Instanton and Sphaleron in QCD and SM

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Outline

1. Topology in QCD and in the Standard Model
2. Instantons vs Sphalerons
3. Applications in QCD and SM
4. How to compute them in SM, QCD?

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1. Instantons vs Sphalerons
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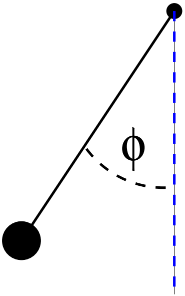
4. How to compute them in SM, QCD?

Instanton, Sphaleron: A Toy Problem

I will borrow a toy problem from Arnold and McLerran,
"The Sphaleron Strikes Back," PRD 37 (1988) 1020

Consider a quantum pendulum

$$\mathcal{L}(\phi) = \frac{m\ell^2}{2}\dot{\phi}^2 - gm\ell(1 - \cos\phi)$$

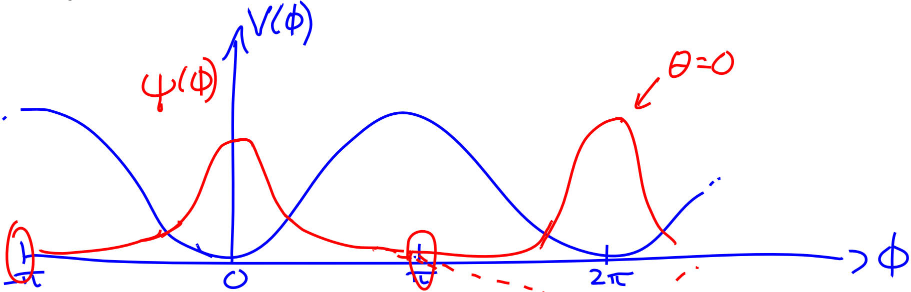


But with a twist (literally!)

$$\psi(\phi + 2\pi) = e^{i\theta}\psi(\phi)$$

- How much does the ground-state energy E_0 or the free energy $F(T)$ depend on θ?
- How much does the pendulum spin around?
  That is, how often, at T, does ϕ change by 2π?

Why does θ matter at all? Think about $\theta = 0$, $\theta = \pi$



For $\theta = 0$: periodic at $\phi = \pi$, ψ has minimum.
For $\theta = \pi$: antiperiodic at $\phi = \pi$, ψ has a zero.
We know the $\theta = 0$ energy is lower. How much?

Path integral

One way (not the easiest, but..) to compute: path integral!
Vacuum energy: for large β,

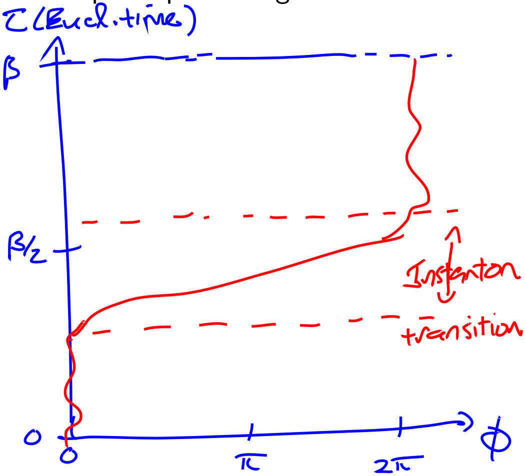
$$\langle 0|e^{-\beta H}|0\rangle = e^{-\beta E_0}$$

approximate $|0\rangle$ as $\phi = 0$.
θ-dependence: transitions

$$\langle \phi = 2\pi|e^{-\beta H}|\phi = 0\rangle$$

Controlled by "instanton" transitions

$\Delta\tau \sim 1/\omega$



Look more at instanton

Vacuum energy:

$$\langle 0|e^{-\beta H}|0\rangle \sim \langle \phi=0|e^{-\beta H}|\phi=0\rangle + \sum_{n=\pm 1} e^{in\theta}\langle \phi = 2n\pi|e^{-\beta H}|\phi=0\rangle$$
$$\sim e^{-\beta E_0} + 2\cos(\theta)e^{-\beta E_0 - S_I} \simeq e^{-\beta E_0}(1+2\cos(\theta)e^{-S_I})$$

Where in $\tau \in [0,\beta]$ does instanton happen? $e^{-S_I} \propto \beta$: $e^{-S_I} = (\beta\omega)e^{-S'_I}$
Only one instanton? Can be more! Exponentiate expression!

$$e^{-\beta E} = e^{-\beta(E_0 - 2\omega\cos(\theta)e^{-S'_I})}$$

Vacuum energy shift: $E = E_0 - 2\omega\cos(\theta)e^{-S_I}$
zero-mode-amputated instanton action gives θ-dependent energy shift.

