Invariant measures for horospherical flows.

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G : conn semisimple real alg gp

(e.g $G = PSL_2\mathbb{R}$, $PSL_2\mathbb{R} \times PSL_2\mathbb{R}$, $PSL_d\mathbb{R}$)

$\Gamma < G$ discrete, Zariski dense

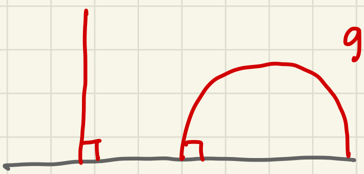
$\Gamma \backslash G \curvearrowleft N$ max horospherical subgp

$\boxed{\text{" What are } N\text{-inv ergodic measures ? "}}$

I. Horocycle flow on hyp surfaces

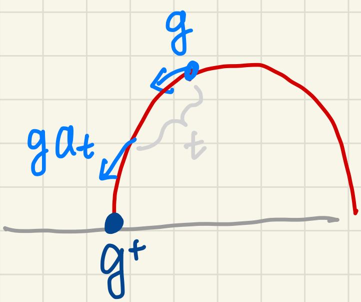
$$\mathbb{H}^2=\{x=(x_1,x_2)\mid x_2>0\} \qquad ds=\frac{dx}{x_2}$$

$\partial\mathbb{H}^2=\mathbb{R}\cup\{\infty\}$

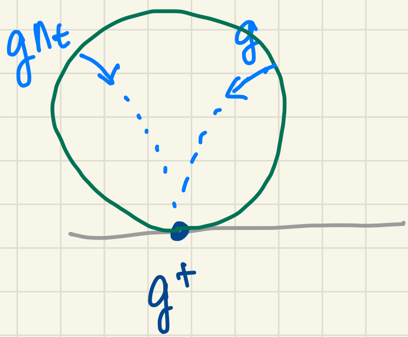


$$\mathrm{Isom}^+(\mathbb{H}^2)=PSL_2\mathbb{R}$$

$$T^1(\mathbb{H}^2)=PSL_2\mathbb{R}$$



geodesic flow $\longleftrightarrow$ $a_t=\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}\end{pmatrix}$

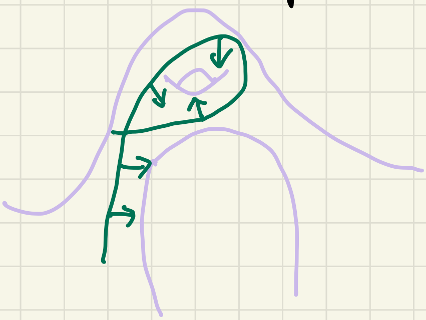


horocycle flow $\longleftrightarrow$ $n_t=\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$

$\Gamma < PSL_2\mathbb{R}$ discrete (torsion-free)

$S = \Gamma \backslash \mathbb{H}^2$ hyp surface

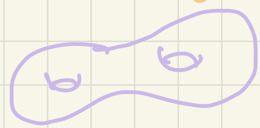
$$T^1(S) = \Gamma \backslash T^1(\mathbb{H}^2) = \Gamma \backslash PSL_2\mathbb{R}$$



$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

horocyclic subgp

Thm (Furstenberg 1973)



$\Gamma < PSL_2\mathbb{R}$ cocpt lattice

The N-action on $\Gamma \backslash PSL_2\mathbb{R}$ is uniquely ergodic.

I.e., $\exists!$ N-inv (loc finite & Borel)

measure on $\Gamma \backslash PSL_2\mathbb{R}$

(= Haar measure)

$\mathbb{II}$. G conn semi-simple real alg gp.

