbb{G}$ has the FKG property *only after conditioning on* $\mathbb{G}$. This is a problem in some arguments (which we work around.)

But: the van den Berg–Kesten–Reimer (BKR) inequality still applies in many cases, because the random graph has a finite range ρ of dependence and is built from a space–time Poisson process, so BKR is valid as long as the disjoint occurrences are separated by distance ρ or more. The Reimer improvement is essential here!

Our core assumptions:

(i) lattice–like random graph as in the preceding;

(ii) speed ω_e is a continuous r.v. with a finite exponential moment;

(iii) unique fluctuation exponent $\chi^- = \chi^+ = \chi \in (0, 1)$.

ψ *subpolynomial* means $r^{-\epsilon} \ll \psi(r) \ll r^{\epsilon}$ as $r \to \infty$, for all ϵ.

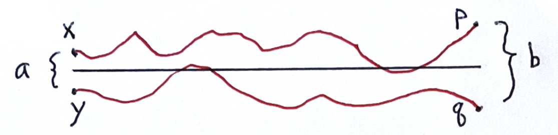
Theorem 2

Under the core assumptions, there exist constants C_i and a subpolynomial function ψ such that for all $C_1 \le a \le b$ and all $L \ge C_2$, for the points

$$x = \left(0, \frac{a}{2}\right), \quad y = \left(0, -\frac{a}{2}\right), \quad p = \left(L, \frac{b}{2}\right), \quad q = \left(L, -\frac{b}{2}\right),$$

we have

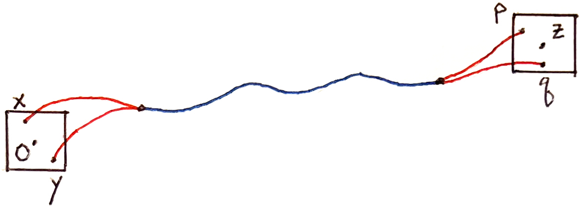
$$\frac{1}{\psi(a)} \frac{a^{\frac{1}{\xi}-1} b}{L} \le P(\Gamma_{xp} \cap \Gamma_{yq} = \emptyset) \le \psi(a) \frac{a^{\frac{1}{\xi}-1} b}{L}. \tag{1}$$



Relation to Dembin–Elboim–Peled (2022). Their result is analogous to: for all $0 < \delta < 8\epsilon$ and all z,

$$P(\text{for some } x, y, p, q \text{ in } \|z\|^{1/8-\epsilon}\text{–boxes, the disjoint pieces have total length more than } \|z\|^{1-\delta}) \leq \frac{C(\log \|z\|)^3}{\|z\|^{\epsilon-\delta/8}}$$

This is on $\mathbb{Z}^2$ (not random graph) with no assumption that χ exists.



The same machinery used for our proof also produces a small–tube theorem:

Theorem 3

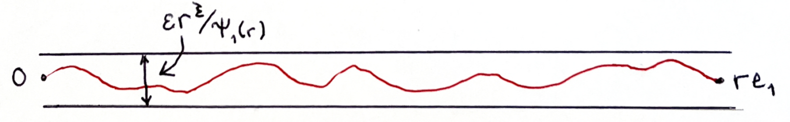
Under the core assumptions, there exist constants C_i and subpolynomial functions ψ_i as follows. For all $r \geq C_1$ and

$$C_2 r^{-\chi} \leq \epsilon \leq \epsilon_1, \tag{2}$$

we have

$$P\left(\max\{|x_2| : x \in \Gamma_{0,re_1}\} \leq \frac{\epsilon r^{\xi}}{\psi_1(r)}\right) \leq C_3 \exp\left(-C_4 \epsilon^{-1/\xi} \psi_2(\epsilon^{-1})\right). \tag{3}$$

A similar result was proved for solvable LPP on $\mathbb{Z}^2$ by Basu and Bhatia (2021), without the subpolynomial corrections ψ_i.



Relation to earlier results:

Chatterjee (2011): (For $\mathbb{Z}^d$) "If there is an exponent χ which characterizes fluctuations of $T(0,x)$ in the above sense of $\chi^- = \chi^+$, and an exponent ξ which characterizes transverse wandering in an analogous way ($\xi^- = \xi^+$), then $\xi = (1+\chi)/2$."

