* Non-vanishing modulo p of values of a modular form, an introduction

Haruzo Hida
UCLA, Los Angeles, CA 90095-1555, U.S.A.

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*Let M/F be CM quadratic extension with integer ring extension R/O. Choose a CM type Σ with $\mathrm{Isom}_{\mathrm{field}}(M,\overline{\mathbb{Q}}) = \Sigma \sqcup \Sigma c$ for complex conjugation c on M, and the p-adic places also split into $\Sigma_p \sqcup \Sigma_p c$. Choose a prime $2 < p \in \mathbb{Z}$ and a O-prime ideal $\mathfrak{l} \nmid (p)$. Let $R_n = O + \mathfrak{l}^n R$ and $Cl_n := \mathrm{Pic}(R_n)$, $Cl_n^- := \mathrm{Coker}(\mathrm{Pic}(O) \to \mathrm{Pic}(R_n))$ and $Cl_\infty^? := \varprojlim_n Cl_n^?$. The image of the fractional R-ideal group prime to $p\mathfrak{l}$ in Cl_∞^- is denoted by Cl^{alg}, namely, for a fractional R-ideal $\mathfrak{A}$, defining R_n-ideal $\mathfrak{A}_n$ so that $\widehat{\mathfrak{A}}_n = R_{n,\mathfrak{l}} \times \widehat{\mathfrak{A}}^{(\mathfrak{l})}$ and $[\mathfrak{A}] = \varprojlim_n [\mathfrak{A}_n] \in Cl_\infty^-$. Decompose $Cl_\infty^- = \Delta^- \times \Gamma : [\mathfrak{Q}] \mapsto ([\mathfrak{Q}]_\Delta, [\mathfrak{Q}]_\Gamma)$ for a finite group Δ^- and $\mathbb{Z}_\ell$-free Γ with $\Gamma_n = \mathrm{Im}(\Gamma \to Cl_n^-)$. Let $\mathcal{Q} \subset Cl^{alg}$ such that $[\mathfrak{Q}]_\Gamma/[\mathfrak{Q}']_\Gamma \notin Cl^{alg}$ as long as $\mathfrak{Q} \neq \mathfrak{Q}'$ in $\mathcal{Q}$. For $[\mathcal{A}] \in \mathrm{Pic}(R_n)$, we define a point $x(\mathcal{A}) = x([\mathcal{A}])$ in the Hilbert modular prime-to-p Shimura variety Sh of carrying the CM abelian variety $X(\mathcal{A})$ of CM-type Σ with $H^1(X(\mathcal{A}),\mathbb{Z}) \cong \mathcal{A}$.

§0. **Fundamental questions.** We embed Pic(R_n) into $Sh^{\mathcal{Q}} := \prod_{\mathcal{Q}} Sh$ by $[\mathcal{A}] \mapsto \mathbf{s}(\mathcal{A}) := (x([\mathcal{A}][\mathfrak{Q}]_{\Gamma}]))_{\mathfrak{Q}\in\mathcal{Q}}$. If a set Ξ of CM points is dense in $Sh^{\mathcal{Q}}$, a $\mathcal{Q}$-tuple of mod p modular forms rarely vanish on Ξ. We ask for a level K quotient $Sh_K = Sh/K$

When a thin set Ξ of CM points is Zariski dense in mod p Hilbert modular Shimura varieties Sh_K and its products?

Assume that $p > 2$ is unramified in $F/\mathbb{Q}$. Fix an algebraic closure $\mathbb{F}$ (resp. $\overline{\mathbb{Q}}_\ell$) of $\mathbb{F}_p$ (resp. $\mathbb{Q}_\ell$). The variety $Sh^{\mathcal{Q}}$ has an action of $GL_2(F_{\mathbb{A}}^{(p\infty)})$ and the stabilizer of $x(R)$ is a torus $M^\times \hookrightarrow GL(2)_{/F}$.