Ex) ① $G = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

② $G = PSL_d\mathbb{R}$ $d \geq 2$

Def $N < G$ horospherical

if $\exists$ diagonalizable $a \in G$ s.t

$$N = \{ g \in G \mid \lim_{k \to \infty} a^{-k} g a^{k} = e \}$$

Ex) ① $G = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

$$N = \left(\begin{array}{c|c} 1 & \mathbb{R} \\ \hline 0 & 1 \end{array}\right) \times \{e\}, \quad \{e\} \times \left(\begin{array}{c|c} 1 & \mathbb{R} \\ \hline 0 & 1 \end{array}\right), \quad \left(\begin{array}{c|c} 1 & \mathbb{R} \\ \hline 0 & 1 \end{array}\right) \times \left(\begin{array}{c|c} 1 & \mathbb{R} \\ \hline 0 & 1 \end{array}\right)$$

up to conjugation

② $G = PSL_d\mathbb{R}$

$$N = \left(\begin{array}{c|c|c|c} I_{d_1} & * & * & * \\ \hline 0 & I_{d_2} & * & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & I_{d_k} \end{array}\right) \qquad d_1 + \cdots + d_k = d$$

up to conjugation

N max horospherical subgp

(uniq. up to conj.)

Ex) ① $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} < PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

② $N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} < PSL_d\mathbb{R}$

$\boxed{\begin{array}{l} \text{Thm (Veech 1977)} \\ \Gamma < G \text{ cocpt lattice} \\ \text{The } N\text{-action on } \frac{G}{\Gamma} \text{ is} \\ \text{uniquely ergodic.} \end{array}}$

Any N-inv measure on $\Gamma \backslash G$

is Haar measure.

Thm (Dani 1978 for G = pdt of rank 1 gps

1981 G = general)

$\Gamma < G$ lattice

Any N-inv ergodic measure σ on $\Gamma \backslash G$

is algebraic / homogeneous,

I.e., $\exists$ conn closed subgp $N < H < G$

s.t σ is an H-inv measure

supported on a closed H-orbit in $\Gamma \backslash G$.

$\rightsquigarrow$ **Dani's conjecture** :

Any invariant ergodic finite measure for a unipotent subgp on $\Gamma\backslash G$ is homogeneous.

: measure theoretic counterpart of Raghunathan's conj. on orbit closures for unip subgp action

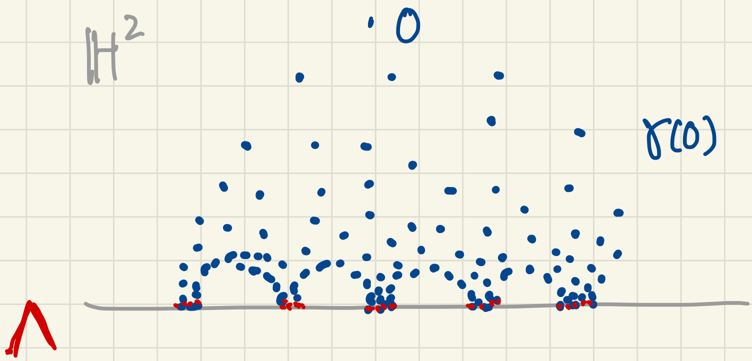
Proved by Ratner 1991.

Q: What can we say about N-inv loc. finite & infinite measures on $\Gamma\backslash G$ when $\mathrm{vol}(\Gamma\backslash G) = \infty$?

III. $\Gamma < PSL_2\mathbb{R}$ discrete, Z. dense

Def (Limit set of Γ)