Preceding theorem plus A. (2021): (For random graph, $d = 2$) "If there is an exponent χ which characterizes fluctuations of $T(0,x)$ in the above sense of $\chi^- = \chi^+$, then the exponent given by $\xi = (1+\chi)/2$ characterizes transverse wandering in an analogous way."

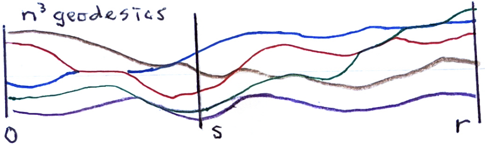
The second removes the need to assume "$\xi^- = \xi^+$."

Common proof ingredient: crossing point bounds. Consider slab geodesics between vertical hyperplanes H_0, H_r at 0 and r, starting in intervals of length $\Delta(r)(\log r)^\nu$. $\frac{r}{3} \leq s \leq \frac{2r}{3}$. How many points where some such geodesic crosses H_s? (Note the geodesics can cross each other.)

Observation: no two geodesics with different crossing points can touch each other on *both* sides of H_s (otherwise a geodesic is nonunique.) Hence if n geodesics all pass through a common point x ("popular site") left of H_s, then they must be mutually disjoint right of H_s. Deterministic result using this:

Lemma 4

Let $\mathcal{G}$ be a collection of n^3 geodesics as above, each with a different crossing point at H_s. There exists a subcollection $\mathcal{G}'$ of n geodesics for which either all geodesics in $\mathcal{G}'$ are disjoint between H_0 and H_s, or all are disjoint between H_s and H_r.



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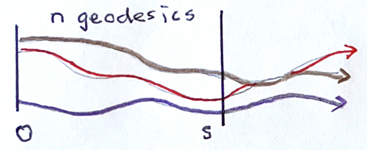
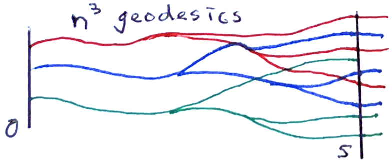
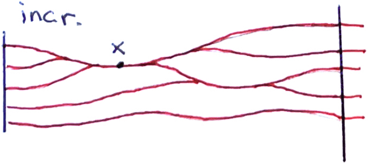


Idea: Order n^3 geodesics in $\mathcal{G}$ top to bottom by starting point in H_0; use H_s crossing point to break ties. Ordering of crossing points, top to bottom, is a permutation π of $\{1, \ldots, n^3\}$.



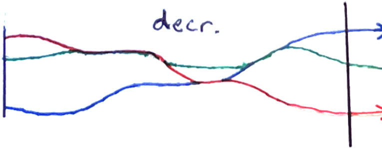
Increasing subsequence $\leftrightarrow$ set of geodesics $\{\Gamma_{(1)} \preceq \cdots \preceq \Gamma_{(m)}\}$ which may touch but don't cross each other, left of H_s.

$\Gamma_{((i-1)n)}, \Gamma_{(in)}$ not disjoint $\implies$ there is a popular site ($n+1$ geodesics through it.)



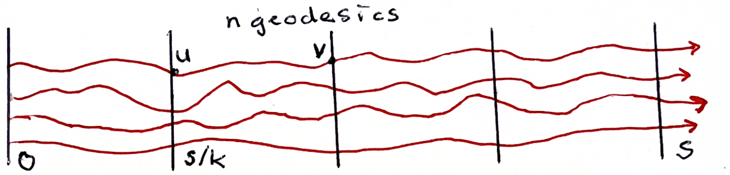
Thus either there is a popular site (so n disjoint geodesics from H_s to H_r) or $\Gamma_{(n)}, \Gamma_{(2n)}, \Gamma_{(3n)}, \ldots$ are disjoint H_0 to H_s.

Decreasing subsequence $\leftrightarrow$ set of geodesics in which every pair crosses, left of H_s...so they must be disjoint to the right of H_s.

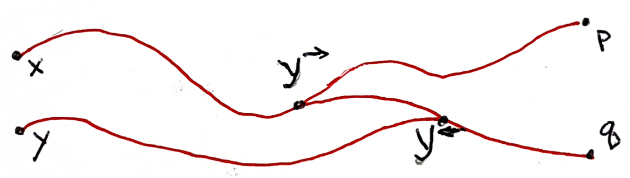


Erdős–Szekeres Theorem: π has either an incr. subsequence of n^2 or decr. of n.

