Parahoric torsors and degenerations

Vikraman Balaji

12th July 2021

Notations

- $k = \mathbb{C}$

Notations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k

Fuchsian groups etc

- Let $\pi := \langle A_1, \ldots, B_g, C_1, \ldots, C_m \rangle$ be a Fuchsian group of elliptic type generated by $2g + m$ elements, modulo the relations

$$\prod_j [A_j, B_j] C_1 \ldots C_m = I. \tag{1}$$

$$C_i^{n_i} = I, \quad (i = 1, 2, \ldots, m). \tag{2}$$

Fuchsian groups etc

- Let $\pi := \langle A_1, \ldots, B_g, C_1, \ldots, C_m \rangle$ be a Fuchsian group of elliptic type generated by $2g + m$ elements, modulo the relations

$$\prod_j [A_j, B_j] C_1 \ldots C_m = I. \tag{1}$$

$$C_i^{n_i} = I, \quad (i = 1, 2, \ldots, m). \tag{2}$$

- Let $q : \mathbb{H} \to X$, assume $g(X) \geq 2$, ramified over $R \subset X$. The isotropy subgroups at the points $z_i \in q^{-1}(R)$ are cyclic of

$$\pi_{z_i} = \langle C_i \rangle$$

order n_i with C_i as generators.

- The *type of a homomorphism* $\rho : \pi \to G$ is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$. Equivalently, the type of ρ is the set of isomorphism classes of the local representations $\rho_{z_i} : \pi_{z_i} \to G, i = 1, \ldots, m$.

- The *type of a homomorphism* $\rho : \pi \to G$ is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$. Equivalently, the type of ρ is the set of isomorphism classes of the local representations $\rho_{z_i} : \pi_{z_i} \to G, i = 1, \ldots, m$.

- The object of the study is: $R^{\boldsymbol{\tau}}(\pi, K_G)$ denote the space of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

- The *type of a homomorphism* $\rho : \pi \to G$ is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$. Equivalently, the type of ρ is the set of isomorphism classes of the local representations $\rho_{z_i} : \pi_{z_i} \to G, i = 1, \ldots, m$.

- The object of the study is: $R^{\boldsymbol{\tau}}(\pi, K_G)$ denote the space of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

- The aim: *describe this space in terms of objects defined on* X. This theme starts with the work of Weil in 1938 and achieved in 1968 by Seshadri for the case $G = GL(n)$ in his paper on π-bundles. The complete picture for the $GL(n)$ case occurs in Mehta-Seshadri in 1980.

- The *type of a homomorphism* $\rho : \pi \to G$ is defined to be the set of conjugacy classes in G of the images $\rho(C_i)$ and is denoted by $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$. Equivalently, the type of ρ is the set of isomorphism classes of the local representations $\rho_{z_i} : \pi_{z_i} \to G, i = 1, \ldots, m$.

- The object of the study is: $R^{\tau}(\pi, K_G)$ denote the space of homomorphisms $\rho : \pi \to K_G$ of type $\boldsymbol{\tau} = \{\boldsymbol{\tau}_i\}$.

- The aim: *describe this space in terms of objects defined on X*. This theme starts with the work of Weil in 1938 and achieved in 1968 by Seshadri for the case $G = GL(n)$ in his paper on π-bundles. The complete picture for the $GL(n)$ case occurs in Mehta-Seshadri in 1980.

- A (π, G)–bundle on $\mathbb{H}$ is defined to be the trivial G–bundle $\mathbb{H} \times G$ on $\mathbb{H}$ with the π–structure given by $\gamma(z, g) = (z, \rho(\gamma).g)$, with ρ a homomorphism $\pi \to G$.

If $G = GL(n)$ is the full-linear group, the (π, G)–bundles on $\mathbb{H}$ have an equivalent description as π–vector bundles on $\mathbb{H}$. We recall that if $V \simeq \mathbb{H} \times \mathbb{C}^n$ is a π–vector bundle on $\mathbb{H}$, the vector bundle $W = q_*^\pi(V)$(invariant direct image by q) on X acquires a *parabolic structure* which consists of the data assigning a *flag to the fibre of W at every ramification point in X for the covering q* together with a tuple of *weights*.

The invariant direct image functor $V \mapsto q_*^\pi(V)$ gives a fully faithful embedding of the category of π–vector bundles on $\mathbb{H}$ into the category of parabolic vector bundles on X (morphisms being taken as isomorphisms). Moreover, we can realise every parabolic bundle with rational weights as $q_*^\pi(V)$ for a suitable π and V.

Towards the general G case: The Weyl alcove

- Recall: The set of conjugacy classes of element in K_G gets identified with T/W which is the *Weyl alcove* since any element of K_G is conjugate to an element in the maximal torus upto an element of the Weyl group.

Towards the general G case: The Weyl alcove

- Recall: The set of conjugacy classes of element in K_G gets identified with T/W which is the *Weyl alcove* since any element of K_G is conjugate to an element in the maximal torus upto an element of the Weyl group.

- Since G is *simple*, then $(Y(T) \otimes \mathbb{Q})/W_{\text{aff}}$ gets identified with the simplex (*the (rational) Weyl alcove*)

$$\mathcal{A} := \{x \in Y(T) \otimes \mathbb{Q} \mid (x, \alpha_0) \leq 1, (x, \alpha_i) \geq 0, \forall \ \textit{positive roots} \ \alpha_i\}$$

where α_0 is the *highest root*.