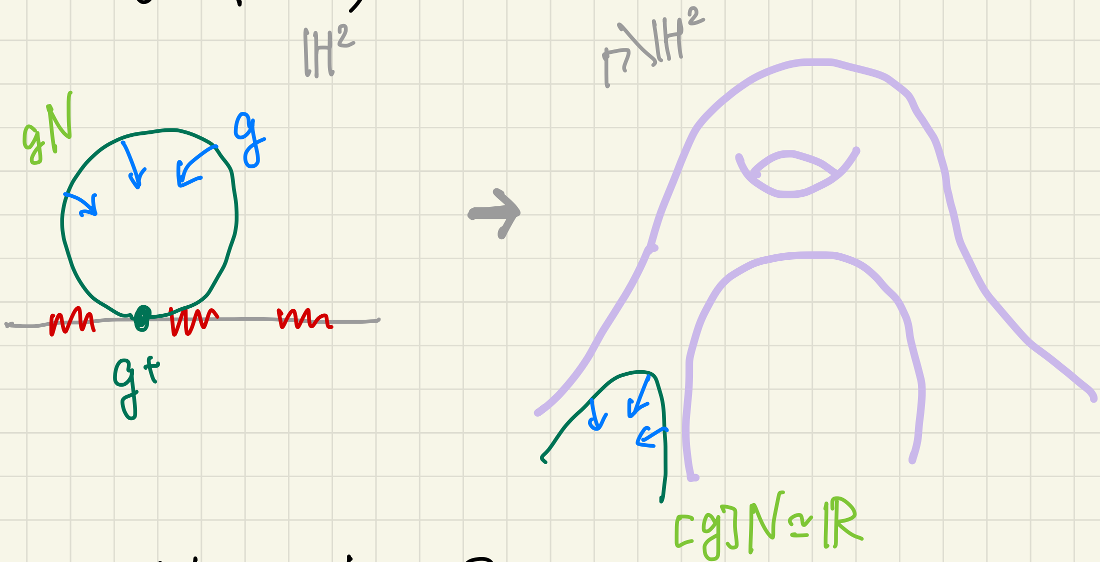
Λ = Accumulation pts of $\Gamma(o)$



uniq Γ-min subset of $\partial\mathbb{H}^2$

If $\Lambda \neq \partial\mathbb{H}^2$, $\exists$ many "trivial" N-inv measures on $\Gamma\backslash SL_2\mathbb{R}$

If $g^+ \notin \Lambda$,

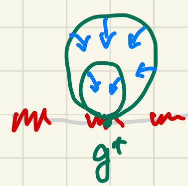


$$[g]N \simeq N \simeq \mathbb{R}$$

dt $\uparrow$: N-inv measure on $\Gamma\backslash G$

Set

$$\mathcal{E}_\Gamma = \{ [g] \in \Gamma\backslash G \mid g^+ \in \Lambda \}$$

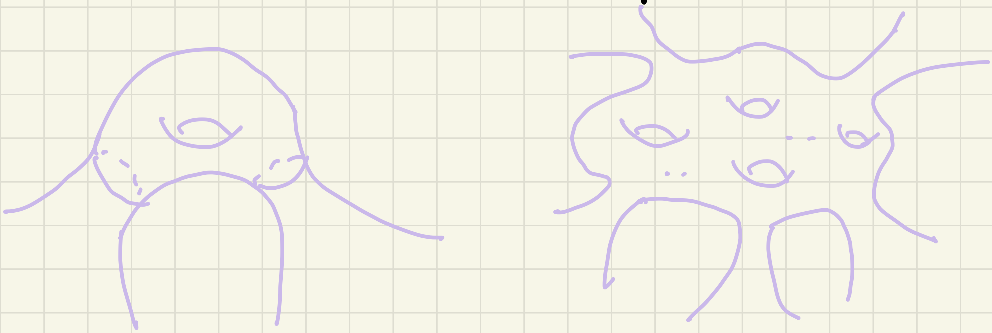


= union of all horocycles based on Λ

: closed N-inv subset in $\Gamma\backslash G$

Q: Is there Unique Ergodicity for the N-action on $\mathcal{E}_\Gamma$?

<u>Def</u> $\Gamma < PSL_2\mathbb{R}$ convex cocompact if $\mathrm{core}(\Gamma \backslash \mathbb{H}^2) = \Gamma \backslash \mathrm{Hull}(\Lambda)$ compact



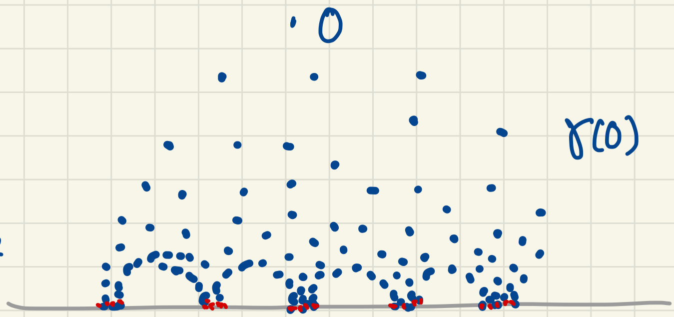
Thm (Burger 1990, Roblin 2003)

$\Gamma < PSL_2\mathbb{R}$ convex cocpt

The N-action on $\mathcal{E}_\Gamma = \{ [g] \mid g^+ \in \Lambda \}$ is uniquely ergodic.