All assumes instantons well separated: $S'_I \gg \omega$

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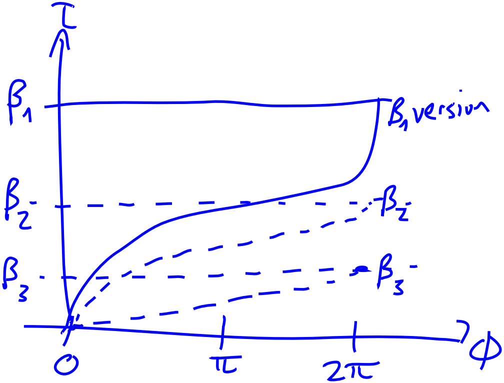
Raise temperature: make $\beta = 1/T$ finite rather than large

$$\mathrm{Tr}\, e^{-\beta H} = e^{-F(T)/T}$$

How does free energy $F(T)$ depend on β (or $T = 1/\beta$)?



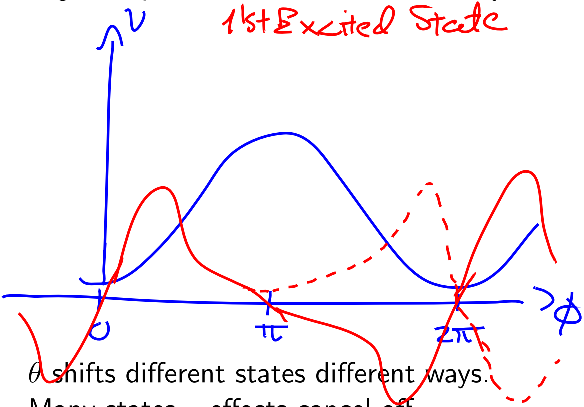
Instanton "natural width" $\sim 1/\omega$
gets "squeezed" if $\beta < 1/\omega$
That is, for $T > \omega$
Where excited states contribute

$$S_I \simeq \frac{m\ell^2}{2} \int d\tau \dot{\phi}^2 = 2\pi^2 m\ell^2\, T$$

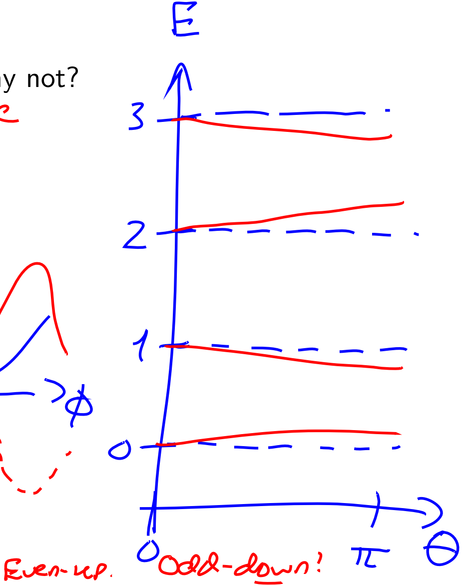
$S_I \propto T$, $\Delta F \propto \exp(-2\pi^2 m\ell^2 T)$

Thermal instanton effects

Instanton determines $F(T)$.
High temp: doesn't care about θ. Why not?



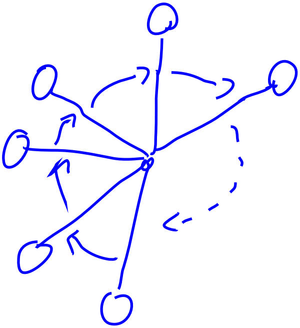
θ shifts different states different ways.
Many states – effects cancel off.



Does ϕ change value in real time?

Logically distinct question – as you turn up T, how often does ϕ loop around by 2π?