- Elements of finite order in K_G modulo conjugacy gets identified with the rational points of the Weyl alcove:

- Elements of finite order in K_G modulo conjugacy gets identified with the rational points of the Weyl alcove:
- Let $K_G \subset G$ be a maximal compact subgroup. For an arbitrary group S, let Torsion(S) denote the subset of elements of finite order in S. We then have the following identifications:

$$\text{Torsion}(K_G)/\textit{conjugation} \simeq \text{Torsion}(T)/W$$

$$(Y(T) \otimes \mathbb{Q}/\mathbb{Z})/W \simeq (Y(T) \otimes \mathbb{Q})/W_{\text{aff}}.$$

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .
- Denote by $(\ ,\ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form.

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .
- Denote by $(\ ,\ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form.
- $A = Spec\ k[[t]]$, $K = Spec\ k((t))$

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .
- Denote by $(\ ,\ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form.
- $A = Spec\ k[[t]]$, $K = Spec\ k((t))$
- For each $\theta \in E$, let $\mathcal{P}_\theta(K) \subset G(K)$ the subgroup

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .
- Denote by $(\ , \ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form.
- $A = Spec\ k[[t]], K = Spec\ k((t))$
- For each $\theta \in E$, let $\mathcal{P}_\theta(K) \subset G(K)$ the subgroup
- $\mathcal{P}_\theta(K) := \langle T(A), U_r(z^{m_r}(\theta)A), r \in R\rangle$

Parahoric groups

- Let $E := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This can be identified with an affine apartment of the Bruhat-Tits building of G.
- Fix (T, B, G) and let ℓ be the rank of G .
- Denote by $(\ ,\ ) : Y(T) \times X(T) \to \mathbb{Z}$ the canonical bilinear form.
- $A = Spec\ k[[t]], K = Spec\ k((t))$
- For each $\theta \in E$, let $\mathcal{P}_\theta(K) \subset G(K)$ the subgroup
- $\mathcal{P}_\theta(K) := \langle T(A), U_r(z^{m_r}(\theta)A), r \in R\rangle$
- where $m_r(\theta) = -[(\theta, r)]$

Since G is simple and simply connected, upto conjugacy by the action of $G(K)$, $\{\mathcal{P}_\theta(K)\}_{\theta \in E}$ is the set of all parahorics and one can take $\theta \in \mathcal{A}_{\mathbb{Q}}$, the rational points of the Weyl alcove.

SL_2

- We take $G = SL(2)$ and let α be the unique simple root. Since it is also the highest root so $\theta_\alpha(\alpha) = 1$.

SL_2

- We take $G = SL(2)$ and let α be the unique simple root. Since it is also the highest root so $\theta_\alpha(\alpha) = 1$.
- $\mathcal{P}_{\theta_\alpha} =< T(A), u_\alpha(z^{-1}A), u_{-\alpha}(z.A) >= \begin{pmatrix} A & z^{-1}A \\ z.A & A \end{pmatrix}$

SL_2

- We take $G = SL(2)$ and let α be the unique simple root. Since it is also the highest root so $\theta_\alpha(\alpha) = 1$.
- $\mathcal{P}_{\theta_\alpha} =< T(A), u_\alpha(z^{-1}A), u_{-\alpha}(z.A) >= \begin{pmatrix} A & z^{-1}A \\ z.A & A \end{pmatrix}$
- Similarly, we have in SL_3:

$$\mathcal{P}_{\theta_{\alpha_1}} = \{ \begin{pmatrix} A & z^{-1}A & z^{-1}A \\ zA & A & A \\ zA & A & A \end{pmatrix} \} \tag{3}$$

SL_2

- We take $G = SL(2)$ and let α be the unique simple root. Since it is also the highest root so $\theta_\alpha(\alpha) = 1$.
- $\mathcal{P}_{\theta_\alpha} =< T(A), u_\alpha(z^{-1}A), u_{-\alpha}(z.A) >= \begin{pmatrix} A & z^{-1}A \\ z.A & A \end{pmatrix}$
- Similarly, we have in SL_3:

$$\mathcal{P}_{\theta_{\alpha_1}} = \{ \begin{pmatrix} A & z^{-1}A & z^{-1}A \\ zA & A & A \\ zA & A & A \end{pmatrix} \} \tag{3}$$

- And

$$\mathcal{P}_{\theta_{\alpha_2}} = \{ \begin{pmatrix} A & A & z^{-1}A \\ A & A & z^{-1}A \\ zA & zA & A \end{pmatrix} \} \tag{4}$$

The standard Iwahori subgroup is:

$$\mathcal{P}_{\theta_{\alpha_2}} = \{ \begin{pmatrix} A & A & A \\ zA & A & A \\ zA & zA & A \end{pmatrix} \} \tag{5}$$

obtained by taking the evaluation map $ev : G(A) \to G(\mathbb{C})$ and taking $ev^{-1}(B)$.

Parahoric group schemes

By the main theorem of Bruhat-Tits, there exist smooth group schemes $\mathcal{G}_\theta$ over *Spec*(A) such that the group $\mathcal{G}_\theta(A) = \mathcal{P}_\theta(K)$ and the group scheme is uniquely determined upto unique isomorphism by its A–valued points.

Our theorem

- The conjugacy classes at the C_i's allows us to define a m-ple of points in the rational Weyl alcove which we denote by $\boldsymbol{\theta_\tau} = \{\theta_i\}$. These correspond to the rational "parabolic" weights at the points of $\mathcal{R}$ in the classical sense of Seshadri and Mehta-Seshadri.