$\exists!$ N-inv measure on $\mathcal{E}_\Gamma$ $(\subset \Gamma\backslash G)$

$=$ **Burger-Roblin measure.**



Patterson-Sullivan
1979

$\exists!$ Γ-conf measure ν_{PS} on Λ (Γ : convex cocpt)

Γ-inv measure on $G \simeq G/P \times P$ $\qquad P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = AN$

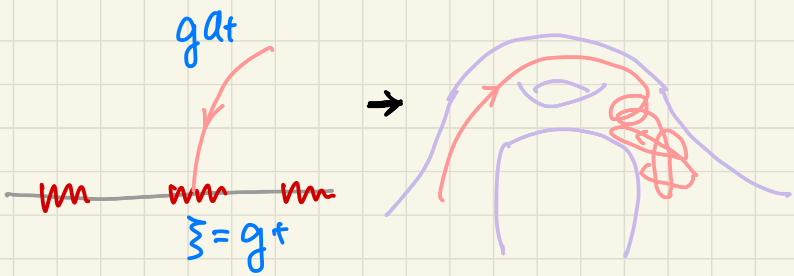
$$= \partial\mathbb{H}^2 \times P$$
$$\cup$$
$$\Lambda \times P$$

Burger-Roblin measure $\quad m^{BR} \approx \nu_{PS} \times dp.$ $\quad$ supported on

$$\mathcal{E}_\Gamma = \{[g] \in \Gamma\backslash G \mid \underset{g^+}{\overset{\shortparallel}{gP}} \in \Lambda\}$$

Main feature of Convex cocpt gps in the Unique Ergodicity of the N-action on $\mathcal{E}_\Gamma$

$$\Lambda = \Lambda_{\text{conical}}$$



Any geodesic ray toward $\xi \in \Lambda$ is <u>recurrent</u> to a cpt subset in $\Gamma \backslash \mathbb{H}^2$

$$\mathcal{E}_\Gamma = \mathcal{R}_\Gamma$$: Recurrent set for the geod. flow

$$\mathcal{R}_\Gamma = \{ [g] \in \Gamma \backslash PSL_2\mathbb{R} \mid \limsup_{t \to +\infty} \Gamma g a_t \neq \phi \}$$

Thm (Roblin 2003, Winter 2015)

G : simple $\mathbb{R}$-alg gp rank $G = 1$

$\Gamma < G$ convex cocompact

N-action on $\mathcal{E}_\Gamma$ is uniq. erg.

(BR-measure is the uniq
N-inv meas. on $\mathcal{E}_\Gamma$)

Q: Higher rank ?

IV. G : conn semisimple real alg gp.

$\Gamma < G$ discrete & Z. dense

"Limit set of Γ"

- P = minimal parabolic subgp of G

$$P = MAN$$

N: max. horospherical subgp

A: max $\mathbb{R}$-split torus $\quad \dim(\mathrm{Lie}\,A) = \mathrm{rank}\,G$

- $\mathcal{F} := G/P$ Furstenberg boundary of G/K

- Cartan decomposition

$$G = K \exp \mathfrak{a}^+ K$$

$\mathfrak{a}^+ < \mathrm{Lie}\,A$ pos. Weyl chamber

- Cartan projection $G \to \mathfrak{a}^+ \quad g \mapsto \mu(g)$

$$g \in K \exp \mu(g) K$$

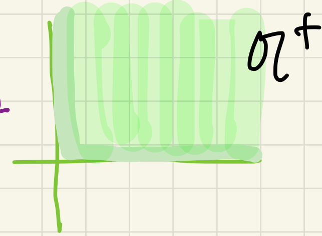
"vector-valued distance"

Ex) ① $G = PSL_2\mathbb{R} \times PSL_2\mathbb{R} = \mathrm{Isom}^\circ(\mathbb{H}^2 \times \mathbb{H}^2)$

$$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \times \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \qquad A = \left\{ \begin{pmatrix} e^{t_1/2} & \\ & e^{-t_1/2} \end{pmatrix} \times \begin{pmatrix} e^{t_2/2} & \\ & e^{-t_2/2} \end{pmatrix} \right\}$$

$$G/P = S^1 \times S^1 = \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$$

$$\mathfrak{a}^+ = \{ (t_1, t_2) \mid t_1, t_2 \geq 0 \} < \mathfrak{a} = \mathbb{R}^2$$