Intuition: if energy $E > 2mg\ell$, pendulum spins around.
Occurs with $e^{-E/T} \sim e^{-2mg\ell/T}$ probability
Thermal activation.
More likely as T gets larger.



Sphaleron rate

Let's couple pendulum to thermal bath (many oscillators, individually weak couplings, range of frequencies..)
Details not important – provides (weak) noise + damping
Pendulum gains energy, spins, loses energy, oscillates ..
$\pm 2\pi$ transitions independent sign if wide-spaced in time.

Very long times: ϕ develops in Brownian "random walk"

$$\left\langle (\phi(t) - \phi(0))^2 \right\rangle \propto t$$

$$\Gamma_{\rm sphal} \equiv \lim_{t\to\infty} \left\langle \frac{(\phi(t) - \phi(0))^2}{t} \right\rangle$$

Sphaleron rate $\Gamma_{\rm sphal}$ controls rate of ϕ-diffusion.

How fast does ϕ change?

What is the rate at which ϕ makes 2π transitions?

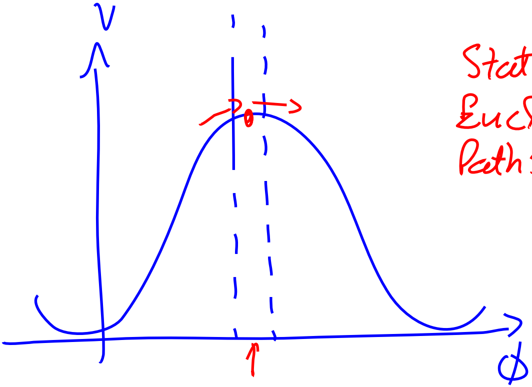
1. How much of the time $\phi \simeq \pi$, **times**
2. How fast ϕ moves across $\phi = \pi$ value, **times**
3. How correlated successive $\phi = \pi$ crossings are.

Last factor is hard to determine, but for "intermediate" damping, it is close to 1.

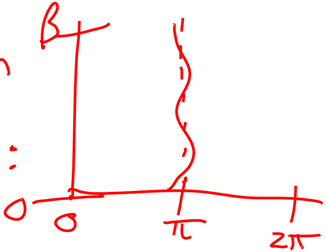
Other factors: $$\Gamma_{\rm sphal} \simeq P(\phi=\pi) \left\langle \left| \frac{d\phi}{dt} \right|_{\phi=\pi} \right\rangle$$

Euclidean picture

How big is $P(\phi = \pi)$? $|d\phi/dt|$? Combination?



State, in Eucl. Path Int:



Sphaleron $\sigma\phi\alpha\lambda\epsilon\rho o\varsigma$ "slippery" (thanks Klinkhamer and Manton PRD 30 (1984) 2212)

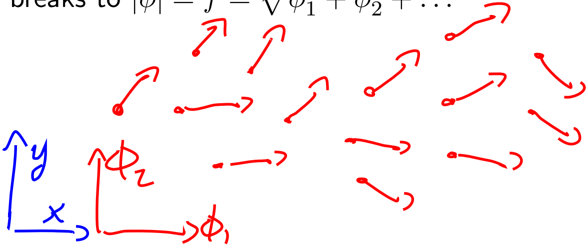
Topology, gauge theory

Change gears: topology in gauge theory.
Warm-up toy problem: N-component scalar theory, symmetry breaking

$$-\mathcal{L}(\phi_i) = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{8}\left(\phi_i\phi_i - f^2\right)^2$$

$\phi(x)$ "like" $\vec{V}$

breaks to $|\phi| = f = \sqrt{\phi_1^2 + \phi_2^2 + \dots}$



Which ϕ component is nonzero can vary through space!

Topology: 2 components in 2 dimensions

Topology: 3 components in 3 dim, 4 in 4 ..

