* Characters of vanishing integral and the thin point set Ξ

Haruzo Hida
UCLA, Los Angeles, CA 90095-1555, U.S.A.

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*Assume that $\mathcal{Q} \cong \Delta^-$ with $\mathfrak{Q}$ split over F. We describe the set $\mathcal{Z} = \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty})$ with $\int_{Cl_n^-} \chi\psi d\varphi_f = 0\}$ and relate it to the point set $\Xi \subset Sh^{\mathcal{Q}}$. Recall $\int_{\Gamma_n} \chi d\varphi_{f_\psi^{\mathcal{Q}}} = \sum_{\mathfrak{Q},\mathcal{A}\in\Gamma_n} \lambda\psi^{-1}(\mathfrak{Q})\chi(\mathcal{A}) f([\mathcal{A}\mathfrak{Q}^{-1}]_n[\mathfrak{Q}]_\Gamma)$. The action of $[\mathfrak{Q}]_\Gamma$ is transcendental and is incorporated into the embedding $C \hookrightarrow Sh^{\mathcal{Q}}$. So we write down $f_\psi^{\mathcal{Q}}$ as a value of a single modular form $f_\psi := \sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f|\langle\mathfrak{Q}\rangle$ first for $f|\langle\mathfrak{Q}\rangle(x(\mathcal{A})) = f(x(\mathcal{A}\mathfrak{Q}^{-1}))$.

§0. The idea when $O_{\mathfrak{l}} = \mathbb{Z}_\ell$. Descend $\mathfrak{Q} \in \mathcal{Q}$ to $\mathfrak{q} = \mathfrak{Q} \cap O$. Then $C_{\mathfrak{q}} := (\varpi_{\mathfrak{q}}^{-1}\widehat{O}w_1 + \widehat{O}w_2)/TX \subset X$ is a O-cyclic subgroup. Define for $\langle\mathfrak{Q}\rangle$ by the action of diag$[\varpi_{\mathfrak{q}}^{-1}, 1]$:

$$f|\langle\mathfrak{Q}\rangle(X, \overline{\Lambda}, w, \omega) := f(X/C_{\mathfrak{q}}, \overline{\Lambda}_{\mathfrak{Q}}, \langle\mathfrak{Q}\rangle w, \omega_{\mathfrak{Q}})\ (f|\langle\mathfrak{Q}\rangle([\mathcal{A}]) = f([\mathcal{A}\mathfrak{Q}^{-1}])),$$

where $\overline{\Lambda}_{\mathfrak{Q}}$ and $\omega_{\mathfrak{Q}}$ are the push-down of $\overline{\Lambda}$ and ω to the quotient $X/C_{\mathfrak{Q}}$. Define $f_\psi = \sum_{\mathfrak{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f|\langle\mathfrak{Q}\rangle$. Recall $v = v(\chi)$ such that $\chi([\mathcal{A}_u]) = \zeta_j^{\mathrm{Tr}(vu)}$ identifying $\Gamma_n[\mathfrak{l}^j] \cong O/\mathfrak{l}^j$ by $x(\mathcal{A}_u) = \alpha(u/\varpi_{\mathfrak{l}}^j)(x(R_n))$. We regard $(f|\langle\mathfrak{Q}\rangle)_{\mathfrak{Q}}$ a modular form on $Sh^{\mathcal{Q}}$ and evaluate it at $\Xi = \Xi_{\underline{n},j}$ defined by the following sequence $\underline{n}$.

$$\underline{n} := \{n | \int_{Cl_n^-} \chi\psi d\varphi_f = 0 \text{ for } n > j \text{ with } cond(\chi) = \mathfrak{l}^n \text{ and } v(\chi) = v\}.$$

Modifying further $f_\psi = \sum_{\mathfrak{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f_v|\langle\mathfrak{Q}\rangle$ with

$$f_v = \sum_{u \in O/\mathfrak{l}^j} \zeta_j^{Tr(vu)} f|\alpha(u/\varpi_{\mathfrak{l}}^j),$$

we show $f_v([\mathcal{A}]) = 0$ for all $\mathrm{s}([\mathcal{A}]) \in \Xi$; so, $f_v = 0$ if Ξ is Zariski dense in $V^{\mathcal{Q}}$. Note $N(\mathfrak{l})^j a(\xi, f) = a(\xi, f_v)$ as long as $\xi \equiv -v$ mod $\mathfrak{l}^j O_{\mathfrak{l}}$. We suppose $j \geq r > 0$ for r with $\ell^r \,\|\, |\mathbb{F}_p[f, \lambda, \psi, \mu_\ell]^\times|$.

