Vanishing viscosity and conserved quantities for 2D incompressible flow

Helena J. Nussenzveig Lopes

Instituto de Matemática, Universidade Federal do Rio de Janeiro

Turbulence: Problems at the interface of Mathematics and Physics (Online)
International Centre for Theoretical Sciences (ICTS
Tata Institute of Fundamental Research
Bangalore, INDIA
December 07-18, 2020.

Collaborators:

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Milton Lopes Filho (Universidade Federal do Rio de Janeiro)

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Milton Lopes Filho (Universidade Federal do Rio de Janeiro)

Anna Mazzucato (Penn State University)

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Milton Lopes Filho (Universidade Federal do Rio de Janeiro)

Anna Mazzucato (Penn State University)

Christian Seis (Universität Münster)

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Milton Lopes Filho (Universidade Federal do Rio de Janeiro)

Anna Mazzucato (Penn State University)

Christian Seis (Universität Münster)

Roman Shvydkoy (Univ. Illinois, Chicago)

Collaborators:

Alexey Cheskidov (Univ. Illinois, Chicago)

Milton Lopes Filho (Universidade Federal do Rio de Janeiro)

Anna Mazzucato (Penn State University)

Christian Seis (Universität Münster)

Roman Shvydkoy (Univ. Illinois, Chicago)

Emil Wiedemann (Universität Ulm)

Anomalous dissipation is a cornerstone of turbulence theory

Anomalous dissipation is a cornerstone of turbulence theory

turbulence $\longleftrightarrow$ irregular flows

Anomalous dissipation is a cornerstone of turbulence theory

turbulence $\longleftrightarrow$ irregular flows

Onsager 1949:

Anomalous dissipation is a cornerstone of turbulence theory

turbulence $\longleftrightarrow$ irregular flows

Onsager 1949:

- anomalous dissipation may occur in inviscid flow with “less than $1/3$ regularity”

Anomalous dissipation is a cornerstone of turbulence theory

turbulence $\longleftrightarrow$ irregular flows

Onsager 1949:

- anomalous dissipation may occur in inviscid flow with "less than $1/3$ regularity"
- inviscid flows with "more than $1/3$ regularity" conserve energy

Wild solutions, anomalous dissipation

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C_x^{0,1/3-\epsilon}$. These are all 3D constructions.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C^{0,1/3-\epsilon}_x$. These are all 3D constructions.
- Choffrut, 2013,

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C_x^{0,1/3-\epsilon}$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C^{0,1/3-\epsilon}_x$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.
- Isett 2018: $C^{0,1/3-\varepsilon}$, compact support in time.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C^{0,1/3-\epsilon}_x$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.
- Isett 2018: $C^{0,1/3-\varepsilon}$, compact support in time.
- Buckmaster-De Lellis-Szekelyhidi-Vicol 2019: $C^{0,1/3-\varepsilon}$ + prescribed energy profile.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C^{0,1/3-\epsilon}_x$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.
- Isett 2018: $C^{0,1/3-\varepsilon}$, compact support in time.
- Buckmaster-De Lellis-Szekelyhidi-Vicol 2019: $C^{0,1/3-\varepsilon}$ + prescribed energy profile.
- Buckmaster-Vicol 2019: $\exists$ viscous flows with prescribed energy profile;

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C_x^{0,1/3-\epsilon}$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.
- Isett 2018: $C^{0,1/3-\varepsilon}$, compact support in time.
- Buckmaster-De Lellis-Szekelyhidi-Vicol 2019: $C^{0,1/3-\varepsilon}$ + prescribed energy profile.
- Buckmaster-Vicol 2019: $\exists$ viscous flows with prescribed energy profile; $\exists$ inviscid limit with anomalous dissipation.

Wild solutions, anomalous dissipation

- Scheffer 93, Shnirelman 95 and De Lellis, Szekelyhidi 2009 - non-uniqueness (compact support in space and time); time-dependent energy.
- Isett 2013; Buckmaster-De Lellis-Isett-Szekelyhidi 2015: $C^{0,1/5-\epsilon}$, Buckmaster, De Lellis, Szekelyhidi 2016, $L^1_t C_x^{0,1/3-\epsilon}$. These are all 3D constructions.
- Choffrut, 2013, $C^{0,1/10}$. Construction works in 2D.
- Isett 2018: $C^{0,1/3-\varepsilon}$, compact support in time.
- Buckmaster-De Lellis-Szekelyhidi-Vicol 2019: $C^{0,1/3-\varepsilon}$ + prescribed energy profile.
- Buckmaster-Vicol 2019: $\exists$ viscous flows with prescribed energy profile; $\exists$ inviscid limit with anomalous dissipation. **3D construction!**

Regularity threshold for conservation of energy

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L^\infty_t H^{5/6}_x$;

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L^\infty_t H^{5/6}_x$;
- Eyink 94: a little more than $L^3_t C^{1/3+\epsilon}_x$;

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.
- State of the art – Cheskidov, Constantin, Friedlander, Shvydkoy 2008: $L_t^3 B_{3,c_0}^{1/3}$,

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.
- State of the art – Cheskidov, Constantin, Friedlander, Shvydkoy 2008: $L_t^3 B_{3,c_0}^{1/3}$, 3D and 2D.

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.
- State of the art – Cheskidov, Constantin, Friedlander, Shvydkoy 2008: $L_t^3 B_{3,c_0}^{1/3}$, 3D and 2D.
- **2D result** – Duchon, Robert 2000: initial vorticity in $W^{1,p}$, for $p > 3/2$ implies conservation of energy.

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.
- State of the art – Cheskidov, Constantin, Friedlander, Shvydkoy 2008: $L_t^3 B_{3,c_0}^{1/3}$, 3D and 2D.
- **2D result** – Duchon, Robert 2000: initial vorticity in $W^{1,p}$, for $p > 3/2$ implies conservation of energy.

  Extension to $p = 3/2$ follows from Cheskidov, Constantin, Friedlander, Shvydkoy.

Regularity threshold for conservation of energy

- Frisch-Sulem 1975: $L_t^\infty H_x^{5/6}$;
- Eyink 94: a little more than $L_t^3 C_x^{1/3+\epsilon}$;
- Constantin, E, Titi 1994: $L_t^3 B_{3,\infty}^{1/3+\epsilon}$.
- State of the art – Cheskidov, Constantin, Friedlander, Shvydkoy 2008: $L_t^3 B_{3,c_0}^{1/3}$, 3D and 2D.
- **2D result** – Duchon, Robert 2000: initial vorticity in $W^{1,p}$, for $p > 3/2$ implies conservation of energy.