Black Box Theorem. *Let V be an irreducible component of $Sh_{/\mathbb{F}}$. If an irreducible component X of the Zariski closure of a CM point set Ξ in $V^{\mathcal{Q}}$ stable under a p-adically open subgroup of $M^\times$ has dimension > 0 and $\Xi \cap X \neq \emptyset$, $X_K = V_K^{\mathcal{Q}}$ for any K.*

My talk is based on this theorem (Theorem 3.20 in my Annals vol.**172** paper). However we do not touch the proof of this theorem in this lecture. Note $Sh_{/Sh_K}$ is an étale pro-variety.

§1. Density Theorem.

Let $\underline{n} = \{n_0, n_1, n_2, \dots\}$ be an infinite sequence of integers, and put $\Xi = \Xi_{\underline{n},j} = \bigsqcup_i \{\mathbf{s}(\mathcal{A}) \in V^{\mathcal{Q}} | [\mathcal{A}] \in K_j^{n_i} :=$ Ker$(Cl_{n_i} \to Cl_j)\}$ for a fixed $0 < j \in \mathbb{Z}$.

Density Theorem. *If $\underline{n}$ contains an arithmetic progression, then Ξ is Zariski dense in $V^{\mathcal{Q}}$.*

For natural numbers $\mathbb{N}$, $\{n \in \mathbb{N} | [\mathcal{A}] \in K_0^n,\ x([\mathcal{A}]_n) \in V\}$ is an arithmetic progression (or empty), as irreducible components of Sh containing CM-points of type Σ is indexed by $N_{M/F}(Cl_0)$. We actually prove if $\underline{n}$ contains an arithmetic progression, $\overline{\Xi}$ has a positive dimensional irreducible component containing a point of Ξ and deduce the density theorem from the black box theorem.

Let $\mathbb{C}_\ell$ be the ℓ-adic completion of $\overline{\mathbb{Q}}_\ell$, and fix a discrete valuation ring $\mathcal{W}$ with residue field $\mathbb{F}$ and completion W.

§2. **A pathologic example.** Here is an example of an affine pro-variety $V = V_{\infty/\mathbb{C}}$ étale over the line $V_0 = \mathrm{Spec}(\mathbb{C}[X])$ such that the Zariski closure of an infinite set $\Xi \subset V(\mathbb{C})$ does not have positive dimensional irreducible component containing a point of Ξ (due to Akshay Venkatesh).

Let $V_n := V_0 \times \mathbb{Z}/2^n\mathbb{Z}$ and the projection $\mathbb{Z}/2^m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2^n\mathbb{Z}$ for $m > n$ induces étale morphism $V_m \twoheadrightarrow V_n$. Regard $(X - j) \subset \mathbb{C}[X]$ for $j \in \mathbb{C}$ as a closed point $j \in V_0(\mathbb{C})$. We define $V := \varprojlim_n V_n \cong V_0 \times \mathbb{Z}_2$ and $\Xi = \{(j, 2^j) \in V | j = 1, 2, \ldots\}$. Write

$$\Xi_n := \{(j, 2^j \mod (2^n)) \in V_n | j = 1, 2, \ldots\}.$$

Then the Zariski closure in V_n of Ξ_n

$$\overline{\Xi}_n = V_0 \sqcup \{(j, 2^j) \in V_n | j = 1, 2, \ldots, n-1\}.$$

The Zariski closure

$$\overline{\Xi} = \varprojlim_n \overline{\Xi}_n = V_0 \sqcup \Xi.$$

One 1-dimensional component $(V_0 \times 0)$ and infinite 0-dimensional components Ξ.

§3. **Shimura variety.** The Shimura variety $Sh_{/\mathcal{W}}$ classifies triples (X, Λ, w) of abelian varieties X with $O \hookrightarrow \mathrm{End}(X)$ and $\dim X = \mathrm{rank}\, O$, level structure $w = {}^t(w_1, w_2)$ which is a basis of the Tate module $V(X) := TX \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(p\infty)}$ over $F_{\mathbb{A}^{(p\infty)}}$, and a polarization class $\Lambda : X^t \cong X \otimes_O \mathfrak{c}$ for a strict ideal class $[\mathfrak{c}] \in Cl_F^+$. For an open subgroup K with $\det(K) = \widehat{O}^\times$, geometrically irreducible components V_K of Sh_K is indexed by $[\mathfrak{c}] \in Cl_F^+$. Put $V = \varprojlim_K V_K$ which is a geometrically irreducible component of Sh.