Our theorem

- The conjugacy classes at the C_i's allows us to define a m-ple of points in the rational Weyl alcove which we denote by $\boldsymbol{\theta_\tau} = \{\theta_i\}$. These correspond to the rational "parabolic" weights at the points of $\mathcal{R}$ in the classical sense of Seshadri and Mehta-Seshadri.
- Since the genus $g \geq 2$, one can prove that there exists a Galois cover $p : Y \to X$, with Galois group Γ, ramified precisely at $\mathcal{R}$ with the prescribed ramification indices d_i.

Our theorem

- The conjugacy classes at the C_i's allows us to define a m-ple of points in the rational Weyl alcove which we denote by $\boldsymbol{\theta}_{\boldsymbol{\tau}} = \{\theta_i\}$. These correspond to the rational "parabolic" weights at the points of $\mathcal{R}$ in the classical sense of Seshadri and Mehta-Seshadri.
- Since the genus $g \geq 2$, one can prove that there exists a Galois cover $p : Y \to X$, with Galois group Γ, ramified precisely at $\mathcal{R}$ with the prescribed ramification indices d_i.
- Let $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}}, X}$ be a parahoric Bruhat-Tits group scheme associated to $\boldsymbol{\tau}$. This is obtained by a gluing of the “constant" group scheme with fibre G with the parahoric group schemes at the points of $\mathcal{R}$ with parahoric structures determined by $\boldsymbol{\theta}_{\boldsymbol{\tau}} = \{\theta_i\}$.

- The set $M_X(\mathcal{G}_{\theta_{\tau},X})$ of S–equivalence classes of semistable $\mathcal{G}_{\theta_{\tau},X}$–torsors on X gets a natural structure of an irreducible normal projective variety of dimension

$$dim(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\theta_{\tau}) \tag{6}$$

In fact, the variety $M_X(\mathcal{G}_{\theta_{\tau},X})$ is the coarse moduli space for the functor of isomorphism classes of $\mathcal{G}_{\theta_{\tau},X}$–torsors on X.

- The set $M_X(\mathcal{G}_{\theta_{\boldsymbol{\tau}},X})$ of S–equivalence classes of semistable $\mathcal{G}_{\theta_{\boldsymbol{\tau}},X}$–torsors on X gets a natural structure of an irreducible normal projective variety of dimension

$$dim(G)(g-1) + \sum_{i=1}^{m} \frac{1}{2} e(\theta_{\boldsymbol{\tau}}) \tag{6}$$

  In fact, the variety $M_X(\mathcal{G}_{\theta_{\boldsymbol{\tau}},X})$ is the coarse moduli space for the functor of isomorphism classes of $\mathcal{G}_{\theta_{\boldsymbol{\tau}},X}$–torsors on X.
- Let $\overline{K}_G = K_G/centre$. There exists a Fuchsian group π and a bijective correspondence between the space $R^{\boldsymbol{\tau}}(\pi, K_G)/\overline{K}_G$ of conjugacy classes of homomorphisms $\rho : \pi \to K_G$ of local type $\boldsymbol{\tau}$ and the set of S–equivalence classes of *semistable* $\mathcal{G}_{\theta_{\boldsymbol{\tau}},X}$–torsors.

- This correspondence induces a homeomorphism

$$R^{\tau}(\pi, K_G)/\overline{K}_G \simeq M_X(\mathcal{G}_{\theta_{\tau},X})$$

of the underlying topological spaces.

- This correspondence induces a homeomorphism

$$R^{\tau}(\pi, K_G)/\overline{K}_G \simeq M_X(\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X})$$

  of the underlying topological spaces.
- Under this correspondence, the subset of irreducible homomorphisms gets identified with isomorphism classes of *stable* $\mathcal{G}_{\boldsymbol{\theta}_{\boldsymbol{\tau}},X}$–torsors.

- This correspondence induces a homeomorphism

$$R^{\tau}(\pi, K_G)/\overline{K}_G \simeq M_X(\mathcal{G}_{\theta_{\tau},X})$$

of the underlying topological spaces.
- Under this correspondence, the subset of irreducible homomorphisms gets identified with isomorphism classes of *stable* $\mathcal{G}_{\theta_{\tau},X}$–torsors.
- The moduli space had several years prior to this work been constructed in a Tannakian manner (with Biswas and Nagaraj) following Nori. But the parahoric theme did not occur there. The stack $Bun(\mathcal{G}_{\theta_{\tau},X})$ was constructed and studied in a paper by J. Heinloth on “Uniformizations" . In the early nineties, there was an attempt by Bhosle and Ramanathan, but this on hindsight gives only a partial answer.

- This correspondence induces a homeomorphism

$$R^{\tau}(\pi, K_G)/\overline{K}_G \simeq M_X(\mathcal{G}_{\theta_{\tau},X})$$

of the underlying topological spaces.
- Under this correspondence, the subset of irreducible homomorphisms gets identified with isomorphism classes of *stable* $\mathcal{G}_{\theta_{\tau},X}$–torsors.
- The moduli space had several years prior to this work been constructed in a Tannakian manner (with Biswas and Nagaraj) following Nori. But the parahoric theme did not occur there. The stack $Bun(\mathcal{G}_{\theta_{\tau},X})$ was constructed and studied in a paper by J. Heinloth on "Uniformizations" . In the early nineties, there was an attempt by Bhosle and Ramanathan, but this on hindsight gives only a partial answer.
- In a paper with Biswas and Yashonidhi Pandey, we give a differential geometric picture of this story in the spirit of Atiyah-Bott-Donaldson-Ramanathan-Subramaniam.

Degenerations

- $k = \mathbb{C}$

Degenerations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k

Degenerations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k
- Fix (T, B, G).

Degenerations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k
- Fix (T, B, G).
- Let ℓ be the rank of G.