Once we're dealing with *n* disjoint geodesics, say between H_0 and H_s, we can hope to use the van den Berg–Kesten inequality to show their existence is unlikely. We must work around the fact that the event "γ is a geodesic" does not in general occur on just the path γ; "*n* disjoint geodesics" are not the same as "*n* disjoint occurrences." We'll skip the details.

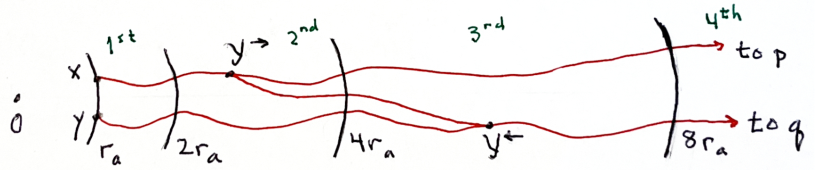


Proof ideas for Theorem 2 (upper bound): What does "disjoint" mean?
Diagonal geodesic Γ_{xq}



Rightward and leftward bifurcation points $\mathcal{Y}^{\rightarrow}, \mathcal{Y}^{\leftarrow}$. Disjoint $\iff$ $\mathcal{Y}^{\rightarrow}$ precedes $\mathcal{Y}^{\leftarrow}$ in Γ_{xq}.

Subdivide the event of disjointness according to the scales $j \leq k$ of $\mathcal{Y}^{\rightarrow}, \mathcal{Y}^{\leftarrow}$ from 0 placed as in the picture ($r_a = \Delta^{-1}(a)$) which has $j = 2, k = 3$:

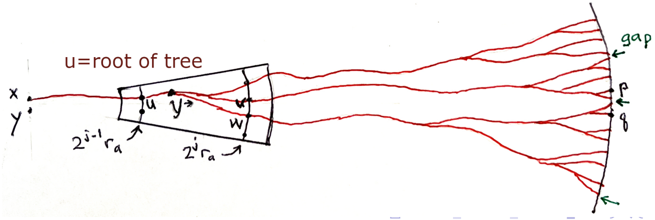


Consider first the rightward bifurcation. Recall the tree of geodesics from some root to the boundary of $B(0, r_a + L)$ (where p, q lie.) Bifurcation of the rightward geodesics Γ_{xp}, Γ_{xq} at $\mathcal{Y}^{\rightarrow}$ means we have crossing points u, v, w as shown. This means there is an arc gap between p and q, for geodesics from some root u (a crossing point), with arcs determined by exits from $B(0, 2^j r_a)$.

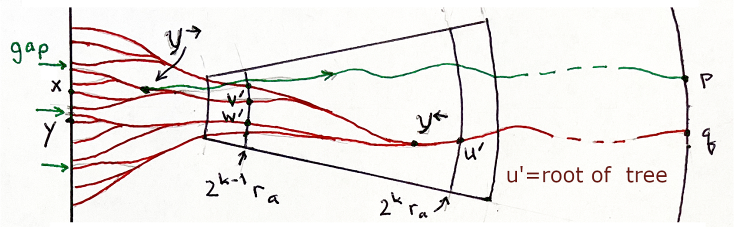
Restrict attention to the jth–scale *annular sector* $\mathbb{S}_{2^j r_a}$ of height of order $\Delta(2^j r_a)(\log(2^j r_a))^{\nu}$ (slightly larger than natural spacing), between radii $\frac{3}{8}2^j r_a$ and $\frac{9}{8}2^j r_a$. Consider geodesics left end of $\mathbb{S}_{2^j r_a}$ to right end, and for s in between, let $\mathcal{E}(s)$ be the set of crossing points at radius s. Thus $u \in \mathcal{E}(2^{j-1} r_a)$ and $v, w \in \mathcal{E}(2^j r_a)$, and with high probability we have

$$|\mathcal{E}(2^{j-1} r_a)| \leq \psi(2^{j-1} r_a), \quad |\mathcal{E}(2^j r_a)| \leq \psi(2^j r_a)$$

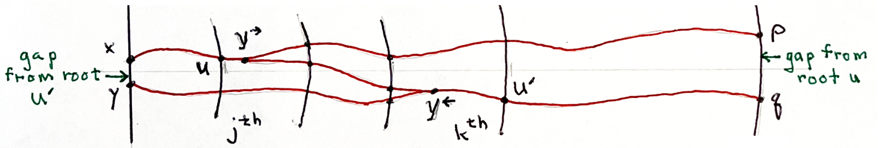
(ψ subpolynomial.) With high prob. there are few choices of u, v, w so few possible arc gaps; it should be unlikely there is one between p and q!



