ELASTIC CLOAKING THEORY

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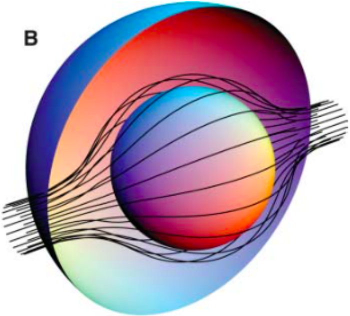
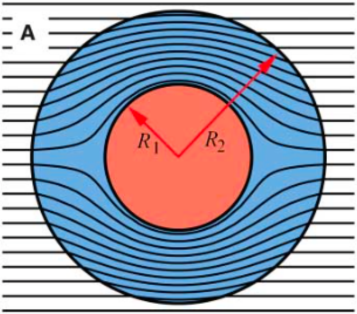
- Illusionists use tricks to render objects invisible.
- 2006 saw scientific breakthrough in this seemingly impossible task:
  [Ref.] J.B.PENDRY, D.SCHURIG, D.R.SMITH, **Science**, **312** *(2006).*
  [Ref.] U.LEONHARDT, **Science**, **312** *(2006).*
- They showed the possibility of designing a coat around an object such that object becomes invisible to electromagnetic radiation.





ELECTROMAGNETIC WAVES

- Governing equations: Maxwell's equations.
- Unknowns: Electric and Magnetic fields: $\mathcal{E}(t,x) \in \mathbb{R}^3$ and $\mathcal{H}(t,x) \in \mathbb{R}^3$

$$\begin{cases} \nabla \times \mathcal{E} = -\mu\, \partial_t \mathcal{H} \\ \nabla \times \mathcal{H} = \ \ \epsilon\, \partial_t \mathcal{E} \end{cases}$$

- Here $\mu, \epsilon \in \mathbb{R}^{3\times 3}$ are material parameters:
magnetic permeability and **electric permittivity**.

AN IDEA TO CLOAK

- The coating material has physical properties $(\mu,\, \epsilon)$ such that the electromagnetic waves are bent around the object.
- Materials with such exotic physical properties are meta-materials.

- Geometric setting: A medium $\Omega \supset B_2 \supset B_1$
- B_1 – region we wish to hide.
  $B_2 \setminus B_1$ – region we wish to apply the meta-material coat.
- Let $\mu_0,\ \epsilon_0$ be constant **magnetic permeability** and **electric permittivity**.
- Let $\mathcal{E}^0$ and $\mathcal{H}^0$ be associated electric and magnetic fields, i.e.

$$\begin{cases} \nabla \times \mathcal{E}^0 = -\mu_0\ \partial_t \mathcal{H}^0 \\ \nabla \times \mathcal{H}^0 = \ \ \epsilon_0\ \partial_t \mathcal{E}^0 \end{cases}$$

♣ Goal: Build magnetic permeability and electric permittivity

$$\mu_{\rm cl}(y),\ \epsilon_{\rm cl}(y) = \begin{cases} \mu_0,\ \epsilon_0 & \text{for } y \in \Omega \setminus B_2 \\ \text{magic parameters} & \text{for } y \in B_2 \setminus B_1 \end{cases}$$

♣ Take the associated Electric and Magnetic fields: $\mathcal{E}^*$ and $\mathcal{H}^*$ satisfy

$$\begin{cases} \nabla \times \mathcal{E}^* = -\mu_{\rm cl}\ \partial_t \mathcal{H}^* \\ \nabla \times \mathcal{H}^* = \ \ \epsilon_{\rm cl}\ \partial_t \mathcal{E}^* \end{cases}$$

♣ Assignment of magic parameters should be such that

$$\mathcal{E}^*(t,\cdot) = \mathcal{E}^0(t,\cdot) \qquad \text{and} \qquad \mathcal{H}^*(t,\cdot) = \mathcal{H}^0(t,\cdot) \qquad \text{in } \Omega \setminus B_2$$

irrespective of the material properties of $\mu_{\rm cl},\ \epsilon_{\rm cl}$ in B_1

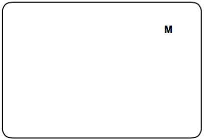
Q1 Can cloaking be translated from electromagnetism to other areas of wave physics? – ACOUSTICS.