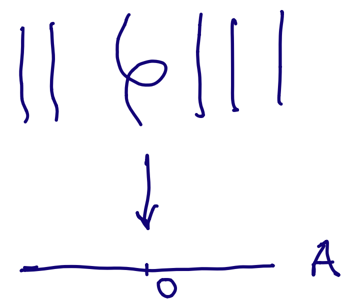
Degenerations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k
- Fix (T, B, G).
- Let ℓ be the rank of G.
- $A = \mathit{Spec}\ k[[t]]$, $K = \mathit{Spec}\ k((t))$ and $o \in A$ the closed point.

Degenerations

- $k = \mathbb{C}$
- G an almost simple, simply-connected, connected algebraic group over k
- Fix (T, B, G).
- Let ℓ be the rank of G.
- $A = Spec\ k[[t]]$, $K = Spec\ k((t))$ and $o \in A$ the closed point.
-

$$\begin{array}{c} C_A \\ \Big\downarrow {\scriptstyle projective, flat} \\ o \in A \end{array} \qquad (7)$$



C_K a smooth projective curve of genus $g \geq 1$ and $C_o = (C, c)$ the closed fiber.
Assume further that C_A is regular over k.

Aim

- $Bun_G(C_K)$- stack of G-bundles on C_K

Aim

- $Bun_G(C_K)$- stack of G-bundles on C_K
- $Bun_G^{ss}(C_K)$ open substack of Ramanathan (semi)stable G-bundles.

Aim

- $Bun_G(C_K)$- stack of G-bundles on C_K
- $Bun_G^{ss}(C_K)$ open substack of Ramanathan (semi)stable G-bundles.
- Aim: To construct a flat degeneration of $Bun_G(C_K)$ and consequently a degeneration of $Bun_G^{ss}(C_K)$.

History

- The question of constructing a relative compactification of $Pic^d(C_K)$ was done in full-generality by Cyril d'Souza in his Ph.D thesis, Oda-Seshadri and Altman-Kleiman.

History

- The question of constructing a relative compactification of $Pic^d(C_K)$ was done in full-generality by Cyril d'Souza in his Ph.D thesis, Oda-Seshadri and Altman-Kleiman.
- All these can be recovered from Simpson's general construction.

History

- The question of constructing a relative compactification of $Pic^d(C_K)$ was done in full-generality by Cyril d'Souza in his Ph.D thesis, Oda-Seshadri and Altman-Kleiman.
- All these can be recovered from Simpson's general construction.
- The idea goes back to a note by Mayer-Mumford.

History

- The question of constructing a relative compactification of $Pic^d(C_K)$ was done in full-generality by Cyril d'Souza in his Ph.D thesis, Oda-Seshadri and Altman-Kleiman.
- All these can be recovered from Simpson's general construction.
- The idea goes back to a note by Mayer-Mumford.
- The limiting objects are "compactified jacobians"

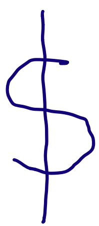
$$\overline{Pic_{d,L}(C_o)}. \tag{8}$$

History

- The question of constructing a relative compactification of $Pic^d(C_K)$ was done in full-generality by Cyril d'Souza in his Ph.D thesis, Oda-Seshadri and Altman-Kleiman.
- All these can be recovered from Simpson's general construction.
- The idea goes back to a note by Mayer-Mumford.
- The limiting objects are "compactified jacobians"

$$\overline{Pic_{d,L}(C_o)}. \tag{8}$$

- Here if C_o is not irreducible (as we have assumed), then one has to choose a polarisation L on C_o. "The" example would be the "dollar curve". This is the starting point of Oda-Seshadri.



- Then what is needed to compactify the $\overline{Pic_d(C_o)}$ are "depth 1" sheaves F of rank 1, i.e. every non-zero subsheaf has support of dim 1, the "pure" sheaves in Simpson.

- Then what is needed to compactify the $\overline{Pic_d(C_o)}$ are "depth 1" sheaves F of rank 1, i.e. every non-zero subsheaf has support of dim 1, the "pure" sheaves in Simpson.
- F is "torsion-free" in the sense that each element of $\mathfrak{m}_c$ which is non-zero on any (analytic) component of C_o through $c \in C_o$ is not a zero-divisor of F.

- Then what is needed to compactify the $\overline{Pic_d(C_o)}$ are "depth 1" sheaves F of rank 1, i.e. every non-zero subsheaf has support of dim 1, the "pure" sheaves in Simpson.
- F is "torsion-free" in the sense that each element of $\mathfrak{m}_c$ which is non-zero on any (analytic) component of C_o through $c \in C_o$ is not a zero-divisor of F.
- The key point is that in the general reducible stable curve cases, one has to work with objects which are "semistable" even in the rank 1 case.

Multidegree

- One can define "multi-degree" $(\lambda_1, \cdots, \lambda_s)$ of the ample L on C_o and the multi-rank $r = (r_1, \cdots, r_s)$ of F where s denotes the number of components.