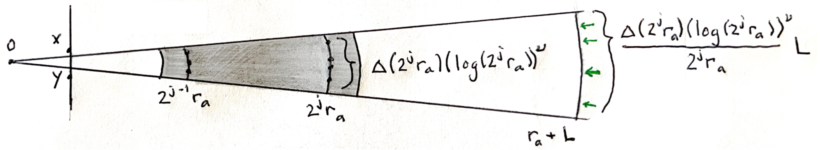
Similar picture for the leftward bifurcation (at least when it occurs in the left half.) For technical convenience, we consider intervals and gaps in a vertical line instead of arcs and arc gaps in a ball boundary. Here $u' \in \mathcal{E}(2^k r_a), v', w' \in \mathcal{E}(2^{k-1} r_a)$, and we consider the geodesic tree with root u', reaching the line H_{r_a}; crossing points at radius $2^{k-1} r_a$ determine intervals in H_{r_a} with gaps between them:



Again with high prob. there are not many choices of u', v', w' so not many possible gaps, and it should be unlikely there is one between x and y.



How unlikely? Rightward:



density of arc gaps is with high prob. at most

$$\psi(2^{j-1}r_a)\psi(2^j r_a)\frac{2^j r_a}{\Delta(2^j r_a)(\log(2^j r_a))^{\nu} L}$$

so the probability of one between p and q (length b) should be at most

$$\psi_1(2^j r_a)\frac{2^j r_a b}{\Delta(2^j r_a)L} \asymp \psi_1(2^j r_a)\frac{2^{j(1-\xi)} r_a b}{\Delta(r_a)L} \asymp \psi_2(2^j r_a)\frac{2^{j(1-\xi)} a^{\frac{1}{\xi}-1} b}{L}$$

(ψ_i subpolynomial, $\Delta(r_a) = a$.)

Leftward is similar: probability of a gap between x and y should be at most

$$\psi_3(2^k r_a)\frac{a}{\Delta(2^k r_a)} \asymp \psi_3(2^k r_a)\frac{1}{2^{k\xi}}.$$

If these leftward and rightward gap events are (roughly) independent, then

$$P(\text{gaps between } x, y \text{ AND between } p, q) \leq \psi_2(2^j r_a)\psi_3(2^k r_a)\frac{2^{j(1-\xi)}a^{\frac{1}{\xi}-1}b}{2^{k\xi}L}.$$

Sum over scales j, k with $j \leq k, 2^k r_a \leq L/4$ (noting $\xi > 1/2$) to get

$$P(\Gamma_{xp}, \Gamma_{yq} \text{ disjoint, bifurcation points left of } L/4) \leq \psi_5(a)\frac{a^{\frac{1}{\xi}-1}b}{L}$$

Switch left and right, a and b to get similar for bifurcation points both right of $3L/4$. Tweak the argument for other bifurcation point locations.

Largest bound is for scales from the left end with $j = k = 1$ suggesting the bifurcations are likeliest to both occur at the left ("small") end when $a \ll b$.

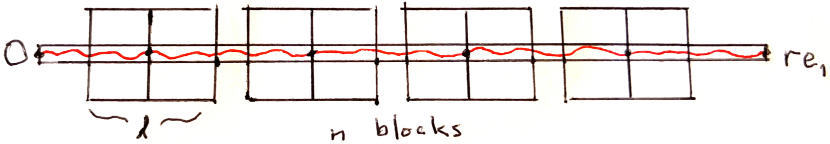
How do we make this sketch rigorous—especially the independence part?

We use averaging. To see how averaging works in a simpler context, let us interrupt this proof to sketch the proof of the small tube theorem.

Proof: To show: for some subpolynomial ψ_1, ψ_2,

$$P\left(\max\{|x_2| : x \in \Gamma_{0,re_1}\} \leq \frac{\epsilon r^\xi}{\psi_1(r)}\right) \leq \exp\left(-C_4 \epsilon^{-1/\xi}\psi_2(\epsilon^{-1})\right). \qquad (4)$$

Cover the thin tube with $n = \epsilon^{-1/\xi}\psi_2(\epsilon^{-1})$ blocks of length $\ell = r/2n$, height $2\Delta(\ell)(\log \ell)^\nu$. Each block has geodesics from end to end, and their crossing points at the middle of the block. If Γ_{0,re_1} stays in the tube, every block must have a crossing point inside the tube; these are independent events. We showed that with high probability, a block has at most a subpolynomial number $\psi(\ell)$ of such crossing points.



