NON-VANISHING MODULO p OF VALUES OF A MODULAR FORM AT CM POINTS

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Abstract. The values at specific CM points of a Hecke eigenform are closely related to modular L-values. Non-vanishing question of a modular measure made out of the values of a modular form is interesting on its own but also has good applications to non-triviality questions of L-values.

Take a CM quadratic extension M/F and a prime ideal $\mathfrak{l}$ of F. For a CM type (M, Σ) and a projective ideal $\mathcal{A}$ of the order R_n of conductor $\mathfrak{l}^n$, we have a CM point $x(\mathcal{A})$ in the Hilbert modular Shimura variety Sh for F. For a Hecke eigenform f of level K on Sh, after modifying the value $f(x(\mathcal{A}))$ into $f_\lambda([\mathcal{A}]) := f_\lambda(x(\mathcal{A}))$ by a suitably chosen Hecke character λ, $f_\lambda([\mathcal{A}])$ only depends on the class $[\mathcal{A}]$ of $\mathcal{A}$ in $Cl_n^- = \mathrm{Coker}(\mathrm{Pic}(O) \to \mathrm{Pic}(R_n))$ (for the integer ring O of F), and we have a "measure" $d\varphi_f$ on the finite group Cl_n^- given by $\int_{Cl_n^-} \chi d\varphi_f := \sum_{\mathcal{A} \in Cl_n^-} \chi(\mathcal{A}) f_\lambda([\mathcal{A}])$ which is a Hecke L-value if f is an Eisenstein series and is a square root of a central critical value of $L(s, f)$ suitably twisted by the character χ (a result of Shimura and Waldspurger).

Decomposing $Cl_\infty^- = \varprojlim_n Cl_n^- = \Gamma \times \Delta^-$ for a torsion-free Γ and a finite group Δ^- is not usually an algebraic operation as prime-to-$\mathfrak{l}$ ideals cannot be torsion in Cl_∞^- except for ambiguous classes. Forgetting about ambiguous classes and choosing a representative set $\mathcal{Q}$ for Δ^- of prime ideals $\mathfrak{Q}$ of M and writing $[\mathfrak{Q}]_\Gamma$ for the projection to Γ, $[\mathfrak{Q}]_\Gamma$ cannot be represented by an ideal. Then we have $\Delta^- = \{[\mathfrak{Q}]_\Gamma [\mathfrak{Q}]^{-1} | \mathfrak{Q} \in \mathcal{Q}\}$. Projecting the measure $d\varphi_\Gamma$ to Γ is a transcendental operation. The projected measure is associated to a linear combination of $f([\mathcal{A}\mathfrak{Q}^{-1}][\mathfrak{Q}]_\Gamma)$, and hence, the projected measure cannot be associated to a modular form. We can embed $x(\mathcal{A}) \mapsto \mathbf{x}(\mathcal{A}) := (x(\mathcal{A}[\mathfrak{Q}]_\Gamma))_\mathfrak{Q} \in Sh^\mathcal{Q}$ for the $\mathcal{Q}$-product $Sh^\mathcal{Q} = \prod_\mathcal{Q} Sh$ as at each step Cl_n^- of the projective limit, $[\mathfrak{Q}]_\Gamma$ is represented by an R_n-ideal. Since the operation $\mathcal{A} \mapsto \mathcal{A}[\mathfrak{Q}]_\Gamma$ is transcendental, one expects $\{\mathbf{x}(\mathcal{A})\}_\mathcal{A}$ is Zariski dense in $Sh^\mathcal{Q}$. On the other hand, we have an automorphism $\langle \mathfrak{Q} \rangle$ of Sh such that $f|\langle \mathfrak{Q} \rangle (x(\mathcal{A})) = f(x(\mathcal{A}\mathfrak{Q}^{-1}))$

Now choose an unramified prime $p > 2$ outside $N(\mathfrak{l})$. Suppose that a thin set $\Xi = \{\mathbf{x}(\mathcal{A})\}_\mathcal{A}$ for some $\mathcal{A}$ of CM points is dense in $Sh^\mathcal{Q}$. Then, if the $\mathcal{Q}$-tuple $(f|\langle \mathfrak{Q} \rangle \mod p)_\mathfrak{Q}$ of mod p modular forms vanish on Ξ, we conclude that $f \mod p = 0$, which rarely happens by the q-expansion principle, and we get a non-vanishing mod p of the L-values. Without touching upon L-values, we study in these lectures the following question for a level K quotient $Sh_K = Sh/K$:

When a thin set of CM points is Zariski dense in mod p Hilbert modular Shimura varieties Sh_K and its products?

The title of each lecture is
(1) An introduction,
(2) Irreducible components of Zariski closure,
(3) Characters of vanishing integral and the thin point set Ξ,
(4) Proof of the non-vanishing theorem.