Q2 Can meta-material ideas be carried over to other physical systems where waves are absent altogether?

- STATIC MECHANICS (elastostatics)
- THERMODYNAMICS (transient heat propagation)

In the absence of wave behaviour, what it means to cloak?

Isotropic homogenous medium



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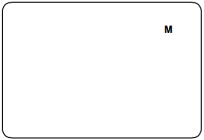
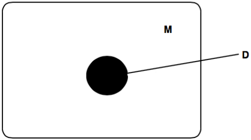


medium with a defect



WHAT ELSE CAN BE CLOAKED?

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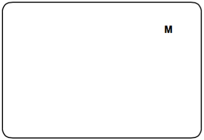
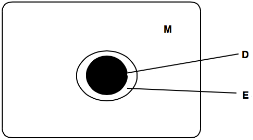


cloak surrounding the defect



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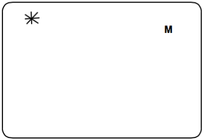

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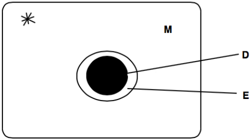

MATERIAL PROPERTIES AND OBSERVATIONS

Context	**Governing equation**	**Material properties**	**Observations**
Electromagnetism	Maxwell's equations	permittivity permeability	Electric field Magnetic field
Thermodynamics	Heat equation	density heat capacity conductivity	Temperature field
Elastostatics	Lamé (Navier) system	Elasticity tensor	Displacement field
Thin plate theory	Linear Kirchhoff equation	Flexural rigidity	Flexural displacement

WE TAKE A LITTLE DETOUR INTO ELECTROSTATICS

ELECTROSTATICS

- Unknown: Electric potential: $u(x) \in \mathbb{R}$

$$\begin{cases} \nabla \cdot (\gamma(x) \nabla u) = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = f \end{cases}$$

- Here $\gamma(x) \in \mathbb{R}^{N\times N}$ denotes the conductivity of Ω.
- We have the notions of

Isotropic: $\gamma(x) = \gamma_0(x)\mathrm{Id}$; Homogeneous: $\gamma_0 \equiv \text{const}$.

Otherwise, anisotropic.

DIRICHLET TO NEUMANN MAP (DtN)

- Given f on the boundary, there exists a unique $u(x)$. Define

$$\Lambda_\gamma(f) = \gamma(x)\nabla u \cdot \mathbf{n}\big|_{\partial\Omega}$$

Here $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$.
- Also referred to as VOLTAGE TO CURRENT MAP.

WHAT IT MEANS TO CLOAK IN ELECTROSTATICS

- ♣ Take $\Omega \subset \mathbb{R}^N$ filled with conductivity Id.
- ♣ Let Λ_{hom} be the associated DtN map.
- ♣ Now take $D \subset \Omega$, a subdomain.
- ♣ Question: Is it possible to prescribe a conductivity

$$\gamma_{\text{cl}}(x) = \begin{cases} \tilde{\gamma} & \text{for } x \in \Omega \setminus D \\ \gamma_1 & \text{for } x \in D \end{cases}$$

such that the associated DtN map $\Lambda_{\gamma_{\text{cl}}}$ satisfies

$$\Lambda_{\text{hom}}(f) = \Lambda_{\gamma_{\text{cl}}}(f) \qquad \forall \text{ voltages } f \text{ on the boundary,}$$

irrespective of the conductivity γ_1 in D.