Let G_i be the event that the ith block has a cossing point inside the thin tube, and let $\mathbb{T}_s$ denote vertical translation by s. Then $P(\mathbb{T}_s G_i) = P(G_i)$. If a given configuration $\mathbb{G}, \tau)$ has $\leq \psi(\ell)$ crossing points in the ith block then

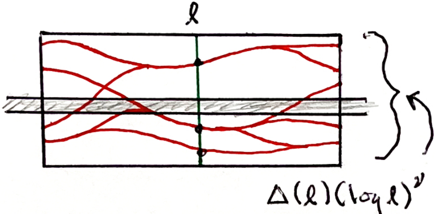
$$|\{s : (\mathbb{G}, \tau) \in \mathbb{T}_s G_i\}| \leq \psi(\ell) \cdot (\text{height of tube})$$

so by choosing the right subpolynomial ψ_1, ψ_2 we get

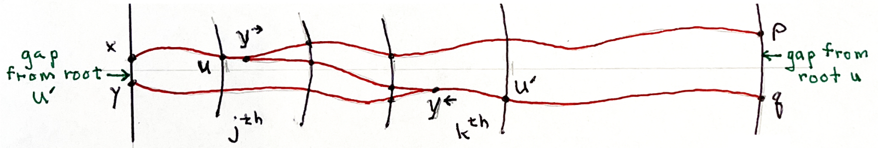
$$\begin{aligned} P(G_i) &= E\left(\frac{1}{2\Delta(\ell)(\log \ell)^\nu} \int_{-\Delta(\ell)(\log \ell)^\nu}^{\Delta(\ell)(\log \ell)^\nu} 1_{\mathbb{T}_s G_i}\, ds \right) \\ &\leq \frac{1}{2\Delta(\ell)(\log \ell)^\nu} \psi(\ell) \frac{2\epsilon r^\xi}{\psi_1(r)} \leq \frac{1}{2}. \end{aligned}$$

Therefore

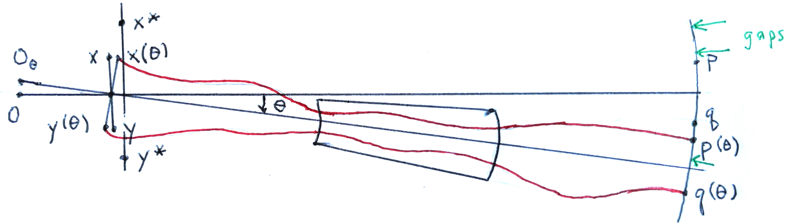
$$\begin{aligned} &P\left(\max\{|x_2| : x \in \Gamma_{0,re_1}\} \leq \frac{\epsilon r^\xi}{\psi_1(r)} \right) \\ &\quad \leq \left(\frac{1}{2}\right)^n = \exp\left(-C_4 \epsilon^{-1/\xi} \psi_2(\epsilon^{-1}) \right). \end{aligned}$$



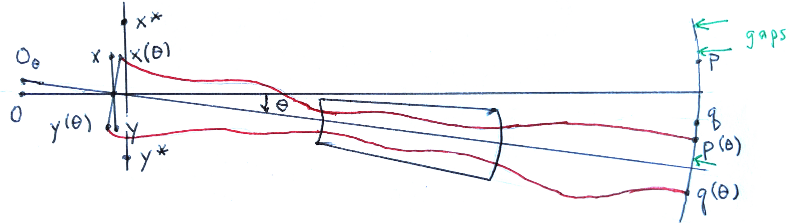
Back to the proof of the disjointness theorem.



Fix scales j, k. We consider the average over rotations by $\theta, |\theta| \leq a(\log a)^{\nu}/r_a$ (width of fattest sector, $j = 1$.) Rotation doesn't change disjointness probability. The rotated annular sector gives different gaps at radius $r_a + L$ for different θ, but we discretize the choice of annular sector (that is, when the system is rotated by θ, the annular sector is rotated by the closest value in some $\{\theta_1, \ldots, \theta_m\}$.)



But this only helps us control the gap–presence probability on the right.



To control gap–presence probability on the left, we also must average over vertical translations $\mathbb{T}_s$ of the whole system, including the ball boundary on the right.