§1. Geometric modular forms.

Geometric modular forms classify quadruples $(X, \overline{\Lambda}, w, \omega)$ with $(X, \overline{\Lambda}, w)_{/A} \in Sh(A)$, where ω is a generator over $O \otimes_{\mathbb{Z}} A$ of $H^0(X, \Omega_{X/A})$. A geometric modular form $f_{/B}$ $(B = W, \mathbb{F})$ is a functorial rule of assigning a value to triples $(X, \overline{\Lambda}, w, \omega)$ to satisfy the following three axioms:

(G1) For a B–algebra homomorphism $\phi : A \to A'$, we have

$$f((X, \overline{\Lambda}, w, \omega) \times_{A,\phi} A') = \phi(f(X, \overline{\Lambda}, w, \omega)).$$

(G2) f is finite at all cusps, that is, the q–expansion of f at every Tate test object does not have a pole at $q = 0$.

(G3) $f(X, \overline{\Lambda}, w, \xi\omega) = \xi^{-k\Sigma} f(X, \overline{\Lambda}, w, \omega)$ for $\xi \in T(A)$ for $T = \mathrm{Res}_{O/\mathbb{Z}}(\mathbb{G}_m)$.

Note $k\Sigma \in \mathrm{Hom}_{\text{alg. gp}}(T, \mathbb{G}_m)$ sending ξ to $\xi^{k\Sigma} = \prod_{\sigma \in \Sigma}(\xi^\sigma)^k$. Only important point about polarization is its ideal $\mathfrak{c}$ such that $\Lambda : X \otimes \mathfrak{c} \cong \mathrm{Pic}^0(X)$, and $[\mathfrak{c}] \in Cl_F^+$ parameterizes geometrically irreducible components of Sh_K if $\det(K) = \widehat{O}^{(p),\times}$. The differential operator d^κ changes k to $k\Sigma + \kappa(1 - c)$. For simplicity, we assume $\kappa = 0$.

§**2. Choice of λ.** For simplicity, assume that f has trivial Neben types. Choose λ so that $\lambda((\xi)) = \xi^{-k\Sigma}$ and $\lambda|_{F^\times_{\mathbb{A}^{(\infty)}}}$ is the central character of f. Fix ω on $X(R)$. Then by the isogeny $\iota : X(R) \to X(\mathcal{A})$ induced by $\mathcal{A} = aR_n$ for $a \in M^\times_{\mathbb{A}^{p\mathfrak{l}\infty}}$, we have $\omega_{\mathcal{A}} = \iota_*\omega$ for all $\mathcal{A}$. Since $\xi : X(\mathcal{A}) \cong X(\xi\mathcal{A})$ for $\xi \in M^\times \cap R^\times_{n,\mathfrak{l}}$ induces $\omega \mapsto \xi_*\omega = \xi\omega$, we find

$$f(x(\xi\mathcal{A})) = f(X(\xi\mathcal{A}), \xi w, \xi\omega_{\mathcal{A}}) = \xi^{-k\Sigma} f(x(\mathcal{A})),$$

and by $\lambda((\xi))\xi^{k\Sigma}$ is the Neben character of f, we find

$$f([\mathcal{A}]) := \lambda(\mathcal{A})^{-1} f(x(\mathcal{A}))$$

only depends on the class $Cl_n^- = M^\times_{\mathbb{A}}/\widehat{R}^\times_n (F^{(\infty)}_{\mathbb{A}})^\times M^\times M^\times_\infty$.