  Extension to $p = 3/2$ follows from Cheskidov, Constantin, Friedlander, Shvydkoy.

  Involves studying optimal conditions for energy flux to vanish.

2D flows

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$,

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t=0) = u_0. \end{cases}$$

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions*

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,for some $p > 1$.

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,for some $p > 1$.

Note:

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,for some $p > 1$.

Note:

- Smooth vorticity transported in 2D,

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p \\ \text{div}\, u = 0 \\ u(t=0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \text{curl}\, u$ is p-th power integrable,for some $p > 1$.

Note:

- Smooth vorticity transported in 2D, L^p bounds preserved by evolution

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,for some $p > 1$.

Note:

- Smooth vorticity transported in 2D, L^p bounds preserved by evolution
- wild solutions:

2D flows

2D Euler equations on the torus $\mathbb{T}^2 \equiv [0, 2\pi]^2$, with initial data $u_0 \in L^2(\mathbb{T}^2)$, no forcing:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p \\ \operatorname{div} u = 0 \\ u(t = 0) = u_0. \end{cases}$$

Interested in *weak solutions* for which vorticity $\omega \equiv \operatorname{curl} u$ is p-th power integrable,for some $p > 1$.

Note:

- Smooth vorticity transported in 2D, L^p bounds preserved by evolution
- wild solutions: no control on integrability of vorticity

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$.

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$.

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\text{div}\Phi(t, \cdot) = 0$ the following identity holds true:

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\text{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\text{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

2. For almost every $t \in (0, T)$,

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \mathrm{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\mathrm{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

2. For almost every $t \in (0, T)$, $\mathrm{div}\, u(t, \cdot) = 0$, in the sense of distributions.

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\text{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

2. For almost every $t \in (0, T)$, $\text{div}\, u(t, \cdot) = 0$, in the sense of distributions.

Existence of such weak solutions is known (DiPerna, Majda 87), but uniqueness is open, except for the case $p = \infty$.

Definition

Fix $T > 0$ and $u_0 \in L^2(\mathbb{T}^2)$ with initial vorticity $\omega_0 = \text{curl}\, u_0 \in L^p(\mathbb{T}^2)$, for some $p \geq 1$. Let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ with $\omega \in L^\infty(0, T; L^p(\mathbb{T}^2))$. We say u is a weak solution of the incompressible Euler equations with initial velocity u_0 if

1. for every test vector field $\Phi \in C^\infty([0, T) \times \mathbb{T}^2)$ such that $\text{div}\Phi(t, \cdot) = 0$ the following identity holds true:

$$\int_0^T \int_{\mathbb{T}^2} \partial_t \Phi \cdot u + u \cdot D\Phi u \, dxdt + \int_{\mathbb{T}^2} \Phi(0, \cdot) \cdot u_0 \, dx = 0.$$

2. For almost every $t \in (0, T)$, $\text{div}\, u(t, \cdot) = 0$, in the sense of distributions.

Existence of such weak solutions is known (DiPerna, Majda 87), but uniqueness is open, except for the case $p = \infty$. We call a weak solution *conservative* if the L^2-norm of velocity is constant in time.

Theorem

Fix $T > 0$ *and let* $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$.

Theorem

Fix $T > 0$ *and let* $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. *Then* u *is conservative.*

Theorem

Fix $T > 0$ *and let* $u \in C_{\mathrm{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. *Then u is conservative. Moreover, the following local energy balance law holds in the sense of distributions:*

Theorem

Fix $T > 0$ *and let* $u \in C_{\mathrm{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. *Then* u *is conservative. Moreover, the following local energy balance law holds in the sense of distributions:*

$$\partial_t \left(\frac{|u|^2}{2} \right) + \mathrm{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right] = 0. \tag{1}$$

Theorem

Fix $T > 0$ and let $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ be a weak solution with $\omega \equiv$ curl $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. Then u is conservative. Moreover, the following local energy balance law holds in the sense of distributions:

$$\partial_t \left(\frac{|u|^2}{2} \right) + \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right] = 0. \tag{1}$$

This result is contained in Cheskidov *et alli* 2008,

Theorem

Fix $T > 0$ and let $u \in C_{\mathrm{weak}}(0, T; L^2(\mathbb{T}^2))$ be a weak solution with $\omega \equiv$ curl $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. Then u is conservative. Moreover, the following local energy balance law holds in the sense of distributions:

$$\partial_t \left(\frac{|u|^2}{2} \right) + \mathrm{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right] = 0. \tag{1}$$

This result is contained in Cheskidov *et alli* 2008, since $L^\infty_t W^{1,3/2}_x \subseteq L^3_t B^{1/3}_{3,c_0}$

Theorem

Fix $T > 0$ *and let* $u \in C_{\text{weak}}(0, T; L^2(\mathbb{T}^2))$ *be a weak solution with* $\omega \equiv$ *curl* $u \in L^\infty(0, T; L^{3/2}(\mathbb{T}^2))$. *Then u is conservative. Moreover, the following local energy balance law holds in the sense of distributions:*

$$\partial_t \left(\frac{|u|^2}{2} \right) + \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right] = 0. \quad (1)$$

This result is contained in Cheskidov *et alli* 2008, since $L^\infty_t W^{1,3/2}_x \subseteq L^3_t B^{1/3}_{3,c_0}$ we outline an elementary proof.

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier.

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε;

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$.

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$. Then:

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$. Then:

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon = -\nabla p^\varepsilon + \mathcal{R}^\varepsilon, \tag{2}$$

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$. Then:

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon = -\nabla p^\varepsilon + \mathcal{R}^\varepsilon, \tag{2}$$

with

$$\mathcal{R}^\varepsilon \equiv (u^\varepsilon \cdot \nabla) u^\varepsilon - \zeta_\varepsilon * [(u \cdot \nabla) u].$$

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$. Then:

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon = -\nabla p^\varepsilon + \mathcal{R}^\varepsilon, \tag{2}$$

with

$$\mathcal{R}^\varepsilon \equiv (u^\varepsilon \cdot \nabla) u^\varepsilon - \zeta_\varepsilon * [(u \cdot \nabla) u].$$

Multiply the equation by u^ε:

Idea of the proof of the Theorem

Let $\zeta_\varepsilon = \zeta_\varepsilon(x)$ be $C^\infty(\mathbb{T}^2)$-mollifier. Take convolution of Euler with ζ_ε; let $u^\varepsilon = \zeta_\varepsilon * u$, $p^\varepsilon = \zeta_\varepsilon * p$. Then:

$$\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon = -\nabla p^\varepsilon + \mathcal{R}^\varepsilon, \tag{2}$$

with

$$\mathcal{R}^\varepsilon \equiv (u^\varepsilon \cdot \nabla) u^\varepsilon - \zeta_\varepsilon * [(u \cdot \nabla) u].$$