Multidegree

- One can define "multi-degree" $(\lambda_1, \cdots, \lambda_s)$ of the ample L on C_o and the multi-rank $r = (r_1, \cdots, r_s)$ of F where s denotes the number of components.
- Seshadri slope

$$\mu_L(F) = \frac{\chi(F)}{\sum \lambda_i r_i}. \tag{9}$$

Multidegree

- One can define "multi-degree" $(\lambda_1, \cdots, \lambda_s)$ of the ample L on C_o and the multi-rank $r = (r_1, \cdots, r_s)$ of F where s denotes the number of components.
- Seshadri slope

$$\mu_L(F) = \frac{\chi(F)}{\sum \lambda_i r_i}. \tag{9}$$

- Torsion-free F of rank 1 is (semi)stable if and only if

$$\forall 0 \neq E \subset F, \mu_L(E)(\leq) < \mu_L(F). \tag{10}$$

Higher rank case $GL(n)$ for $n > 1$

- Gieseker in 1982 for rank 2 and degree 1

Higher rank case $GL(n)$ for $n > 1$

- Gieseker in 1982 for rank 2 and degre 1
- Seshadri in Asterisque

Higher rank case $GL(n)$ for $n > 1$

- Gieseker in 1982 for rank 2 and degree 1
- Seshadri in Asterisque
- Later in 1999 Nagaraj-Seshadri generalized for all rank and all degree

Higher rank case $GL(n)$ for $n > 1$

- Gieseker in 1982 for rank 2 and degree 1
- Seshadri in Asterisque
- Later in 1999 Nagaraj-Seshadri generalized for all rank and all degree
- Around 1999 Ivan Kausz constructed the stack and also obtained a modular compactification of GL(n) by these methods.

Seshadri's approach

Seshadri's approach was to compactify *à* la Picard

- by adding torsion-free sheaves

Seshadri's approach

Seshadri's approach was to compactify *à* la Picard

- by adding torsion-free sheaves
- 'semistable' via GIT.

Seshadri's approach

Seshadri's approach was to compactify *à* la Picard

- by adding torsion-free sheaves
- 'semistable' via GIT.
- Flatness of degeneration

$$\begin{array}{c} \mathcal{M}_A^{tf} \\ \downarrow \\ A \end{array} \tag{11}$$

gets reduced to a question over A of *reducedness* of

$$Z_{r,A} = \{(X, Y, t) \in M_{r\times r} \times M_{r\times r} \times A | XY - YX = t\} \tag{12}$$

at $(0, 0)$.

Faltings' approach

- In the Asterisque, Seshadri gives a proof (due to Cowsik) for rank 2. The general case is a consequence of a paper by E. Strickland in 1987.

Faltings' approach

- In the Asterisque, Seshadri gives a proof (due to Cowsik) for rank 2. The general case is a consequence of a paper by E. Strickland in 1987.
- Faltings reproved this in 1994 and also generalized these results to Sp_{2r} and O_r.

Faltings' approach

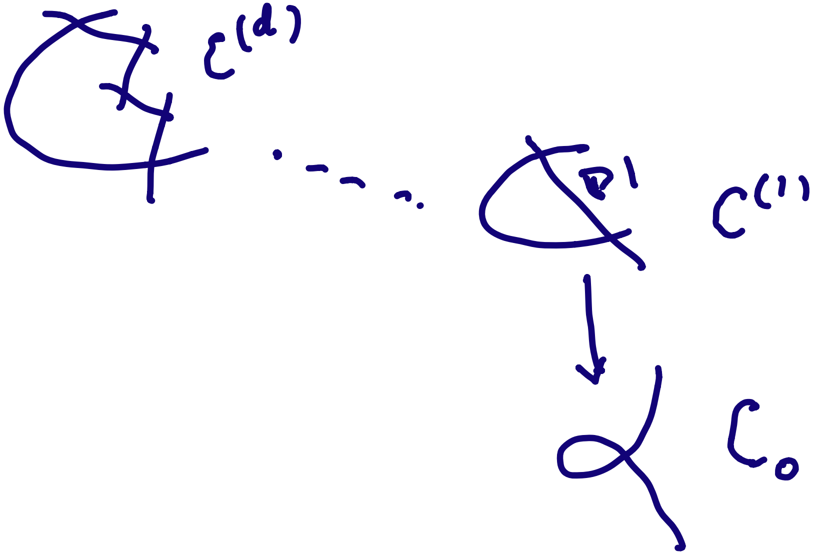
- In the Asterisque, Seshadri gives a proof (due to Cowsik) for rank 2. The general case is a consequence of a paper by E. Strickland in 1987.
- Faltings reproved this in 1994 and also generalized these results to Sp_{2r} and O_r.
- These questions are connected to singularities of "local models" for Shimura Varieties of the PEL type. See work of Pappas, Rapoport.

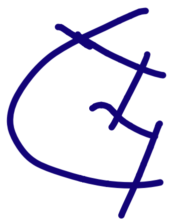
Faltings' approach

- In the Asterisque, Seshadri gives a proof (due to Cowsik) for rank 2. The general case is a consequence of a paper by E. Strickland in 1987.
- Faltings reproved this in 1994 and also generalized these results to Sp_{2r} and O_r.
- These questions are connected to singularities of "local models" for Shimura Varieties of the PEL type. See work of Pappas, Rapoport.
- For $G = \mathrm{SL}(n)$, Faltings' approach works only for $\mathrm{SL}(2) = Sp_2$ and there has been no progress for general G.