② $G = PSL_d\mathbb{R}$

$$P = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \qquad A = \left\{ \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_d} \end{pmatrix} \;\middle|\; \sum t_i = 0 \right\}$$

G/P = Space of full flags in $\mathbb{R}^d$

$$\mathfrak{a}^+ = \left\{ (t_1, \dots, t_d) \;\middle|\; \begin{array}{c} t_1 \geq t_2 \geq \dots \geq t_d \\ \sum t_i = 0 \end{array} \right\}$$

Limit set of Γ

$o \in G/K$

$\mathcal{F} = G/P \supset \Lambda$ = Accumulation pts of $\Gamma(o)$ in "$\mathcal{F}$"

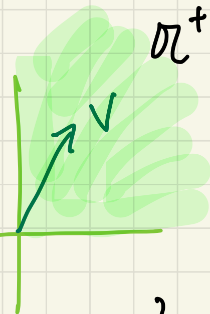
= uniq Γ-min subset of $\mathcal{F}$

$$\mathcal{E}_\Gamma = \{ [g] \in \Gamma\backslash G \mid gP \in \Lambda \}$$

$$\begin{array}{cc} g & G \\ \downarrow & \downarrow \\ gP & G/P \end{array}$$

uniq P min. subset of $\Gamma\backslash G$

For each $v \in \mathfrak{a}^+ \setminus \{0\}$,



v-directional recurrent subset

$$\mathcal{R}(v) = \{ [g] \in \Gamma\backslash G \mid \limsup_{t\to+\infty} \Gamma g \exp(tv) \neq \emptyset \}$$

P-invariant Borel subset of $\mathcal{E}_\Gamma$

If $\Gamma < G$ cocpt, $\mathcal{R}(v) = \Gamma\backslash G$ $\forall v$.

Q: Is there Unique Ergodicity for N-action on $\mathcal{R}(v)$ for certain class of discrete subgps?

$\Pi = \{\alpha_1, \dots, \alpha_r\}$ set of all simple roots

$(r = \operatorname{rank} G)$

$$\mathfrak{a} \simeq \mathbb{R}^{\operatorname{rank} G}$$

$$u \longmapsto (\alpha_1(u), \dots, \alpha_r(u))$$

$\boxed{\begin{array}{l}\underline{\text{Def}} \text{ A finitely gen. } \Gamma < G \text{ is} \\ \text{Anosov } (\text{w.r.t } \Pi) \\ \text{if } (\Gamma, |\cdot|_{\text{word}}) \underset{\text{Q.I}}{\sim} (\Gamma, \alpha_i \circ \mu) \ \forall i\end{array}}$