The action $\langle\mathfrak{Q}\rangle = \mathrm{diag}[\varpi_{\mathfrak{q}}^{-1}, 1]$ is at the place $\mathfrak{q} = \mathfrak{Q} \cap O$ and the action $\alpha(u/\varpi_{\mathfrak{l}}^r)$ is at $\mathfrak{l} \neq \mathfrak{q}$; so, they commute. Thus

$$f|\langle\mathfrak{Q}\rangle([\mathcal{A}]) \text{ and } f_v([\mathcal{A}]) \text{ are well defined for } [\mathcal{A}] \in Cl_n^-.$$

§3. Shimura's reciprocity law.

Let (M', Σ') be the reflex of (M, Σ). We suppose that $f_{/\mathbb{F}}$ is the reduction modulo p of $f_{/\mathcal{W}}$ and write E over M' be the field of rationality of ψ, $f_{/\mathcal{W}}$ and λ. Let E_f be the field of rationality over $E[\mu_{\ell^\infty}]$ of $x(\mathcal{A}) \in Sh$ for all $[\mathcal{A}] \in Cl^{alg}$. Then E_f is an abelian extension over E. Then for an idele b of $M'^{\times}_{\mathbb{A}}$, we have $b^{\Sigma'} = \prod_{\sigma' \in \Sigma'} b^{\sigma'} \in M^{\times}_{\mathbb{A}}$, and hence we have an Artin symbol $[N(b)^{\Sigma'}, E]$ acting on E_f for the norm map $N := N_{E/M'}$, whose ideal version, we write as $\sigma = \sigma_b = [N(b)^{\Sigma'}, E]$.

Here is a reciprocity law of Shimura:

$$f([\mathcal{A}])^{\sigma} = f([N(b)^{-\Sigma'}\mathcal{A}]), \qquad \text{(R)}$$

which implies

$$\left(\int_{\Gamma_n} \chi d\varphi^{\mathcal{Q}}_{f_\psi}\right)^{\sigma} = \chi^{\sigma}(N(b)^{\Sigma}) \int_{\Gamma_n} \chi^{\sigma} d\varphi^{\mathcal{Q}}_{f_\psi}.$$

§4. **Trace relation.** Let $\mathbb{F}_P = \mathbb{F}_p[f_{/\mathbb{F}}, \psi, \lambda_{/\mathbb{F}}, \mu_\ell]$ (the field of rationality of $f_{/\mathbb{F}}, \psi, \lambda_{/\mathbb{F}}$ and μ_ℓ). Define $r > 0$ by $\ell^r \,\|\, |\mathbb{F}_P^\times|$.

Lemma. *For a generator $\zeta_n \in \mu_{\ell^n}$, if $\mathbb{F}_P[\chi] = \mathbb{F}_P[\zeta_n]$ with $n > j \geq r$, we have*

$$\mathrm{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_{\ell^j}]}(\zeta_n^s) = \begin{cases} [\mathbb{F}_P[\zeta_n] : \mathbb{F}_P[\zeta_j]]\zeta_n^s & \text{if } \zeta_n^s \in \mu_{\ell^j}, \\ 0 & \text{otherwise.} \end{cases}$$

Note $[\mathbb{F}_P[\zeta_n] : \mathbb{F}_P[\zeta_j]] = \ell^{n-j} \neq 0$ in $\mathbb{F}$.
Proof. By our assumption, $j > 0$. Then the minimal equation of $\mathbb{F}_P[\chi]$ of ζ_n^s over $\mathbb{F}_P[\mu_{\ell^j}]$ is, if $\zeta_n^s \notin \mu_{\ell^j}$, for $m = n - j$

$$\begin{aligned} &X^{\ell^m(\ell-1)} + X^{\ell^m(\ell-2)} + \cdots + 1 \\ &\qquad = X^{\ell^m(\ell-1)} - \mathrm{Tr}_{\mathbb{F}_P[\zeta_n^s]/\mathbb{F}_P[\mu_{\ell^j}]}(\zeta_n^s) X^{\ell^m(\ell-1)-1} + \cdots . \end{aligned}$$