Gieseker’s approach

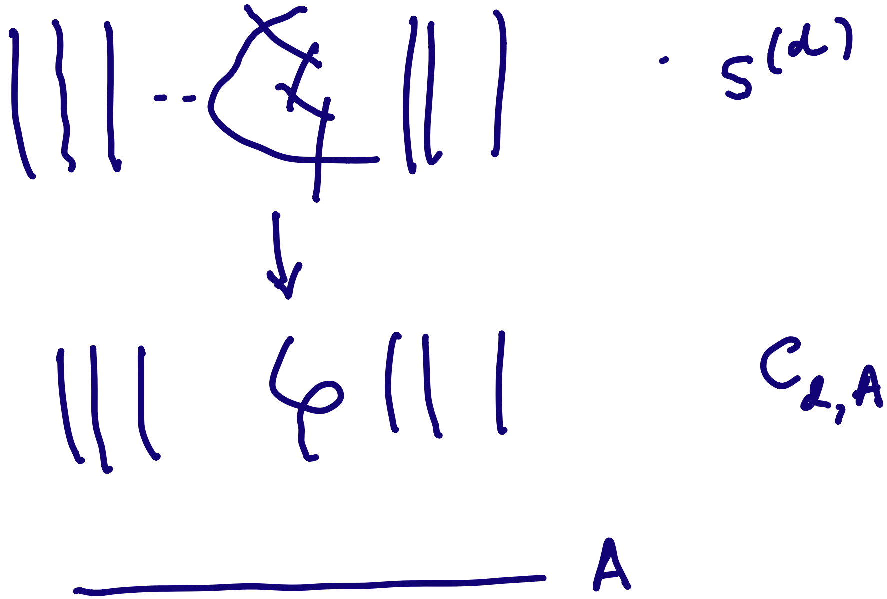
The basic idea is to get "limiting objects" i.e. compactify the moduli of vector bundles on the nodal curve by not adding torsion-free sheaves but by adding semistable curves $C^{(d)}$ with fixed stable model $= (C, c)$ and vector bundles on the curves $C^{(d)}$ with restrictions (called "admissible bundles").



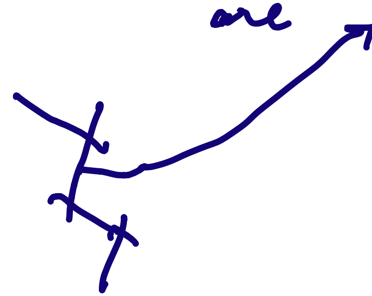
$C^{(d)} =$  is a "semistable" curve with fixed stable model C_0.

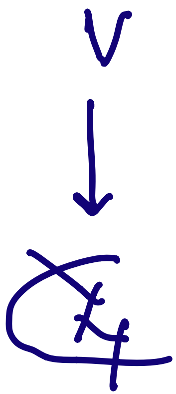
Diagrams illustrating Gieseker's approach

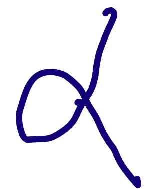
Work with surfaces:



– Work with "admissible" bundles on $C^{(d)}$

– V s.t $V|$  are $\oplus \mathcal{O}^{a_i} \oplus \mathcal{O}(1)^{b_i}$



and s.t $p_*(V)$ is torsion-free on 

Basic picture

Consider $k[[t]] \to k[[t]]$ given by $t \mapsto t^d$. This induces a map $A \to A$ and a diagram:

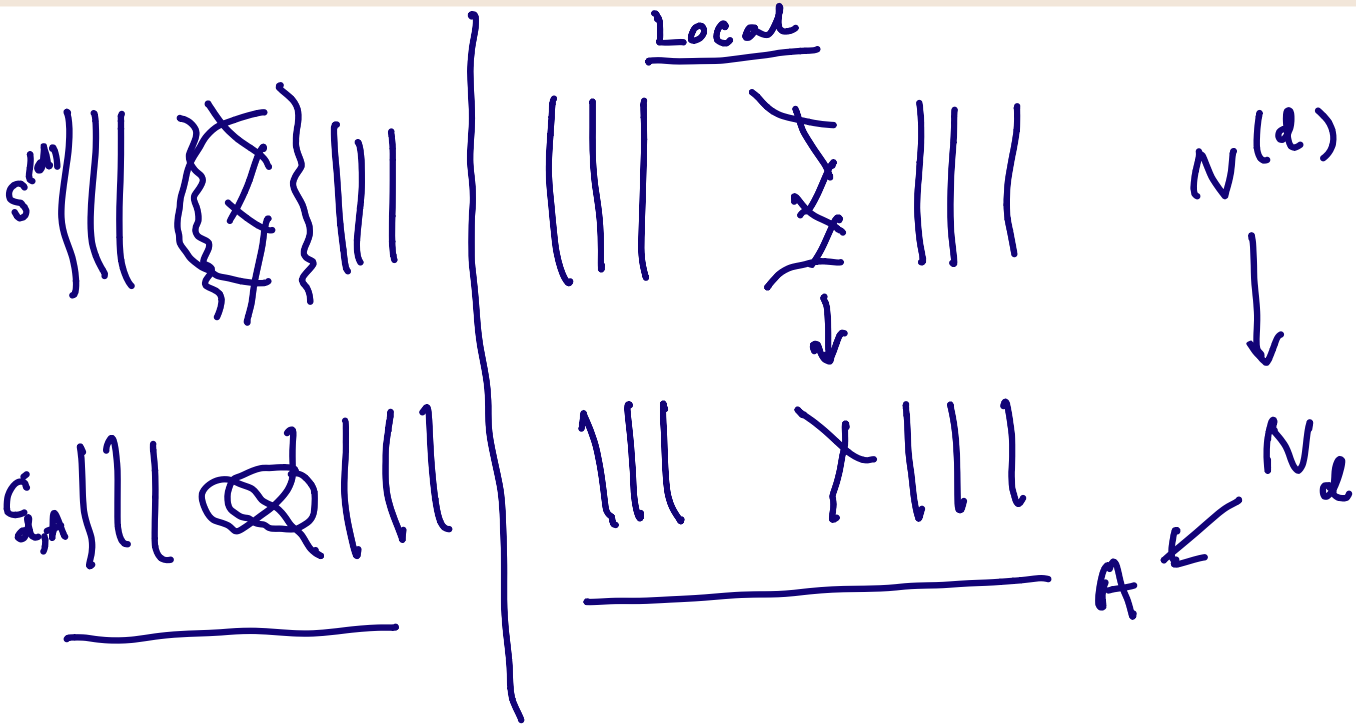
$$\begin{array}{ccc} C_{d,A} & \longrightarrow & C_A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \end{array} \tag{13}$$

where $C_{d,A}$ in a neighbourhood of the node is

$$N_d := Spec \frac{k[[x,y,t]]}{(xy - t^d)} \tag{14}$$

N_d is an affine normal surface which has A_d-type of singularities.