Multiply the equation by u^ε:

$$\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) + \operatorname{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] = u^\varepsilon \cdot \mathcal{R}^\varepsilon. \tag{3}$$

As $\varepsilon \to 0$, we have:

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|u|^2}{2} \right)$ in the sense of distributions;

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|u|^2}{2} \right)$ in the sense of distributions;

(B) $\text{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] \to \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right]$ in the sense of distributions;

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2}\right) \to \partial_t \left(\frac{|u|^2}{2}\right)$ in the sense of distributions;

(B) $\operatorname{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon\right)\right] \to \operatorname{div} \left[u \left(\frac{|u|^2}{2} + p\right)\right]$ in the sense of distributions;

(C) $u^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0, T; L^1(\mathbb{T}^2))$.

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|u|^2}{2} \right)$ in the sense of distributions;

(B) $\text{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] \to \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right]$ in the sense of distributions;

(C) $u^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0, T; L^1(\mathbb{T}^2))$.

(A) and (B) are subcritical for $\omega \in L^{3/2}$.

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|u|^2}{2} \right)$ in the sense of distributions;

(B) $\text{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] \to \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right]$ in the sense of distributions;

(C) $u^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0, T; L^1(\mathbb{T}^2))$.

(A) and (B) are subcritical for $\omega \in L^{3/2}$. In fact, they require $\omega \in L^{6/5}$.

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2}\right) \to \partial_t \left(\frac{|u|^2}{2}\right)$ in the sense of distributions;

(B) $\text{div}\left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon\right)\right] \to \text{div}\left[u\left(\frac{|u|^2}{2} + p\right)\right]$ in the sense of distributions;

(C) $u^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0, T; L^1(\mathbb{T}^2))$.

(A) and (B) are subcritical for $\omega \in L^{3/2}$. In fact, they require $\omega \in L^{6/5}$. It is the convergence of the energy flux term, which is (C), that requires $\omega \in L^{3/2}$.

As $\varepsilon \to 0$, we have:

(A) $\partial_t \left(\frac{|u^\varepsilon|^2}{2} \right) \to \partial_t \left(\frac{|u|^2}{2} \right)$ in the sense of distributions;

(B) $\text{div} \left[u^\varepsilon \left(\frac{|u^\varepsilon|^2}{2} + p^\varepsilon \right) \right] \to \text{div} \left[u \left(\frac{|u|^2}{2} + p \right) \right]$ in the sense of distributions;

(C) $u^\varepsilon \cdot \mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0, T; L^1(\mathbb{T}^2))$.

(A) and (B) are subcritical for $\omega \in L^{3/2}$. In fact, they require $\omega \in L^{6/5}$. It is the convergence of the energy flux term, which is (C), that requires $\omega \in L^{3/2}$. (Good behavior of the energy flux term is the key point in all results along these lines.)

Convergence of the flux term:

Convergence of the flux term: we show $\mathcal{R}^{\varepsilon} \to 0$ strongly in $L^{\infty}(0, T; L^{6/5}(\mathbb{T}^2))$.

Convergence of the flux term: we show $\mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0,T;L^{6/5}(\mathbb{T}^2))$. This is enough, since u^ε is bounded in $L^\infty(0,T;L^6(\mathbb{T}^2))$.

Convergence of the flux term: we show $\mathcal{R}^{\varepsilon} \to 0$ strongly in $L^{\infty}(0, T; L^{6/5}(\mathbb{T}^2))$. This is enough, since u^{ε} is bounded in $L^{\infty}(0, T; L^{6}(\mathbb{T}^2))$. We have:

Convergence of the flux term: we show $\mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0,T;L^{6/5}(\mathbb{T}^2))$. This is enough, since u^ε is bounded in $L^\infty(0,T;L^6(\mathbb{T}^2))$. We have:

$$\|\mathcal{R}^\varepsilon\|_{L^\infty(L^{6/5})} = \|(u^\varepsilon \cdot \nabla)u^\varepsilon - \zeta_\varepsilon * [(u \cdot \nabla)u]\|_{L^\infty(L^{6/5})}$$

Convergence of the flux term: we show $\mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0,T;L^{6/5}(\mathbb{T}^2))$. This is enough, since u^ε is bounded in $L^\infty(0,T;L^6(\mathbb{T}^2))$. We have:

$$\|\mathcal{R}^\varepsilon\|_{L^\infty(L^{6/5})} = \|(u^\varepsilon\cdot\nabla)u^\varepsilon - \zeta_\varepsilon * [(u\cdot\nabla)u]\|_{L^\infty(L^{6/5})}$$

$$\leq \|(u^\varepsilon\cdot\nabla)(u^\varepsilon - u)\|_{L^\infty(L^{6/5})} + \|(u^\varepsilon - u)\cdot\nabla u\|_{L^\infty(L^{6/5})} +$$

$$+ \|(u\cdot\nabla)u - \zeta_\varepsilon * [(u\cdot\nabla)u]\|_{L^\infty(L^{6/5})}$$

Convergence of the flux term: we show $\mathcal{R}^{\varepsilon} \to 0$ strongly in $L^{\infty}(0,T;L^{6/5}(\mathbb{T}^2))$. This is enough, since u^{ε} is bounded in $L^{\infty}(0,T;L^{6}(\mathbb{T}^2))$. We have:

$$\|\mathcal{R}^{\varepsilon}\|_{L^{\infty}(L^{6/5})} = \|(u^{\varepsilon}\cdot\nabla)u^{\varepsilon} - \zeta_{\varepsilon} * [(u\cdot\nabla)u]\|_{L^{\infty}(L^{6/5})}$$

$$\leq \|(u^{\varepsilon}\cdot\nabla)(u^{\varepsilon}-u)\|_{L^{\infty}(L^{6/5})} + \|(u^{\varepsilon}-u)\cdot\nabla u\|_{L^{\infty}(L^{6/5})} +$$

$$+ \|(u\cdot\nabla)u - \zeta_{\varepsilon} * [(u\cdot\nabla)u]\|_{L^{\infty}(L^{6/5})}$$

$$\leq \|u^{\varepsilon}\|_{L^{\infty}(L^{6})}\|\nabla u^{\varepsilon} - \nabla u\|_{L^{\infty}(L^{3/2})} + \|u^{\varepsilon}-u\|_{L^{\infty}(L^{6})}\|\nabla u\|_{L^{\infty}(L^{3/2})}$$

$$+ \|(u\cdot\nabla)u - \zeta_{\varepsilon} * [(u\cdot\nabla)u]\|_{L^{\infty}(L^{6/5})} \to 0,$$