So, we get the above formula. □

§**5.** f_ψ **to** f_v**.** Recall

$$\left(\int_{\Gamma_n[\mathcal{B}]}\chi([\mathcal{A}])d\varphi_{f_\psi^{\mathcal{Q}}}([\mathcal{A}][\mathcal{B}])\right)^\sigma = \chi^\sigma([N(b)^{\Sigma'}])\int_{\Gamma_n}\chi([\mathcal{A}])d\varphi_{f_\psi^{\mathcal{Q}}}([\mathcal{A}][\mathcal{B}])$$

by Shimura's reciprocity law (R), and

$$\int_{\Gamma_n}\chi([\mathcal{A}])d\varphi_{f_\psi^{\mathcal{Q}}}([\mathcal{A}][\mathcal{B}]) = 0 \Leftrightarrow \int_{\Gamma_n}\sigma(\chi([\mathcal{A}]))d\varphi_{f_\psi^{\mathcal{Q}}}([\mathcal{A}][\mathcal{B}]) = 0.$$

Thus for $n \in \underline{n}$ and any $[\mathcal{B}] \in \Gamma_n$, we find for $\mathrm{Tr} := \mathrm{Tr}_{\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_{\ell^j}]}$,

$$\begin{aligned} 0 &= \sum_{\sigma\in\mathrm{Gal}(\mathbb{F}_P[\chi]/\mathbb{F}_P[\mu_{\ell^j}])}\sum_{\mathcal{A}\in\Gamma_n}\sum_{\mathfrak{Q}\in\mathcal{Q}}\lambda\psi^{-1}(\mathfrak{Q})\chi^\sigma(\mathcal{A})f|\langle\mathfrak{Q}\rangle([\mathcal{A}\mathcal{B}][\mathfrak{Q}]_\Gamma) \\ &= \sum_{\mathcal{A}}\sum_{\mathfrak{Q}}\lambda\psi^{-1}(\mathfrak{Q})\mathrm{Tr}(\chi(\mathcal{A}))f|\langle\mathfrak{Q}\rangle([\mathcal{A}\mathcal{B}][\mathfrak{Q}]_\Gamma) \\ &\overset{\text{Trace rel}}{=} \ell^{n-j}\sum_{\mathfrak{Q}\in\mathcal{Q}}\lambda\psi^{-1}(\mathfrak{Q})\zeta_j^{\mathrm{Tr}(vu)}\sum_{u \bmod \mathfrak{l}^j} f|\langle\mathfrak{Q}\rangle|\alpha(u/\varpi_{\mathfrak{l}}^j)([\mathcal{B}][\mathfrak{Q}]_\Gamma) \\ &= \ell^{n-j}\sum_{\mathfrak{Q}\in\mathcal{Q}}\lambda\psi^{-1}(\mathfrak{Q})f_v|\langle\mathfrak{Q}\rangle([\mathcal{B}][\mathfrak{Q}]_\Gamma). \end{aligned}$$

§6. Conclusion.

Let $\tilde{f} := \sum_{\mathfrak{Q}\in\mathcal{Q}} 1 \otimes \cdots \otimes \overset{\mathfrak{Q}}{\lambda\psi^{-1}(\mathfrak{Q}) f_v|\langle\mathfrak{Q}\rangle} \otimes \cdots \otimes 1$ as a function on $V^{\mathcal{Q}}$. Then for the embedding $\mathbf{s} : C \cap V^{\mathcal{Q}} \to V^{\mathcal{Q}}$ given by $\mathbf{s}(x(\mathcal{A})) = \mathbf{s}(\mathcal{A}) = (x(\mathcal{A}[\mathfrak{Q}_\Gamma]))_{\mathfrak{Q}\in\mathcal{Q}}$,

$$\sum_{\mathfrak{Q}\in\mathcal{Q}} \lambda\psi^{-1}(\mathfrak{Q}) f_v|\langle\mathfrak{Q}\rangle([\mathcal{B}][\mathfrak{Q}]_\Gamma) = \lambda(\mathcal{B})^{-1}\tilde{f}(\mathbf{s}(\mathcal{B})).$$

Thus if Ξ is Zariski-dense in $V^{\mathcal{Q}}$, we conclude $f_v = 0$. By computation, $a(\xi, f) \neq 0$ for $\xi \in -v$ is equivalent to $a(\xi, f_v) \neq 0$, a contradiction.