Local analysis



Admissibility

- Let $p_d : C^{(d)} \to C$. A vector bundle V is admissible if the restriction to each rational curve in the preimage of c, i.e. $V|_{\mathbb{P}^1}$ is such that

Admissibility

- Let $p_d : C^{(d)} \to C$. A vector bundle V is admissible if the restriction to each rational curve in the preimage of c, i.e. $V|_{\mathbb{P}^1}$ is such that
- it is a direct sum of only $\mathcal{O}$'s and $\mathcal{O}(1)$'s with at least one $\mathcal{O}(1)$ and

Admissibility

- Let $p_d : C^{(d)} \to C$. A vector bundle V is admissible if the restriction to each rational curve in the preimage of c, i.e. $V|_{\mathbb{P}^1}$ is such that
- it is a direct sum of only $\mathcal{O}$'s and $\mathcal{O}(1)$'s with at least one $\mathcal{O}(1)$ and
- further, $p_{d*}(V)$ is torsion-free on C.

Admissibility

- Let $p_d : C^{(d)} \to C$. A vector bundle V is admissible if the restriction to each rational curve in the preimage of c, i.e. $V|_{\mathbb{P}^1}$ is such that
- it is a direct sum of only $\mathcal{O}$'s and $\mathcal{O}(1)$'s with at least one $\mathcal{O}(1)$ and
- further, $p_{d*}(V)$ is torsion-free on C.
- *I would like to view this differently, and this will be the starting point towards the general G case*.

The NS-morphism

- We have seen the local description of the moduli space of torsion-free sheaves

The NS-morphism

- We have seen the local description of the moduli space of torsion-free sheaves
- I view the Nagaraj-Seshadri morphism $\begin{array}{c}\mathcal{M}_A^{Gies}(n)\\ \downarrow{\scriptstyle \eta}\\ \mathcal{M}_A^{tf}(n)\end{array}$ as a resolution of singularities.

The NS-morphism

- We have seen the local description of the moduli space of torsion-free sheaves
- I view the Nagaraj-Seshadri morphism $\begin{array}{c}\mathcal{M}_A^{Gies}(n)\\ \big\downarrow \eta \\ \mathcal{M}_A^{tf}(n)\end{array}$ as a resolution of singularities.
- In particular if $F \in \mathcal{M}_o^{tf}$ is such that at $c \in C$

$$F|_c \simeq \mathfrak{m}^n,$$

then the exceptional $\eta^{-1}(F) \simeq \overline{PGL(n)}$ is isomorphic to the wonderful compactification by Nagaraj-Seshadri.

Moduli problems for G almost simple, simply-connected

- There is no good "intrinsic" analogue of $M_A^{tf}(n)$ for a general G.
- When G is "classical" this was carried out by Faltings in 1994. This is a generalization of the torsion-free approach with limiting objects obtained by using "orthogonal" or "symplectic" torsion-free sheaves.

Jump to G case

The basic question: Given a G-torsor

$$\begin{array}{c} E_K \\ \downarrow G \\ C_K \end{array}$$

what would be a candidate to extend E_K to an object E_A over C_A?

- There is no torsion-free analogue now.

Murphy's First Law: Anything that can go wrong will go wrong!

Jump to G case

The basic question: Given a G-torsor

$$\begin{array}{c} E_K \\ \downarrow{\scriptstyle G} \\ C_K \end{array}$$

what would be a candidate to extend E_K to an object E_A over C_A?

- There is no torsion-free analogue now.
- Can something be said about the kind of singularity of the extended moduli, even if it be, heuristic?

Murphy's First Law: Anything that can go wrong will go wrong!

Jump to G case

The basic question: Given a G-torsor

$$\begin{array}{c} E_K \\ \Big\downarrow G \\ C_K \end{array}$$

what would be a candidate to extend E_K to an object E_A over C_A?

- There is no torsion-free analogue now.
- Can something be said about the kind of singularity of the extended moduli, even if it be, heuristic?
- Gieseker moduli is better since one could hope that the limits are G-torsors (following the vector bundle case.

Murphy's First Law: Anything that can go wrong will go wrong!

Jump to G case

The basic question: Given a G-torsor

$$\begin{array}{c} E_K \\ \downarrow G \\ C_K \end{array}$$

what would be a candidate to extend E_K to an object E_A over C_A?

- There is no torsion-free analogue now.
- Can something be said about the kind of singularity of the extended moduli, even if it be, heuristic?
- Gieseker moduli is better since one could hope that the limits are G-torsors (following the vector bundle case.
- How does one recover the "Gieseker" list.

Murphy's First Law: Anything that can go wrong will go wrong!