A problem: the points $x(\theta), y(\theta)$ are not all in the same vertical line, where we observe intervals and gaps. Solution: replace all $x(\theta), y(\theta)$ with x^*, y^* as in the picture. Modulo low–probability events, we have

$$\Gamma_{x(\theta)p(\theta)} \cap \Gamma_{y(\theta)q(\theta)} = \emptyset \implies \Gamma_{x^*p(\theta)} \cap \Gamma_{y^*q(\theta)} = \emptyset$$

and all points $\mathbb{T}_s x^*, \mathbb{T}_s y^*$ lie in the same line. As with rotations, we need to discretize the vertical translations of (just) the annular sectors: translate by $s \implies$ annular sector shifts by closest element of some $\{s_1, \ldots, s_g\}$. This way we can control the number of crossing points for all translations, rotations, and scales of annular sectors, at once (e.g. jth scale, rotated by θ_i, translated by s_ℓ.)

The full averaging: $\mathbb{U}_\theta =$ rotation around $(x+y)/2$ by θ.

$$\theta_a = \frac{a(\log a)^\nu}{\Delta^{-1}(a)}, \quad s_k = \Delta(2^k r_a)(\log(2^k r_a))^\nu.$$

θ_a is the "natural angular spacing" at radius $r_a = \Delta^{-1}(a)$. Event

$$D_{jk} : \Gamma_{xp} \cap \Gamma_{yq} = \emptyset, \text{ bifurcations on scales } j, k.$$

Then

$$P(\Gamma_{xp} \cap \Gamma_{yq} = \emptyset) = \sum_{1 \le j \le k \le J_{max}} E\left(\frac{1}{2s_k} \frac{1}{2\theta_a} \int_{-s_k}^{s_k} \int_{-\theta_a}^{\theta_a} 1_{\mathbb{T}_s \mathbb{U}_\theta D_{jk}} \, d\theta \, ds \right).$$

Well...not quite, actually. This glosses over some (actually many) things—

Aberrations: Throughout, we have referred to events occurring "with high probability," sometimes implicitly, and then assumed they did occur. Examples:

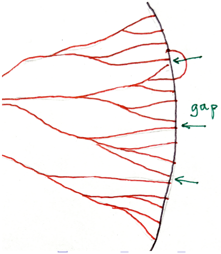
(i) geodesics of interest pass through an annular sector
(ii) limits on the number of crossing points
(iii) no or minimal backtracking by geodesics.

Generically we call failures of such events *aberrations*. Other than (ii), all can be reduced to instances of fat triangles in geodesics, that is, points u, v, w in some geodesic in some $B(0, r)$ where $u \to v \to w$ involves extra distance by at least $\asymp \log r$ standard deviations:

$$g(v - u) + g(w - v) \geq g(w - u) + C\sigma(|w - u|) \log r.$$

A backtrack of order more than $\sigma(|w - u|) \log r$ is an example:

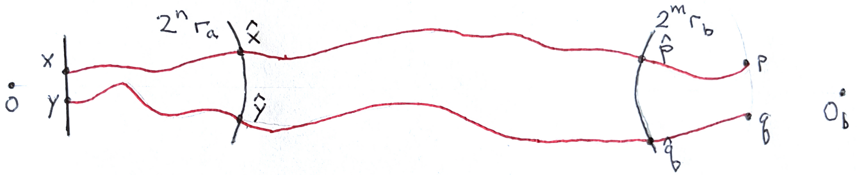
Evading a gap:



Dealing with aberrations: A fat triangle occurs on the *nth scale* if it occurs for $r = 2^n r_a$; similar for excess crossing points in an annular sector. Suppose the largest scale with an aberration, going rightward from 0, is the nth, and going leftward from 0_b it's the mth. Probability of this can be bounded by about

$$P(\text{aberration on } n\text{th scale}) \cdot P(\Gamma_{\hat{x}\hat{p}} \cap \Gamma_{\hat{y}\hat{q}} = \emptyset) \cdot P(\text{aberration on } m\text{th scale})$$

and the middle probability is for an aberration–free context, where the averaging argument works. Bounds for the probability of a fat triangle existing somewhere in a given ball were established in A. (2021).



Open problem: Hammond's problem with $k \geq 3$. What is the probabilistic/geometric reason for

$$P(\exists k \text{ disjoint geodesics}) \asymp \epsilon^{(k^2-1)/2}?$$

k geodesics means $2k-2$ bifurcation points, but the exponent grows as k^2—why?