Convergence of the flux term: we show $\mathcal{R}^\varepsilon \to 0$ strongly in $L^\infty(0,T;L^{6/5}(\mathbb{T}^2))$. This is enough, since u^ε is bounded in $L^\infty(0,T;L^6(\mathbb{T}^2))$. We have:

$$\|\mathcal{R}^\varepsilon\|_{L^\infty(L^{6/5})} = \|(u^\varepsilon\cdot\nabla)u^\varepsilon - \zeta_\varepsilon * [(u\cdot\nabla)u]\|_{L^\infty(L^{6/5})}$$

$$\leq \|(u^\varepsilon\cdot\nabla)(u^\varepsilon - u)\|_{L^\infty(L^{6/5})} + \|(u^\varepsilon - u)\cdot\nabla u\|_{L^\infty(L^{6/5})} +$$

$$+\|(u\cdot\nabla)u - \zeta_\varepsilon * [(u\cdot\nabla)u]\|_{L^\infty(L^{6/5})}$$

$$\leq \|u^\varepsilon\|_{L^\infty(L^6)}\|\nabla u^\varepsilon - \nabla u\|_{L^\infty(L^{3/2})} + \|u^\varepsilon - u\|_{L^\infty(L^6)}\|\nabla u\|_{L^\infty(L^{3/2})}$$

$$+\|(u\cdot\nabla)u - \zeta_\varepsilon * [(u\cdot\nabla)u]\|_{L^\infty(L^{6/5})} \to 0,$$

because $u^\varepsilon \to u$ in $L^\infty(L^6(\mathbb{T}^2))$, $\nabla u^\varepsilon = \zeta^\varepsilon * \nabla u \to \nabla u$ in $L^\infty(L^{3/2}(\mathbb{T}^2))$ and $u\cdot\nabla u \in L^\infty(L^{6/5}(\mathbb{T}^2))$.

$p = \frac{3}{2}$ is optimal

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations.

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

Introduce the Littlewood-Paley truncation S_q by

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

Introduce the Littlewood-Paley truncation S_q by

$$S_q[f] = \widehat{f}_{(0,0)} + \sum_{p \leq q-1} \Delta_p f = \sum_{\alpha \in \mathbb{Z}^2} \chi(\lambda_q^{-1}\alpha)\widehat{f}(\alpha)e^{2\pi i \alpha \cdot x}.$$

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

Introduce the Littlewood-Paley truncation S_q by

$$S_q[f] = \widehat{f}_{(0,0)} + \sum_{p \leq q-1} \Delta_p f = \sum_{\alpha \in \mathbb{Z}^2} \chi(\lambda_q^{-1}\alpha)\widehat{f}(\alpha)e^{2\pi i \alpha \cdot x}.$$

S_q is a convolution with a mollifier,

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

Introduce the Littlewood-Paley truncation S_q by

$$S_q[f] = \widehat{f}_{(0,0)} + \sum_{p \leq q-1} \Delta_p f = \sum_{\alpha \in \mathbb{Z}^2} \chi(\lambda_q^{-1}\alpha)\widehat{f}(\alpha)e^{2\pi i \alpha \cdot x}.$$

S_q is a convolution with a mollifier, hence smooth. Can argue easily that energy flux for $S_q[f]$ vanishes if $f \in W^{1,3/2}$

$p = \frac{3}{2}$ is optimal

Conservation of energy for weak solutions hinges upon a scaling argument that has very little to do with the dynamics of the Euler equations. Therefore, to show that the exponent $3/2$ is optimal in the argument above, we construct a vector field which just fails to be $W^{1,3/2}$ for which the energy flux does not vanish.

Introduce the Littlewood-Paley truncation S_q by

$$S_q[f] = \widehat{f}_{(0,0)} + \sum_{p \leq q-1} \Delta_p f = \sum_{\alpha \in \mathbb{Z}^2} \chi(\lambda_q^{-1}\alpha)\widehat{f}(\alpha)e^{2\pi i \alpha \cdot x}.$$

S_q is a convolution with a mollifier, hence smooth. Can argue easily that energy flux for $S_q[f]$ vanishes if $f \in W^{1,3/2}$ – easy adaptation of argument for $\omega \in L^{3/2}$ with S_q in place of the convolution with a mollifier.

Testing Euler with $S_q[S_q[u]]$,

Testing Euler with $S_q[S_q[u]]$, it is easy to see that

Testing Euler with $S_q[S_q[u]]$, it is easy to see that proof of energy conservation reduces to showing

Testing Euler with $S_q[S_q[u]]$, it is easy to see that proof of energy conservation reduces to showing energy flux

$$\Pi_q[u] = \int_{\mathbb{T}^2} S_q[u] \cdot S_q[(u \cdot \nabla)u] \, dx$$

vanishes on average in time as $q \to \infty$.

Testing Euler with $S_q[S_q[u]]$, it is easy to see that proof of energy conservation reduces to showing energy flux

$$\Pi_q[u] = \int_{\mathbb{T}^2} S_q[u] \cdot S_q[(u \cdot \nabla) u] \, dx$$

vanishes on average in time as $q \to \infty$. This holds, in fact, pointwise in time for any divergence-free field with curl in $L^{3/2}$.

Testing Euler with $S_q[S_q[u]]$, it is easy to see that proof of energy conservation reduces to showing energy flux

$$\Pi_q[u] = \int_{\mathbb{T}^2} S_q[u] \cdot S_q[(u \cdot \nabla)u]\, dx$$

vanishes on average in time as $q \to \infty$. This holds, in fact, pointwise in time for any divergence-free field with curl in $L^{3/2}$.

Theorem (Cheskidov, Lopes Filho, N-L, Shvydkoy; 2016)

There exists a divergence free vector field $u \in B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, for any $1 \leq p < 3/2$, such that $\limsup_{q\to\infty} \Pi_q[u] \neq 0$.

Note.

Note. The div-free vector field u

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \le p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION:

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution,

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D,

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity,

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative?

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported?