Fix a choice of $\widehat{O}$-basis $w_R = {}^t(1, w_2)$ of $\widehat{R}$ and resulting regular representation $\rho : M_{\mathbb{A}} \to M_2(F_{\mathbb{A}})$. Choose $\mathcal{W}$-model $X(\mathcal{A})$ of the abelian variety $X(\mathcal{A})(\mathbb{C}) = \mathbb{C}/\mathcal{A}^\Sigma$ of CM type Σ; so, $TX(A) = \widehat{\mathcal{A}}^{(p)}$. Here is the level structure $w_{\mathcal{A}}$ on $TX(\mathcal{A})$:
Outside $\mathfrak{l}$: $w_{\mathcal{A}}^{(\mathfrak{l})} = \rho(a) w_R^{(\mathfrak{l})}$ for $a \in M_{\mathbb{A}}^{(\mathfrak{l}),\times}$ with $\mathcal{A}^{(\mathfrak{l})} = a\widehat{R}^{(\mathfrak{l})}$,
At $\mathfrak{l}$, $w_{\mathcal{A},\mathfrak{l}} = {}^t(1, \varpi_{\mathfrak{l}}^n w_{2,\mathfrak{l}}) = \alpha_{\mathfrak{l}}^n w_{R,\mathfrak{l}}$ for $\alpha_{\mathfrak{l}} = \mathrm{diag}[1, \varpi_{\mathfrak{l}}]$.

A polarization $\Lambda(\mathcal{A})$ is given by a Riemann form $(x, y) \mapsto \mathrm{Tr}_{M/\mathbb{Q}}(\delta x y^c)$ for a fixed $\delta \in M$ prime to p with $\delta^c = -\delta$, which we forget about.

§4. GL(2)-action.

The triple $(X(\mathcal{A}), \Lambda(\mathcal{A}), w_{\mathcal{A}})$ gives a point $x(\mathcal{A}) \in Sh$. Each $g \in \mathrm{GL}_2(F_{\mathbb{A}})$ acts on Sh by $(X, w) \mapsto (X, gw)$ (left action). We have $\rho(a)(x(\mathcal{A})) = x(a\mathcal{A}))$ for $a \in M^{\times}_{\mathbb{A}^{(\mathfrak{l}p\infty)}}$; so, $\alpha \in M^{\times}$ with $\alpha \equiv 1 \mod \mathfrak{l}^n R_{\mathfrak{l}}$ fixes $x(\mathcal{A}) \in Sh$ if $\mathcal{A}$ is a proper R_n-ideal prime to $\mathfrak{l}$, and $a \in M^{\times}_{\mathbb{A}^{(\mathfrak{l}p\infty)}}$ acts on $C := \bigsqcup_n \{\mathbf{s}(\mathcal{A}) | \mathcal{A} \in Cl_n\} \subset Sh^{\mathcal{Q}}$.

Prepare the product $V_K^{\mathcal{Q}}$ of $\mathcal{Q}$ copies of an irreducible component V_K of the Shimura variety $Sh_{K/\mathbb{F}}$, embed C into $Sh_K^{\mathcal{Q}}$ by $x(\mathcal{A}) \mapsto \mathbf{s}(\mathcal{A}) = (x(\mathcal{A}[\mathfrak{Q}]_{\Gamma}))_{\mathfrak{Q} \in \mathcal{Q}} \in Sh^{\mathcal{Q}}$. Since $[\mathfrak{Q}]_{\Gamma} \in \mathcal{Q}$ is given by $\varprojlim_n [\delta_n]_n$ for a proper R_n-ideal δ_n, we have $[\mathcal{A}[\mathfrak{Q}]_{\Gamma}]_n = [\mathcal{A}\delta_n]_n$.