♣ Integration by parts yields

$$\int_{\partial\Omega} \Lambda_\gamma(f)(x) v(x) \, \mathrm{d}\sigma(x) = \int_\Omega \gamma(x) \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x.$$

♣ Take a C^1 diffeomorphism

$$F : \Omega \to \Omega \quad \text{such that} \quad F(x) = x \ \text{ for } x \in \partial\Omega.$$

♣ Making a change of variable $y = F(x)$ inside the integral yields

$$\int_\Omega \gamma(x) DF^\top(x) \nabla_y u^*(y) \cdot DF^\top(x) \nabla_y v^*(y) \frac{\mathrm{d}y}{\det DF(x)}$$

where $u^*(y) := u(F^{-1}(y))$ and $v^*(y) := v(F^{-1}(y))$.

TRANSFORMATION INVARIANCE

♣ The conductivities

$$\gamma(x) \qquad \text{and} \qquad F_*\gamma(y) = \frac{DF(x)\gamma(x)DF^\top(x)}{\det DF(x)} \quad \text{with } x = F^{-1}y.$$

have the same DtN maps, i.e.

$$\Lambda_\gamma = \Lambda_{F_*\gamma}.$$

♣ An appropriate choice of F will yield the coefficient γ_{cl}.

- Blow-up a point (origin): Take $\Omega = B_2$ and

$$F(x) := \left(1 + \frac{1}{2}|x|\right)\frac{x}{|x|} \qquad \text{for } x \in B_2 \setminus \{0\}$$

- F maps 0 to B_1 and maps $B_2 \setminus \{0\}$ to $B_2 \setminus B_1$
- F is smooth except at 0
- Cloaking coefficient:

$$\gamma_{\rm cl}(x) = \begin{cases} F_* \mathrm{Id} & \text{for } x \in \Omega \setminus D \\ \gamma_1 & \text{for } x \in D \end{cases}$$

  with $D = B_1$, i.e. the domain to which the point gets mapped to.
- $\gamma_{\rm cl}$ has SINGULARITY when approaching boundary of B_1

- Regularised transform

$$\mathcal{F}_\varepsilon(x) := \begin{cases} x & \text{for } x \in \Omega \setminus B_2 \\ \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon}\right) \frac{x}{|x|} & \text{for } x \in B_2 \setminus B_\varepsilon \\ \frac{x}{\varepsilon} & \text{for } x \in B_\varepsilon \end{cases}$$

- Taking $\varepsilon = 0$ yields singular transform.
- $\mathcal{F}_\varepsilon$ maps B_ε to B_1 and maps $B_2 \setminus B_\varepsilon$ to $B_2 \setminus B_1$
- $\mathcal{F}_\varepsilon$ is smooth everywhere
- Corresponding cloaking coefficients are NOT singular

- Unknown displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$ solves

$$\begin{cases} \nabla \cdot \big(\mathbb{C} : \nabla \mathbf{u}\big) = \mathbf{0} & \text{in } \Omega, \\ \big(\mathbb{C} : \nabla \mathbf{u}\big)\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

- Here

$$\big(\mathbb{C} : \nabla \mathbf{u}\big)_{ij} := \sum_{k,l=1}^{d} C_{ijkl} \big(\nabla \mathbf{u}\big)_{kl} \qquad \text{for} \quad 1 \le i, j \le d$$

- Here $\mathbb{C} = (C_{ijkl})$ is the fourth-order elasticity tensor satisfying

$$(\mathbb{C} : A) : A = \sum_{i,j,k,l=1}^{d} C_{ijkl} A_{ij} A_{kl} \ge \mathfrak{a} \sum_{i,j=1}^{d} |A_{ij}|^2 =: \mathfrak{a} \, \|A\|^2$$

for any symmetric matrix A.