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Kraichnan 2D turbulence theory:

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Kraichnan 2D turbulence theory: forward enstrophy cascade

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Kraichnan 2D turbulence theory: forward enstrophy cascade $\rightarrow$ regularizing effect in 2D

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Kraichnan 2D turbulence theory: forward enstrophy cascade $\rightarrow$ regularizing effect in 2D

Suggests exists dynamical mechanism preventing anomalous dissipation in 2D

Note. The div-free vector field u in $B^{1/3}_{3,\infty} \cap W^{1,p}(\mathbb{T}^2)$, $1 \leq p < 3/2$, **not** a dynamical example not solution of Euler

QUESTION: Is there an Euler (weak) solution, in 2D, with some control on (integrability of) vorticity, which is not conservative? For which vorticity is transported? Lagrangian structure?

Kraichnan 2D turbulence theory: forward enstrophy cascade $\rightarrow$ regularizing effect in 2D

Suggests exists dynamical mechanism preventing anomalous dissipation in 2D even for supercritical (less than $1/3$ regular) flows

Vanishing viscosity solutions

Vanishing viscosity solutions

Definition

Let $u \in C(0, T; L^2(\mathbb{T}^2))$.

Vanishing viscosity solutions

Definition

Let $u \in C(0, T; L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

Vanishing viscosity solutions

Definition

Let $u \in C(0,T;L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;

Vanishing viscosity solutions

Definition

Let $u \in C(0, T; L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;
2. there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that, as $\nu \to 0$,

Vanishing viscosity solutions

Definition

Let $u \in C(0, T; L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;
2. there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that, as $\nu \to 0$,
   - $u^\nu \rightharpoonup u$ weakly* in $L^\infty(0, T; L^2(\mathbb{T}^2))$;

Vanishing viscosity solutions

Definition

Let $u \in C(0,T;L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;
2. there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that, as $\nu \to 0$,
   - $u^\nu \rightharpoonup u$ weakly* in $L^\infty(0,T;L^2(\mathbb{T}^2))$;
   - $u^\nu(0,\cdot) \equiv u_0^\nu \to u_0$ strongly in $L^2(\mathbb{T}^2)$.

Vanishing viscosity solutions

Definition

Let $u \in C(0,T;L^2(\mathbb{T}^2))$. We say that u is a *physically realizable weak solution of the incompressible 2D Euler equations* with initial velocity $u_0 \in L^2(\mathbb{T}^2)$ if the following conditions hold.

1. u is a weak solution of the Euler equations;
2. there exists a family of solutions of the incompressible 2D Navier-Stokes equations with viscosity $\nu > 0$, $\{u^\nu\}$, such that, as $\nu \to 0$,
   - $u^\nu \rightharpoonup u$ weakly* in $L^\infty(0,T;L^2(\mathbb{T}^2))$;
   - $u^\nu(0,\cdot) \equiv u_0^\nu \to u_0$ strongly in $L^2(\mathbb{T}^2)$.

Energy

Energy

Theorem (Cheskidov,Lopes Filho, N-L, Shvydkoy; 2016)

Let $u \in C(0, T; L^2(\mathbb{T}^2))$ be a physically realizable weak solution of the incompressible 2D Euler equations.

Energy

Theorem (Cheskidov,Lopes Filho, N-L, Shvydkoy; 2016)

Let $u \in C(0, T; L^2(\mathbb{T}^2))$ *be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that* $u_0 \in L^2$ *is such that* $\text{curl}\, u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$*, for some* $p > 1$*.*

Energy

Theorem (Cheskidov,Lopes Filho, N-L, Shvydkoy; 2016)

Let $u \in C(0, T; L^2(\mathbb{T}^2))$ *be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that* $u_0 \in L^2$ *is such that* $\text{curl}\, u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$*, for some* $p > 1$*. Then* u *conserves energy.*

Energy

Theorem (Cheskidov,Lopes Filho, N-L, Shvydkoy; 2016)

Let $u \in C(0, T; L^2(\mathbb{T}^2))$ *be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that* $u_0 \in L^2$ *is such that* $\text{curl}\, u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$*, for some* $p > 1$*. Then* u *conserves energy.*

Obs.

Energy

Theorem (Cheskidov,Lopes Filho, N-L, Shvydkoy; 2016)

Let $u \in C(0, T; L^2(\mathbb{T}^2))$ be a physically realizable weak solution of the incompressible 2D Euler equations. Suppose that $u_0 \in L^2$ is such that curl $u_0 \equiv \omega_0 \in L^p(\mathbb{T}^2)$*, for some $p > 1$. Then u conserves energy.*

Obs. $1 < p < 3/2$ 'Onsager supercritical'.

Proof:

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial.

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions.

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \operatorname{curl} u^\nu$.

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \mathrm{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Gagliardo-Nirenberg $\Longrightarrow$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \operatorname{curl} u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Gagliardo-Nirenberg $\Longrightarrow$ for any $1 < p < 2$:

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \mathrm{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Gagliardo-Nirenberg $\Longrightarrow$ for any $1 < p < 2$:

$$\|\omega^\nu\|_{L^2} \leq \|\nabla \omega^\nu\|_{L^2}^{1-\frac{p}{2}} \|\omega^\nu\|_{L^p}^{\frac{p}{2}}.$$

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Gagliardo-Nirenberg $\Longrightarrow$ for any $1 < p < 2$:

$$\|\omega^\nu\|_{L^2} \leq \|\nabla \omega^\nu\|_{L^2}^{1-\frac{p}{2}} \|\omega^\nu\|_{L^p}^{\frac{p}{2}}.$$

Then

Proof: Assume $\omega_0 \in L^p(\mathbb{T}^2)$ for some $p < 2$, and $\omega_0 \notin L^2(\mathbb{T}^2)$ otherwise, the result is trivial. u is physically realizable $\Longrightarrow \exists$ family $\{u^\nu\}$ of solutions of Navier-Stokes satisfying the corresponding conditions. $\omega^\nu = \text{curl}\, u^\nu$. The vorticity equation given by:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu.$$

Multiply by ω^ν and integrate on torus:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 = -2\nu \|\nabla \omega^\nu\|_{L^2}^2.$$

Gagliardo-Nirenberg $\Longrightarrow$ for any $1 < p < 2$:

$$\|\omega^\nu\|_{L^2} \leq \|\nabla \omega^\nu\|_{L^2}^{1-\frac{p}{2}} \|\omega^\nu\|_{L^p}^{\frac{p}{2}}.$$

Then

$$-2\nu \|\nabla \omega^\nu\|_{L^2}^2 \leq -2\nu \|\omega^\nu\|_{L^2}^{\frac{4}{2-p}} \|\omega^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$.