Question: When Ξ_K in $V_K^{\mathcal{Q}}$ is Zariski dense for an infinite set $\Xi \subset C \cap V^{\mathcal{Q}}$.

If $[\mathfrak{Q}]_{\Gamma} = [\mathfrak{Q}']_{\Gamma}\mathfrak{A}$ with $\mathfrak{A} \in Cl^{alg}$, $x(\mathcal{A}[\mathfrak{Q}']_{\Gamma}) = \langle\mathfrak{A}\rangle(x(\mathcal{A}[\mathfrak{Q}]_{\Gamma}))$ for a morphism $\langle\mathfrak{A}\rangle : Sh \to Sh$ such that ; so, the answer is negative.

§**5. Goal.** Identify $\mu_{\ell^\infty} = \mu_{\ell^\infty}(\mathbb{F}) = \mu_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$. We regard the set of continuous characters $\widehat{\Gamma} := \mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ as a subset of $\mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$ by sending a character χ to $(\chi(\gamma_1), \ldots, \chi(\gamma_d)) \in \mu_{\ell^\infty}^d(\overline{\mathbb{Q}}_\ell) \subset \mathbb{G}_m^d(\overline{\mathbb{Q}}_\ell)$ for a basis $\gamma_1, \ldots, \gamma_d$ of Γ over $\mathbb{Z}_\ell$. A subset $\mathcal{Z}$ of $\widehat{\Gamma}$ is said to be **Zariski dense** if $\mathcal{Z}$ is Zariski dense in $\mathbb{G}_m^d$ over $\overline{\mathbb{Q}}_\ell$.

Fix a $U(\mathfrak{l})$-eigenform $g_{/\mathcal{W}}$ of weight k and put $f_{/W} = d^\kappa g$ for $d^\kappa : \sum_\xi a_\xi q^\xi \mapsto \sum_\xi a_\xi \xi^\kappa q^\xi$ with $\kappa_\sigma \geq 0$ and $\xi^\kappa := \prod_\sigma \xi^{\kappa_\sigma \sigma}$. Let $f_{/\mathbb{F}} := f \mod \mathfrak{m}_W$. Put $f([\mathcal{A}]) := \lambda^{-1}(\mathcal{A}) f(x(\mathcal{A}))$ choosing (once and for all) a Hecke character λ of M such that $f([\mathcal{A}])$ only depends on the class $[\mathcal{A}] \in Cl_n^-$ for all n. Define a measure $d\varphi_f = d\varphi_{f,n}$ on Cl_n^- for each n so that it has volume $f([\mathcal{A}])$ at $[\mathcal{A}] \in Cl_n^-$. Fix a character $\psi : \Delta^- \to \mathbb{F}^\times$.

Non-vanishing theorem: *Suppose that there exists $\xi \in F \cap O_\mathfrak{l}$ in each class $v \in (O/\mathfrak{l}^j)^\times$ for a sufficiently large $j \gg 0$ such that the q-expansion coefficient $a(\xi, f) \neq 0$ in $\mathbb{F}$. Then the set of characters $\chi \in \widehat{\Gamma}$ such that $\int_{Cl_n^-} \chi\psi d\varphi_f \neq 0$ in $\mathbb{F}$ for some n is Zariski dense. If $O_\mathfrak{l} \cong \mathbb{Z}_\ell$, j can be taken to be r such that $\ell^r \,\|\, |\mathbb{F}_p[f, \lambda, \psi, \mu_\ell]| - 1$.*

§6. Transcendence of $Cl_\infty^- = \Gamma \times \Delta^-$.