The sequence

$$\underline{n} := \{n | cond(\chi) = \mathfrak{l}^n \text{ and } \chi \in \mathcal{Z}\}$$

defines $\Xi = \{\mathbf{s}(\mathcal{A}) | [\mathcal{A}] \in \bigsqcup_{n\in\underline{n}} \mathrm{Ker}(\Gamma_n \to \Gamma_j)\}$ as we took the trace to $\mathbb{F}_P[\mu_{\ell j}]$. Therefore if $\underline{n}$ contains an arithmetic progression, then $f_v = 0$ by the density theorem.

§**7. Rigidity of torus.** On the contrary to the assertion of the non-vanishing theorem, we assume that

$$\mathcal{X} := \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) | \int_{Cl_n^-} \chi\psi d\varphi_f \neq 0,\ v(\chi) = v\}$$

has Zariski closure $\overline{\mathcal{X}}$ with $\dim \overline{\mathcal{X}} < d$. Since $\mathcal{X}$ is stable by p-Frobenius $t \mapsto t^P$ for a p-power P, $\overline{\mathcal{X}}$ is stable under $t \mapsto t^{P^m}$ for all m. Let W_ℓ be a discrete valuation ring finite flat over $W(\overline{\mathbb{F}}_\ell)$. We apply to the formal completion $\widehat{\mathcal{X}}$ of $\overline{\mathcal{X}}$ the following

Rigidity Theorem. *Let $X = \mathrm{Spf}(\mathcal{T})$ be a closed formal subscheme of $\widehat{G} = \widehat{\mathbb{G}}^n_{m/W_\ell}$ flat geometrically irreducible over W_ℓ (i.e., $\mathcal{T} \cap \overline{\mathbb{Q}}_\ell = W_\ell$). Suppose there exists an open subgroup U of $\mathbb{Z}_\ell^\times$ such that X is stable under the action $\widehat{G} \ni t \mapsto t^u \in \widehat{G}$ for all $u \in U$. If X contains a Zariski dense subset $\Omega \subset X(\mathbb{C}_\ell) \cap \mu^n_{\ell^\infty}(\mathbb{C}_\ell)$, then there exist $\omega \in \Omega$ and a formal subtorus T such that $X = T\omega$.*

§8. The strategy.

A key point is the use of a rigidity theorem asserting a formal subscheme of $\widehat{\mathbb{G}}_{m/W_\ell}$ stable under $t \mapsto t^P$ for a p-power P is a union of formal subtorus up to making finite quotient. Define $\mathcal{X} := \{\chi \in \mathrm{Hom}(\Gamma, \mu_{\ell^\infty}) | \int_{Cl_\infty^-} \chi\psi d\varphi_f \neq 0\}$, and regard $\mathcal{Z}$ and $\mathcal{X}$ as a subset of $\widehat{\mathbb{G}}_{m/W_\ell}$ for a sufficiently large W_ℓ. Stability of $\widehat{\mathcal{X}} \subset \widehat{\mathbb{G}}_m^d$ under a suitable power of p-Frobenius implies stability of $\widehat{\mathcal{X}}$ under an open subgroup $U \subset \mathbb{Z}_\ell^\times$ generated by P. Assume $\dim \widehat{\mathcal{X}} < d$ for $d = [F : \mathbb{Q}]$. By the rigidity theorem applied to $\widehat{\mathcal{X}}$, we find an arithmetic progression $\underline{n}$ such that χ with conductor $\mathfrak{l}^n$ for all $n \in \underline{n}$ is in $\mathbb{G}_m^d - \widehat{\mathcal{X}}$ to conclude $f_v = 0$, a contradiction against $a(\xi, f_v) = N(\mathfrak{l})^j a(\xi, f) \neq 0$ for $\xi \in -v$. Thus the non-vanishing theorem follows. The details will be discussed in the last lecture.