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time,

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time, starting from $\delta > 0$:

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time, starting from $\delta > 0$:

$$[y(t)]^{\frac{-p}{2-p}} - [y(\delta)]^{\frac{-p}{2-p}} \geq \frac{2\nu C_0 p}{2-p}(t-\delta).$$

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time, starting from $\delta > 0$:

$$[y(t)]^{\frac{-p}{2-p}} - [y(\delta)]^{\frac{-p}{2-p}} \geq \frac{2\nu C_0 p}{2-p}(t-\delta).$$

In limit $\delta \to 0$,

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time, starting from $\delta > 0$:

$$[y(t)]^{\frac{-p}{2-p}} - [y(\delta)]^{\frac{-p}{2-p}} \geq \frac{2\nu C_0 p}{2-p}(t-\delta).$$

In limit $\delta \to 0$, since $\lim_{\delta\to 0}\|\omega^\nu(\delta,\cdot)\|_{L^2}^2 = +\infty$, have:

Multiply the vorticity equation by $|\omega^\nu|^{p-2}\omega^\nu$ and integrate on torus $\Longrightarrow$ maximum principle for L^p norm of vorticity:

$$\|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0^\nu\|_{L^p},$$

for any $t \geq 0$.
Therefore:

$$\frac{d}{dt}\|\omega^\nu\|_{L^2}^2 \leq -2\nu\|\omega^\nu\|_{L^2}^{\frac{4}{2-p}}\|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}.$$

Write $y = y(t) = \|\omega^\nu\|_{L^2}^2$ and $C_0 = \|\omega_0^\nu\|_{L^p}^{-\frac{2p}{2-p}}$. Then, integrating in time, starting from $\delta > 0$:

$$[y(t)]^{\frac{-p}{2-p}} - [y(\delta)]^{\frac{-p}{2-p}} \geq \frac{2\nu C_0 p}{2-p}(t-\delta).$$

In limit $\delta \to 0$, since $\lim_{\delta\to 0}\|\omega^\nu(\delta,\cdot)\|_{L^2}^2 = +\infty$, have:

$$\|\omega^\nu(t,\cdot)\|_{L^2}^2 \leq \left(\frac{2\nu p C_0 t}{2-p}\right)^{-\frac{2-p}{p}}.$$

Energy identity for 2D Navier-Stokes:

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{5}$$

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{5}$$

Integrating in time and using the estimate for vorticity

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{5}$$

Integrating in time and using the estimate for vorticity we get

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{5}$$

Integrating in time and using the estimate for vorticity we get

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \quad \geq \quad -2\nu\int_0^t \left(\frac{2\nu p C_0 s}{2-p}\right)^{-\frac{2-p}{p}} ds$$

Energy identity for 2D Navier-Stokes:

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\nabla u^\nu\|_{L^2}^2. \tag{4}$$

Rewriting in terms of vorticity yields

$$\frac{d}{dt}\|u^\nu\|_{L^2}^2 = -2\nu\|\omega^\nu\|_{L^2}^2. \tag{5}$$

Integrating in time and using the estimate for vorticity we get

$$\begin{aligned} 0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \quad &\geq \quad -2\nu \int_0^t \left(\frac{2\nu p C_0 s}{2-p}\right)^{-\frac{2-p}{p}} ds \\ &= \quad -2\nu \left(\frac{2\nu p C_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}, \end{aligned}$$

Hence,

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$.

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

DiPerna-Majda 1987 result $\Longrightarrow \lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 = \|u^0(t,\cdot)\|_{L^2}^2$,

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

DiPerna-Majda 1987 result $\Longrightarrow \lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 = \|u^0(t,\cdot)\|_{L^2}^2$, uniformly in time.

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu \to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

DiPerna-Majda 1987 result $\Longrightarrow \lim_{\nu \to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 = \|u^0(t,\cdot)\|_{L^2}^2$, uniformly in time. *Non-concentration result*.

Hence,

$$0 \geq \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 \geq -(2\nu)^{\frac{2(p-1)}{p}} \left(\frac{pC_0}{2-p}\right)^{-\frac{2-p}{p}} \frac{p}{2(p-1)} t^{\frac{2(p-1)}{p}}.$$

Since $p > 1$ the right-hand-side of this inequality vanishes as $\nu \to 0$. Therefore,

$$\lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 - \|u_0^\nu\|_{L^2}^2 = 0.$$

DiPerna-Majda 1987 result $\Longrightarrow \lim_{\nu\to 0} \|u^\nu(t,\cdot)\|_{L^2}^2 = \|u^0(t,\cdot)\|_{L^2}^2$, uniformly in time. *Non-concentration result*.
Using strong convergence of initial data, together with the known fact that there are no energy concentrations for the vanishing viscosity limit with vorticity in L^p, $p > 1$, we complete the proof.

L^p-norms of vorticity

We consider conserved quantities for *vorticity*

L^p-norms of vorticity

We consider conserved quantities for *vorticity*

ω transported by div-free vector field:

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

L^p-norms of vorticity

We consider conserved quantities for *vorticity*

ω transported by div-free vector field:

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

Natural question:

L^p-norms of vorticity

We consider conserved quantities for *vorticity*

ω transported by div-free vector field:

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

Natural question: regularity conditions for conservation of $\|\omega(t,\cdot)\|_{L^p}$?

L^p-norms of vorticity

We consider conserved quantities for *vorticity*

ω transported by div-free vector field:

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

Natural question: regularity conditions for conservation of $\|\omega(t, \cdot)\|_{L^p}$?

More generally, regularity conditions for ω to be *renormalized solution* of the transport equation?

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C_b^1(\mathbb{R})$

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C_b^1(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved,

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C^1_b(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved, e.g. L^p norms.

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C^1_b(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved, e.g. L^p norms.

Also:

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C_b^1(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved, e.g. L^p norms.

Also: uniqueness for linear transport equation,

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C_b^1(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved, e.g. L^p norms.

Also: uniqueness for linear transport equation, Lagrangian formulation of transport,

Renormalized solutions

Consider the transport equation

$$\partial_t w + b \cdot \nabla w = 0.$$

Definition (DiPerna-Lions)

A measurable function w is a renormalized solution of the transport equation if

$$\partial_t \beta(w) + b \cdot \beta(w) = 0,$$

for every $\beta \in C_b^1(\mathbb{R})$

One consequence of being renormalized is that, for divergence-free b, rearrangement-invariant norms of w are conserved, e.g. L^p norms.

Also: uniqueness for linear transport equation, Lagrangian formulation of transport, (notion of Lagrangian solution).

- Mazzucato, Lopes Filho, N-L 2005:

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution.

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015:

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized.