$\exists\, C_1, C_2 \geq 1$ s.t. $\forall \gamma \in \Gamma$,

$$(C_1|\gamma| + C_2) \geq \alpha_i(\mu(\gamma)) \geq C_1^{-1}|\gamma| - C_2$$

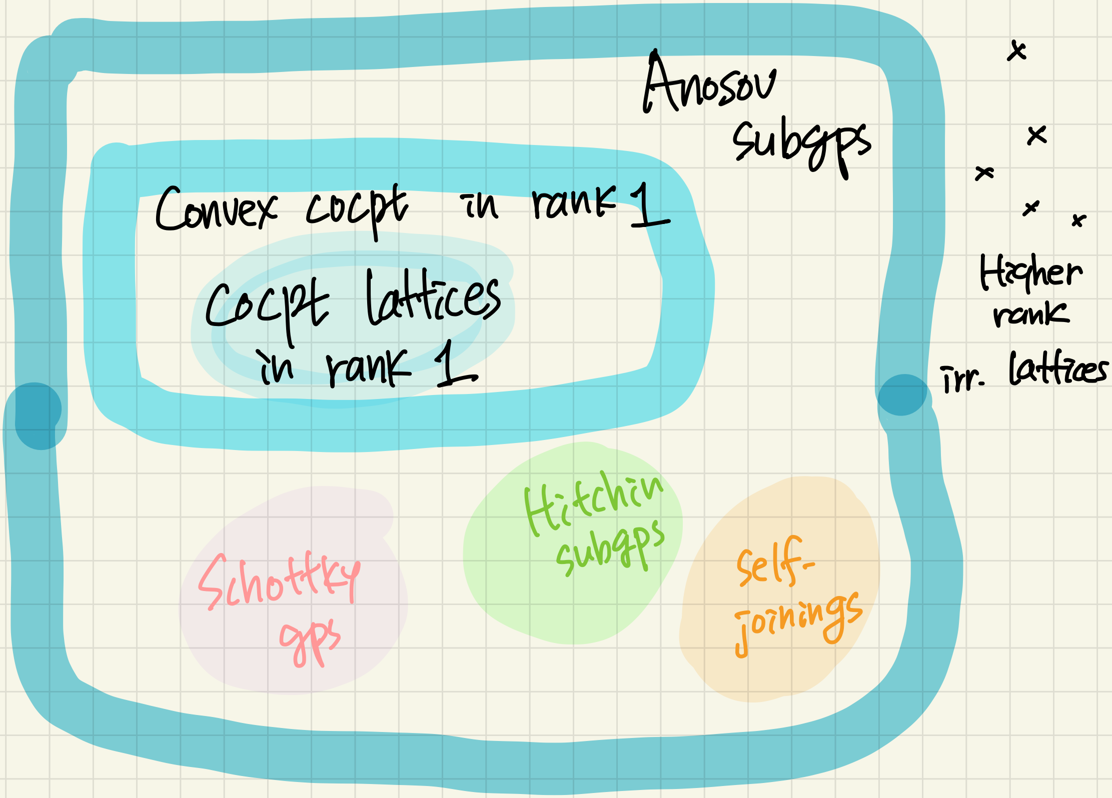
Labourie 2006

Guichard-Wienhard 2012

Kapovich-Leeb-Porti 2017

If $\operatorname{rank} G = 1$, Anosov $\Leftrightarrow (\Gamma, |\cdot|) \overset{\text{Q.I}}{\sim} (\Gamma o, d)$ Riem metric

$\Leftrightarrow$ convex cocpt.



Ex) Self-joining

$\Sigma < PSL_2\mathbb{R}$ cocpt lattice

$\rho \in \mathrm{Teich}(\cancel{\mathbb{H}^2}\ \Sigma) \iff \rho : \Sigma \to PSL_2\mathbb{R}$ disc. faithful

$\Sigma_\rho = (\mathrm{Id} \times \rho)(\Sigma)$
$= \{ (g, \rho(g)) \mid g \in \Sigma \} < PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

Anosov.

Anosov subgps in $G = \prod_{i=1}^{r} G_i$ $\operatorname{rank} G_i = 1$

are self-joinings of Convex Cocpt gps.

Σ : f.g $\rho_i : \Sigma \longrightarrow G_i$ convex cocpt rep with finite ker.

$$\Gamma = \left(\prod_{i=1}^{r} \rho_i\right)(\Sigma)$$
$$= \{ (\rho_1(g), \dots, \rho_r(g)) \mid g \in \Sigma \} \underset{\text{Anosov}}{<} G$$

Higher-rank analogues of Burger-Roblin measures on $\mathcal{E}_\Gamma = \{ [g] \in \Gamma \backslash G \mid gP \in \Lambda \}$

$\Gamma < G$ Z.d discrete

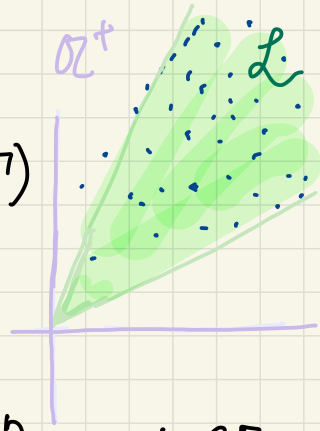
$$G \simeq G/P \times P$$
$$\cup$$
$$\Lambda \times P \qquad m^{BR} \approx \nu \otimes dp$$

limit set of Γ

ν : Γ-conf measure on Λ

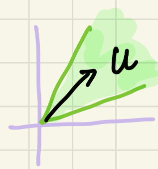
Def The limit cone of Γ

$$\mathcal{L} = \text{the asymptotic cone of } \mu(\Gamma)$$
$$= \left\{ \lim_{i \to \infty} t_i \mu(\gamma_i) \;\middle|\; \begin{array}{l} t_i \to 0 \\ \gamma_i \in \Gamma \end{array} \right\}$$