Let $f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma) : C \to \mathbb{F}$. Then

$$\int_{Cl_n} \chi\psi d\varphi_f = \int_{\Gamma_n} \chi d\varphi_{f_\psi} = \sum_{\mathcal{A}\in\Gamma_n} \chi(\mathcal{A}) f_\psi^{\mathcal{Q}}([\mathcal{A}]).$$

The function $f_\psi^{\mathcal{Q}}([\mathcal{A}]) = \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma)$ cannot be a modular form, as the decomposition $Cl_\infty^- = \Gamma \times \Delta^-$ is transcendental ($Cl^{alg} \cap \Delta^-$ is 2-torsion). For simplicity, we assume Cl_F covers $Cl_0^+ := H^0(\mathrm{Gal}(M/F), Cl_0)$ (no 2-torsion) and $\kappa = 0$.

Application:

If g is an Eisenstein series, non-vanishing modulo p of the canonical algebraic part of $L(0, \psi^{-1}\chi^{-1}\lambda)$ for densely populated anticyclotomic χ except for the case where the value vanishes by mod p root number $= -1$. A similar result holds for cusp forms (see Hsieh's paper in Documenta **19**, 2014).

§7. Basic properties.

1. For a proper R_n-ideal $\mathcal{A}_n$ prime to $p\mathfrak{l}$, define a proper $R_{n'}$-ideal by $\widehat{\mathcal{A}}_{n'} := \widehat{\mathcal{A}}_n^{(\mathfrak{l})} \times R_{n',\mathfrak{l}}$. Put $\Gamma_n[\ell^j] := \{\gamma \in \Gamma_n | \gamma^{\ell^j} = [R_n]\}$ on $n > j$. We have three identities for $\alpha(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$:

$\alpha_{\mathfrak{l}}(x(\mathcal{A}_n)) = x(\mathcal{A}_{n+1})$ for $\alpha_{\mathfrak{l}} = \text{diag}[1, \varpi_{\mathfrak{l}}]$,

$O_{\mathfrak{l}}/\mathfrak{l}^i \cong \Gamma_n[\ell^j]$ by $u \mapsto \alpha(u/\varpi_{\mathfrak{l}}^j)x(R_n) =: x(\mathcal{A}_{u,n})$ if $n > j$,

$\begin{pmatrix} 1 & u \\ 0 & \varpi_{\mathfrak{l}}^i \end{pmatrix} (x(R_n)) = \alpha(u/\varpi_{\mathfrak{l}}^i)\alpha_{\mathfrak{l}}^i(x(R_n)) = x(\mathcal{A}_{u,n+i})$.

So choosing a generator ζ_j of μ_{ℓ^j}, we have $v = v(\chi) \in O/\mathfrak{l}^j$ such that $\chi(\mathcal{A}_u) = \zeta_j^{\text{Tr}(vu)}$. Fix $v \in O/\mathfrak{l}^j$.

2. Put $\underline{n} := \{n | \int_{\Gamma_n} \chi d\varphi_{f_\psi} = 0$ with $v(\chi) = v$ and $cond(\chi) = \mathfrak{l}^n\}$, and $\Xi = \Xi_{\underline{n},j} := \{\mathbf{s}(\mathcal{A}) | \mathcal{A} \in K_j^n, n \in \underline{n}\}$, where $\mathfrak{l}^n$ is the conductor of χ. If $\mathcal{X} = \{n | \int_{\Gamma_n} \chi d\varphi_{f_\psi} \neq 0$ with $v(\chi) = v\}$ is not Zariski dense, we show $\underline{n}$ contains a arithmetic progression. Then $f_\psi^{\mathcal{Q}}([\mathcal{A}]) = 0$ for all $x(\mathcal{A}) \in \Xi$ implies $f = 0$ on Sh, contradictory to $a(\xi, f) \neq 0$ for $\xi \in -v$. We prove this point in the third lecture.

§8. What is the topic in the next lecture?

In the next lecture, we give a description of a proof of the density theorem, and at the end, we come back to the proof of the non-vanishing theorem.

One might optimistically hope that the condition $X \cap \Xi \neq \emptyset$ might not be essential. However, as already discussed, we have a counter example of a pro-curve V with infinite set Ξ such that $X \cap \Xi = \emptyset$.