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution;

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Crippa, Nobili, Seis, Spirito 2018:

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \implies$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Crippa, Nobili, Seis, Spirito 2018: every *physically realizable* weak solution of Euler,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Crippa, Nobili, Seis, Spirito 2018: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^1)$,

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Crippa, Nobili, Seis, Spirito 2018: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^1)$, is renormalized.

- Mazzucato, Lopes Filho, N-L 2005: Let $p \geq 2 \Longrightarrow$ every weak solution of 2D Euler, with $\omega \in L^\infty(L^p)$, is a renormalized solution. Proof is straightforward consistency from DiPerna-Lions.

- Crippa, Spirito 2015: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^p)$, $p > 1$, is renormalized. Proof is by considering *adjoint problem*; existence for adjoint, uniqueness of renormalized solution; duality proofs from DiPerna-Lions.

- Crippa, Nobili, Seis, Spirito 2018: every *physically realizable* weak solution of Euler, with $\omega \in L^\infty(L^1)$, is renormalized. Proof involves extension of DiPerna-Lions theory to encompass L^1 vorticity and establishing uniform integrability

Summary:

Summary: if u^E is physically realizable weak solution

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit)

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1$

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1 \implies$ energy conserved

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L_t^\infty L_x^p$ then:

- if $p > 1 \Longrightarrow$ energy conserved
- if $p \geq 1$

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1 \implies$ energy conserved

- if $p \geq 1$ L^p-norm of ω^E conserved

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1 \implies$ energy conserved
- if $p \geq 1$ L^p-norm of ω^E conserved
- if $p > 1$ then $u^\nu \to u^E$ $C_t(L^2_x)$

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1 \implies$ energy conserved
- if $p \geq 1$ L^p-norm of ω^E conserved
- if $p > 1$ then $u^\nu \to u^E$ $C_t(L^2_x)$
- if $p \geq 1$ then $\omega^\nu \rightharpoonup \omega^E$ w-$*$ $L^\infty_t L^p_x$.

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L^\infty_t L^p_x$ then:

- if $p > 1 \implies$ energy conserved
- if $p \geq 1$ L^p-norm of ω^E conserved
- if $p > 1$ then $u^\nu \to u^E$ $C_t(L^2_x)$
- if $p \geq 1$ then $\omega^\nu \rightharpoonup \omega^E$ w-$*$ $L^\infty_t L^p_x$.

Question:

Summary: if u^E is physically realizable weak solution (vanishing viscosity limit) with $\omega^E \in L_t^\infty L_x^p$ then:

- if $p > 1 \implies$ energy conserved
- if $p \geq 1$ L^p-norm of ω^E conserved
- if $p > 1$ then $u^\nu \to u^E$ $C_t(L_x^2)$
- if $p \geq 1$ then $\omega^\nu \rightharpoonup \omega^E$ w-$*$ $L_t^\infty L_x^p$.

Question: convergence of vorticity only weak or can it be improved?

First addressed by Constantin, Drivas, Elgindi 2019,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$.

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart $+$ new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$,
$\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L^1_t L^p$.

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart $+$ new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L^1_t L^p$. Then,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart $+$ new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$,
$\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L^1_t L^p$. Then,

passing to subsequences as needed,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart $+$ new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$,
$\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L^1_t L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L^\infty_t L^p_x$

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L_t^1 L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$

Nearly simultaneously

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L_t^1 L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$

Nearly simultaneously Ciampa, Crippa, Spirito 2020,

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart $+$ new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L_t^1 L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$

Nearly simultaneously Ciampa, Crippa, Spirito 2020, virtually same result

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \leq p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L_t^1 L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$

Nearly simultaneously Ciampa, Crippa, Spirito 2020, virtually same result but $1 \leq p < \infty$

First addressed by Constantin, Drivas, Elgindi 2019, $p = \infty$:
$\omega_0 \in L^\infty(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^2, **forcing** $g^\nu \in L^\infty L^\infty$. Then

$\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$, any $1 \le p < \infty$.

Proof is complicated, uses borderline regularity for Biot-Savart + new uniform short time estimates on vorticity gradients, intermediate linear problem

Recently N-L, Seis, Wiedemann 2020, $1 < p < \infty$: $\omega_0 \in L^p(\mathbb{T}^2)$, $\omega_0^\nu \to \omega_0$ in L^p, **forcing** $g^\nu \in L_t^1 L^p$. Then,

passing to subsequences as needed, $\omega^\nu \to \omega^E$ strongly in $L_t^\infty L_x^p$

Nearly simultaneously Ciampa, Crippa, Spirito 2020, virtually same result but $1 \le p < \infty$ and $g^\nu = 0$.

Discuss simpler case,

Discuss simpler case, $g^\nu = 0$

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$*,* $\omega_0 \in L^p(\mathbb{T}^2)$*,* $1 < p < \infty$*,*

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ *strong* L^p.

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ strong L^p. Let u^E be physically realizable Euler solution,

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ strong L^p. Let u^E be physically realizable Euler solution, $\operatorname{curl} u^E = \omega^E$,

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$*,* $\omega_0 \in L^p(\mathbb{T}^2)$*,* $1 < p < \infty$*,* $\omega_0^\nu \to \omega_0$ *strong* L^p*. Let* u^E *be physically realizable Euler solution,* $\operatorname{curl} u^E = \omega^E$*,* $\omega^E(0,\cdot) = \omega_0$*.*

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ strong L^p. Let u^E be physically realizable Euler solution, $\operatorname{curl} u^E = \omega^E$, $\omega^E(0, \cdot) = \omega_0$. Then

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ strong L^p. Let u^E be physically realizable Euler solution, $\mathrm{curl}\, u^E = \omega^E$, $\omega^E(0,\cdot) = \omega_0$. Then

$$\omega^\nu \to \omega^E \text{ strongly in } C(0,T;L^p(\mathbb{T}^2)),$$

Discuss simpler case, $g^\nu = 0$

Theorem (N-L, Seis, Wiedemann 2020)

Let $T > 0$, $\omega_0 \in L^p(\mathbb{T}^2)$, $1 < p < \infty$, $\omega_0^\nu \to \omega_0$ *strong* L^p. *Let* u^E *be physically realizable Euler solution,* $\operatorname{curl} u^E = \omega^E$, $\omega^E(0,\cdot) = \omega_0$. *Then*

$$\omega^\nu \to \omega^E \text{ strongly in } C(0,T;L^p(\mathbb{T}^2)),$$

where $\omega^\nu = \operatorname{curl} u^\nu$ *and* $u^\nu \rightharpoonup u^E$ *weak-*$*$ $L^\infty(0,T;L^2)$.