- Typical elastic material has the following symmetries:

$$\text{major symmetry: } C_{ijkl} = C_{klij},$$

$$\text{minor symmetries: } C_{ijkl} = C_{jikl} = C_{ijlk},$$

  for $i, j, k, l = 1, ..., d$
- Isotropic homogeneous elastic medium has

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

- λ bulk modulus (first Lamé parameter) and μ shear modulus
- In this setting, elasticity system reads as

$$\begin{cases} \mu\,\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla.\mathbf{u}) = \mathbf{0} & \text{in } \Omega, \\ \lambda\,(\nabla.\mathbf{u})\mathbf{n} + \mu\,\mathbf{e}(\mathbf{u})\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega. \end{cases}$$

  where $\mathbf{e}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)$ is symmetrized gradient

♣ Objective: Given $0 < \delta \ll 1$, construct a fourth order tensor $\mathbb{C}^\delta_{\rm cl}(x)$ with the structure

$$\mathbb{C}^\delta_{\rm cl}(x) = \begin{cases} \mathbb{C}_0 & \text{for } x \in \Omega \setminus B_2, \\ \mathbb{D}^\delta & \text{for } x \in B_2 \setminus B_1, \\ \mathbb{C}_1 & \text{for } x \in B_1, \end{cases}$$

where $\mathbb{C}_0$ and $\mathbb{C}_1$ are isotropic fourth order tensors of the form with Lamé parameters λ_0, μ_0 and λ_1, μ_1 respectively

♣ The above construction should be such that the solution $\mathbf{u}^\delta_{\rm cl}$ to elasticity system with $\mathbb{C}^\delta_{\rm cl}$ as the elasticity tensor satisfies

$$\left\| \mathbf{u}^\delta_{\rm cl} - \mathbf{u}_{\rm hom} \right\|_{\mathrm{H}^{\frac{1}{2}}\left(\partial\Omega;\mathbb{R}^d\right)} \lesssim h(\delta)$$

$\mathbf{u}_{\rm hom}$ being solution to elasticity system with elasticity tensor $\mathbb{C}_0$

- ♣ Let $\mathbb{C}$ be a fourth order tensor.
- ♣ Consider a Lipschitz invertible map $\mathbb{F} : \Omega \mapsto \Omega$ such that $\mathbb{F}(x) = x$ for each $x \in \Omega \setminus B_2$.
- ♣ Then $\mathbf{u}(x)$ is a solution to

$$\nabla \cdot \Big(\mathbb{C}(x) : \nabla \mathbf{u} \Big) = \mathbf{0} \qquad \text{in } \Omega$$

if and only if $\mathbf{v} = \mathbf{u} \circ \mathbb{F}^{-1}$ is a solution to

$$\nabla \cdot \Big(\mathbb{F}^* \mathbb{C}(y) : \nabla \mathbf{v} \Big) = \mathbf{0} \qquad \text{in } \Omega,$$

where the tensor $\mathbb{F}^*\mathbb{C}$ is defined as

$$\left[\mathbb{F}^* \mathbb{C} \right]_{ijkl}(y) = \frac{1}{\det(D\mathbb{F})(x)} \sum_{p,q=1}^{d} \frac{\partial \mathbb{F}_i}{\partial x_p}(x) \frac{\partial \mathbb{F}_k}{\partial x_q}(x) \mathbb{C}_{pjql}(x),$$

for all $i, j, k, l \in \{1, \ldots, d\}$.
- ♣ Moreover we have,

$$\mathbf{u}(x) = \mathbf{v}(x) \qquad \text{for all } x \in \Omega \setminus B_2.$$

SOME OPEN QUESTIONS IN INVISIBILITY CLOAKING

- Design of meta-materials.
  - Exotic physical properties are not known for ordinary materials.
  - Design assemblies of small components which may have desired properties as a whole.
  - Inverse homogenization: given a property, look for a microstructure.
- Invisibility cloaking in nonlinear models.
- Rigorous results on invisibility cloaking for wave phenomena.
- Is it possible to cloak **gravitational waves**?
  - study Einstein's field equations
  - transformation media theory for this very geometric equation?
  - LIGO's achievement in detecting gravitational waves

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THANK YOU FOR YOUR ATTENTION