Basic idea

Let $C_{d,A} \to C_A$ be the base change by $k[[t]] \to k[[t]]$ given by $t \mapsto t^d$. Let

$$\begin{array}{c} S^{(d)} \\ \downarrow {\scriptstyle p_d} \\ C_{d,A} \end{array} \tag{15}$$

be the minimal resolution of singularities. The pre-image of the curve $C_o = (C, c)$ is $C^{(d-1)}$. Locally we have:

$$\begin{array}{c} N^{(d)} \\ \downarrow {\scriptstyle p_d} \\ N_d \end{array} \tag{16}$$

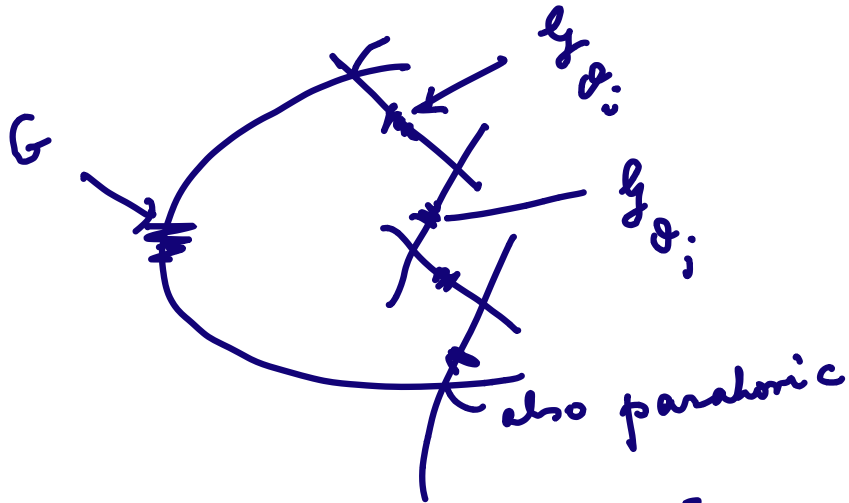
Here $N^{(d)}$ is the minimal resolution.

- On each semi-stable curve $C^{(d)}, 0 \leq d \leq \ell$, we define smooth affine group schemes $\{\mathcal{H}^{G}_{\tau, C^{(d)}}\}_{\tau}$ indexed by “types" which gives the list of “admissible" group schemes.

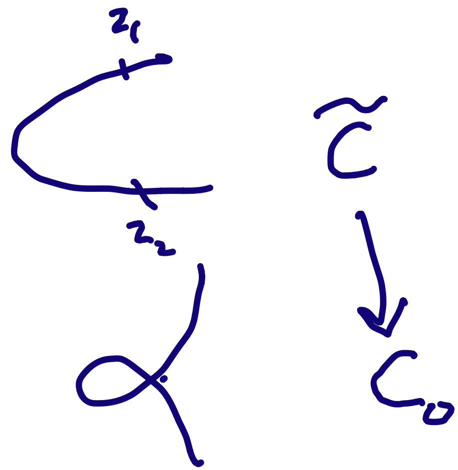
- On each semi-stable curve $C^{(d)}, 0 \leq d \leq \ell$, we define smooth affine group schemes $\{\mathcal{H}^G_{\tau, C^{(d)}}\}_\tau$ indexed by "types" which gives the list of "admissible" group schemes.
- One can then define an admissible torsor E for these group schemes on $C^{(d)}$, where the local triviality of E is constrained by the condition $E|_{N^{(d)}} \simeq \mathcal{H}^G_{\tau, N^{(d)}}$.

- On each semi-stable curve $C^{(d)}, 0 \leq d \leq \ell$, we define smooth affine group schemes $\{\mathcal{H}^G_{\tau, C^{(d)}}\}_\tau$ indexed by “types" which gives the list of “admissible" group schemes.
- One can then define an admissible torsor E for these group schemes on $C^{(d)}$, where the local triviality of E is constrained by the condition $E|_{N^{(d)}} \simeq \mathcal{H}^G_{\tau, N^{(d)}}$.
- A Gieseker G-torsor is a triple $(C^{(d)}, \mathcal{H}^G_{\tau, C^{(d)}}, E)$.

List of Gieseker Bundles



<u>Normalizations</u>.



Main Theorem

- The stack $\mathrm{Gies}_G(C_A)$ of Gieseker torsors is an algebraic stack locally of finite type, which is regular and flat over A. Over K we have an identification $\mathrm{Gies}_G(C_K) = \mathrm{Bun}_G(C_K)$ with the stack of G-torsors on the smooth projective curve C_K. Further the closed fibre $\mathrm{Gies}_G(C_o) \subset \mathrm{Gies}_G(C_A)$ is a divisor with normal crossings with $\ell + 1$ smooth components indexed by the extended Dynkin diagram.

In a recent work, Belkale and Gibney have constructed a global moduli for $SL(n)$-torsors over $\bar{M}_g$ using "conformal blocks". The space does not have a modular description but conforms to the more general philosophy of degenerations.

Main Theorem

- The stack $\mathrm{Gies}_G(C_A)$ of Gieseker torsors is an algebraic stack locally of finite type, which is regular and flat over A. Over K we have an identification $\mathrm{Gies}_G(C_K) = \mathrm{Bun}_G(C_K)$ with the stack of G-torsors on the smooth projective curve C_K. Further the closed fibre $\mathrm{Gies}_G(C_o) \subset \mathrm{Gies}_G(C_A)$ is a divisor with normal crossings with $\ell + 1$ smooth components indexed by the extended Dynkin diagram.
- The open substack $\mathrm{Gies}_G(C_A)^{\mathtt{L}-ss}$ of $\mathtt{L}$-(semi)stable Gieseker torsors has a coarse space which parametrizes S-equivalence classes of Gieseker torsors and which provides a *proper flat degeneration* of the moduli scheme of μ-(semi)stable G-torsors on C_K.

In a recent work, Belkale and Gibney have constructed a global moduli for $SL(n)$-torsors over $\bar{M}_g$ using "conformal blocks". The space does not have a modular description but conforms to the more general philosophy of degenerations.