Proof:

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0, T; L^p)$,

Proof:

<u>Step 1</u> $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0, T; L^p)$, ω^ν equicontinuous $[0, T]$ to $\mathcal{D}'$

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0, T; L^p)$, ω^ν equicontinuous $[0, T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0, T; w - L^p)$ (Aubin-Lions)

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Proof:

<u>Step 1</u> $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0, T; L^p)$, ω^ν equicontinuous $[0, T]$ to $\mathcal{D}'$

<u>Step 2</u> $\omega^\nu \rightharpoonup \omega^E$ $C(0, T; w - L^p)$ (Aubin-Lions)

<u>Step 3</u> $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0, T)$

Indeed,

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p}$$

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p}$$

weak lower semicontinuity of norm

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p}$$

weak lower semicontinuity of norm

$$\leq \limsup_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0\|_{L^p}$$

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p}$$

weak lower semicontinuity of norm

$$\leq \limsup_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0\|_{L^p}$$

parabolic maximum principle

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_\nu \|\omega^\nu(t,\cdot)\|_{L^p}$$

weak lower semicontinuity of norm

$$\leq \limsup_\nu \|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0\|_{L^p}$$

parabolic maximum principle

$$= \|\omega(t,\cdot)\|_{L^p}!$$

Proof:

Step 1 $\omega^\nu \rightharpoonup \omega^E$ weak-$*$ $L^\infty(0,T;L^p)$, ω^ν equicontinuous $[0,T]$ to $\mathcal{D}'$

Step 2 $\omega^\nu \rightharpoonup \omega^E$ $C(0,T;w-L^p)$ (Aubin-Lions)

Step 3 $\|\omega^\nu(t,\cdot)\|_{L^p} \to \|\omega^E(t,\cdot)\|_{L^p}$ in $C(0,T)$

Indeed,

$$\|\omega(t,\cdot)\|_{L^p} \leq \liminf_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p}$$

weak lower semicontinuity of norm

$$\leq \limsup_{\nu} \|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega_0\|_{L^p}$$

parabolic maximum principle

$$= \|\omega(t,\cdot)\|_{L^p}!$$

$$0 \leq \|\omega(t,\cdot)\|_{L^p} - \|\omega^\nu(t,\cdot)\|_{L^p} \leq \|\omega(T,\cdot)\|_{L^p} - \|\omega^\nu(T,\cdot)\|_{L^p} \to 0$$

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p,

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed,

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence + convergence of norm $\Longrightarrow$ strong convergence.

Step 4 $\omega^{\nu_n}(t, \cdot) \to \omega(t, \cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence $+$ convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence $+$ convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

Step 4 $\omega^{\nu_n}(t, \cdot) \to \omega(t, \cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence + convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence + convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument/maximum principle

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence $+$ convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument/maximum principle/conservation of L^p-norm.

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence $+$ convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument/maximum principle/conservation of L^p-norm.

Obs

Step 4 $\omega^{\nu_n}(t,\cdot) \to \omega(t,\cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence $+$ convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument/maximum principle/conservation of L^p-norm.

Obs Proof is somewhat more complicated if there is forcing.

Step 4 $\omega^{\nu_n}(t, \cdot) \to \omega(t, \cdot)$ strong L^p, pointwise in $[0, T]$

Indeed, in L^p weak convergence + convergence of norm $\Longrightarrow$ strong convergence. Need $p > 1$

Step 5 Convergence is uniform in time:

use equicontinuity and a repeat of weak lower semicontinuity argument/maximum principle/conservation of L^p-norm.

Obs Proof is somewhat more complicated if there is forcing. Use intermediate linear problem.

Comments on Ciampa, Crippa, Spirito 2020

Comments on Ciampa, Crippa, Spirito 2020

- No forcing

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs:

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian,

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C_t^0 L_x^q$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$),

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C_t^0 L_x^q$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$), $1 \leq q < \infty$

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C^0_t L^q_x$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$), $1 \leq q < \infty$
- Eulerian includes $p = 1$,

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C^0_t L^q_x$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$), $1 \leq q < \infty$
- Eulerian includes $p = 1$, fluid domain is full plane;

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C^0_t L^q_x$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$), $1 \leq q < \infty$
- Eulerian includes $p = 1$, fluid domain is full plane; proof uses intermediate linear problem, uniform integrability of ω^ν, and an extension of DiPerna-Lions.

Comments on Ciampa, Crippa, Spirito 2020

- No forcing
- Two proofs: Lagrangian, Eulerian. Lagrangian is on $\mathbb{T}^2$, $p > 1$. Eulerian is on $\mathbb{R}^2$, $p \geq 1$. Claim $p = 1$ works on $\mathbb{T}^2$ also.
- Lagrangian uses stochastic Lagrangian representation of viscous vorticity. Quantitative comparison of distance between trajectories
- If $p = \infty$ get rate for $C^0_t L^q_x$ convergence (rate depends on L^1-modulus of continuity of $\omega_0 \in L^\infty$), $1 \leq q < \infty$
- Eulerian includes $p = 1$, fluid domain is full plane; proof uses intermediate linear problem, uniform integrability of ω^ν, and an extension of DiPerna-Lions.
- Also extend energy conservation to full plane fluid domain.

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation.

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation?

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? 'Yes' in 2D

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? ’Yes’ in 2D
- Vorticity transport is a relevant physical restriction on incompressible flow behavior that is ignored by wild solutions

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? 'Yes' in 2D
- Vorticity transport is a relevant physical restriction on incompressible flow behavior that is ignored by wild solutions – too irregular for vorticity transport.

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? 'Yes' in 2D
- Vorticity transport is a relevant physical restriction on incompressible flow behavior that is ignored by wild solutions – too irregular for vorticity transport.
- Energy conservation in the case $p = 1$?

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? 'Yes' in 2D
- Vorticity transport is a relevant physical restriction on incompressible flow behavior that is ignored by wild solutions – too irregular for vorticity transport.
- Energy conservation in the case $p = 1$? No tools.

Conclusions

- The Onsager scaling is not the last word on inviscid dissipation. Dynamical mechanism to avoid anomalous dissipation? 'Yes' in 2D
- Vorticity transport is a relevant physical restriction on incompressible flow behavior that is ignored by wild solutions – too irregular for vorticity transport.
- Energy conservation in the case $p = 1$? No tools.
- Vorticity weak solutions obtained as limits of smooth approximations or the vortex blob method are also renormalized.

Thank you!