Convex cone with $\operatorname{int} \mathcal{L} \neq \phi$ Benoist 97

For any unit vector $u \in \operatorname{Int} \mathcal{L}$,
Quint constructed a Γ-conf measure ν_u on Λ



$$\mathbb{P}(\operatorname{Int} \mathcal{L}) \longrightarrow \{\Gamma\text{-conf. measures on } \Lambda\}$$
$$u \longmapsto \nu_u$$

Ex) If $\Gamma < G$ cocpt, $\nu_u = $ Leb. measure on $\mathcal{F} = G/P$
$\forall\, u \in \operatorname{Int} \mathcal{L} = \operatorname{Int} \mathfrak{a}^+$

Γ : Anosov, Z dense

Classification of Γ-conformal measures

Thm (Lee-O. 2020)

$$\mathbb{P}(\mathrm{Int}\,\mathcal{L}) \overset{\text{homeo.}}{\Longleftrightarrow} \{\text{all } \Gamma\text{-conf measures on } \Lambda\}$$

$\mathbb{P}(\mathrm{Int}\,\mathcal{L}) \cong \mathbb{R}^{\mathrm{rank}\,G - 1}$

$$u \longmapsto \nu_u$$

mutually singular to each other

$G = G/P \times P$

$\cup$

$\Lambda \times P$

For each direction $u \in \mathrm{Int}\,\mathcal{L}$,

$$m_u^{BR} \simeq \nu_u \otimes dp \; : \; P \text{ quasi-inv measure on } \mathcal{E}_\Gamma$$

Burger-Roblin measure

$\mathcal{E}_\Gamma = \{ [g] \in \Gamma \backslash G \mid gP \in \Lambda \}$

Ergodicity of Burger-Roblin measures

Thm (Lee-O. 2020) $u \in \operatorname{Int} \mathcal{L}$

(1) Each m_u^{BR} is MN-ergodic

(2) $\exists\ 1 \le k = k(\Gamma) \le [P : P^\circ] = [M : M^\circ]$

m_u^{BR} has precisely k # of N-erg comp.

Ex. If M is connected, (e.g. pdt of rank one gps center-free)

then m_u^{BR} is N-ergodic.

For each unit $u \in \mathfrak{a}^+$,

$\mathcal{R}_u$ = the u-directional recurrent set

$= \{ [g] \in \Gamma \backslash G \mid \limsup_{t \to +\infty} \Gamma g \exp(tu) \neq \phi \}$

: P-inv Borel subset of $\mathcal{E}_\Gamma$

Rank dichotomy on the relation between m_u^{BR} & $\mathcal{R}_v$

Thm (Burger-Landesberg-Lee-O. 2021)

- If $\operatorname{rank} G \leq 3$ & $u \in \operatorname{Int} \mathcal{L}$,

  $\mathcal{R}_u$ is co-null for m_u^{BR}

- In all other cases, i.e., $\operatorname{rank} G \geq 4$ or $v \notin \operatorname{Int} \mathcal{L}$ or $u \neq v$,

  $\mathcal{R}_v$ is null for m_u^{BR}

Unique Ergodicity on $\mathcal{R}_u$

Thm (Landesberg-Lee-Lindenstrauss-O.)
2021

center-free

G = pdt of rank one gps $= \prod_{i=1}^{r} G_i$

$\Gamma < G$ Anosov (= self-joining gps)

- If $r \leq 3$, $\forall\, u \in \operatorname{Int} \mathcal{L}$,

  m_u^{BR} is the uniq N-inv measure supported on $\mathcal{R}_u$

- Otherwise, i.e, if $r \geq 4$ or $u \notin \operatorname{Int} \mathcal{L}$,

  No N-inv measure supported on $\mathcal{R}_u$

HAPPY BIRTHDAY

DAVI !!