Weakly integrable modules over affine Kac–Moody algebras

Algebraic and Combinatorial Methods in Representation Theory
(in honour of Vyjayanthi Chari's 65th birthday)
ICTS Bangalore

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- Fix any $\lambda \in \mathfrak{h}^*$.
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Example: The adjoint representation of $\mathfrak{g}$ is isomorphic to $V(\theta)$, where θ is the highest root of $\mathfrak{g}$.

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Unlike the general Kac-Moody algebras, the affine Kac-Moody algebras also admit an alternative definition which is quite explicit. It is the interplay between these two definitions that makes the study of these algebras and their representations tractable. The explicit realization is what we need for this talk, which I shall now describe.

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- $Z(\widehat{\mathfrak{g}}) = \{x \in \widehat{\mathfrak{g}} \mid [x,y] = 0\ \forall\ y \in \widehat{\mathfrak{g}}\} = \mathbb{C}K.$

Structure theory of affine Kac–Moody algebras

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Theorem

$\widehat{V}(\Lambda)$ is an integrable $\widehat{\mathfrak{g}}$-module if and only if $\Lambda \in \widehat{P}^+_{\widehat{\mathfrak{g}}}$.

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Definition

A $\widehat{\mathfrak{g}}$-module V is called integrable if

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Evaluation modules

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Remark: $V(\underline{\lambda}, \underline{b}, m)$ is irreducible $\iff$ all the b_i's are distinct non-zero complex numbers.

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Classification of irreducible integrable modules

Theorem (Chari, Invent. Math., 1986 and Chari–Pressley, Math. Ann., 1986)

Let V be an irreducible integrable $\widehat{\mathfrak{g}}$-module with finite-dimensional $\widehat{\mathfrak{h}}$-weight spaces. Then:

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Observation: All the above modules have finite-dimensional $\mathbb{Z}$-graded components, i.e. they have finite-dimensional weight spaces with respect to $\mathbb{C}d$.

Quasi-finite modules

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A $\widehat{\mathfrak{g}}$-module V is said to be quasi-finite if

1. V is a weight module with respect to $\mathbb{C}d$, i.e. $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $V_n = \{v \in V \mid d.v = nv\}$;
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  Define $\Lambda \in \widehat{\mathfrak{h}}^*$ by setting $\Lambda(\alpha_i^{\vee}) = i \; \forall \; 1 \leqslant i \leqslant l$ and $\Lambda(K) = \lambda(d) = 0$ and consider the irreducible highest weight module $\widehat{V}(\Lambda)$.

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1. V is a weight module with respect to $\mathbb{C}d$, i.e. $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where $V_n = \{v \in V \mid d.v = nv\}$;
2. $\dim V_n < \infty \ \forall\, n \in \mathbb{Z}$.

- **Question:** Does there exist quasi-finite modules which are not integrable?

  Example: Recall that $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ and $\Pi_{\widehat{\mathfrak{g}}}^{\vee} = \{\alpha_i^{\vee}\}_{i=0}^{l}$ is the set of all simple co-roots of $\widehat{\mathfrak{g}}$, where $l = \dim \mathfrak{h}$ and $\alpha_0^{\vee} = K - \theta^{\vee}$.
  Define $\Lambda \in \widehat{\mathfrak{h}}^*$ by setting $\Lambda(\alpha_i^{\vee}) = i \ \forall\ 1 \leqslant i \leqslant l$ and $\Lambda(K) = \lambda(d) = 0$ and consider the irreducible highest weight module $\widehat{V}(\Lambda)$.
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Problem

Classify irreducible quasi-finite modules over $\widehat{\mathfrak{g}}$.

Weakly integrable modules

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Lemma

Let V be a $\widehat{\mathfrak{g}}$-module. Then V is weakly integrable with finite-dimensional $\widehat{\mathfrak{h}}$-weight spaces if and only if V is quasi-finite.

Classification of irreducible weakly integrable modules

Theorem (P.)

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Conversely, all the above mentioned irreducible modules are weakly integrable.

References

[1] Quasi-finite modules over extended affine Lie algebras,
https://arxiv.org/abs/2308.10665, 2023 (P.)

[2] Integrable representations of affine Lie-algebras,
Invent. Math., 1986 (V. Chari).

[3] New unitary representations of loop groups,
Math. Ann., 1986 (V. Chari, A. Pressley).

[4] Integrable highest weight modules over affine Lie superalgebras and Appell's function,
Comm. Math. Phys., 2001 (V. Kac, M. Wakimoto).