Now, SU(2) gauge theory

Gauge group $\mathrm{SU}(2) \cong S^3$. Most general element:

$$\Lambda = \exp\left(i\theta_i \frac{\sigma_i}{2}\right)$$
$$= \cos\frac{|\theta|}{2}\mathbf{1} + i\sin\frac{|\theta|}{2}\frac{\theta_i}{|\theta|}\sigma_i$$

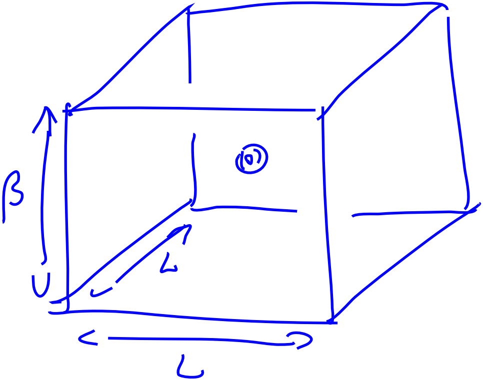
Think of $\mathbf{1}$ and σ_i as 4 independent directions.
Sum-squared of coefficients is 1.
So SU(2) is 3-sphere in space with axes $\mathbf{1}, i\sigma_1, i\sigma_2, i\sigma_3$
and points in SU(2) can "wind" over an S^3 the same as $\phi_{1,2,3,4}$.

Gauge fields on T^4

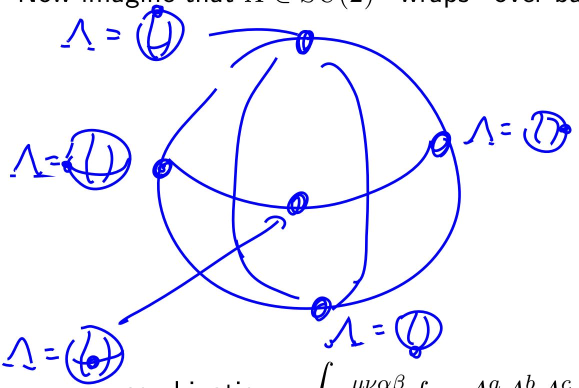
Consider a box L on a side, periodic boundaries, time extent β.
Lattice people live here. Gauge theory well defined.
If L big enough, could this be our universe?

You cannot always define A_μ^a over whole space. (Like coord on Earth – leave out the poles.)
You *can* define A_μ^a in "chart" consisting of space, minus a small ball, radius $r \ll L$.
A_μ pure gauge on this surface,

$$A_\mu^a = \mathrm{Tr}\,\Lambda^{-1} T^a \partial_\mu \Lambda$$



Now imagine that $\Lambda \in \mathrm{SU}(2)$ "wraps" over ball:



combination $\int \epsilon^{\mu\nu\alpha\beta} f_{abc} A^a_\mu A^b_\nu A^c_\alpha \hat{n}_\beta \, d^3\Sigma = 6 \times 16\pi^2$

It just so happens that:

$$K^\mu \equiv \frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\left(F^a_{\nu\alpha}A^a_\beta - \frac{1}{3}f_{abc}A^a_\nu A^b_\alpha A^c_\beta\right)$$

$$\partial_\mu K^\mu = \frac{1}{64\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} \equiv \frac{1}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu}$$

The thing which integrates to a winding number over the ball is the boundary-term when you integrate

$$\int d^4x\,\frac{1}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu} = \int_\Sigma n_\mu K^\mu = N_I$$

The combination $\int d^3\vec{x} K^0 \equiv N_{\rm CS}$ Chern-Simons Number

Instanton action

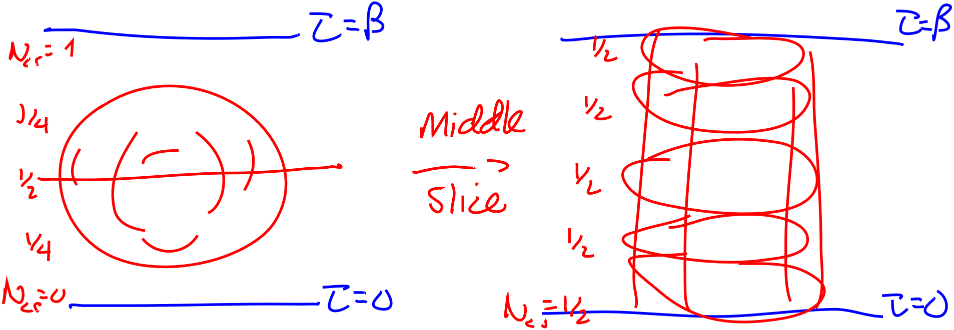
Suppose $\int F\tilde{F} = 32\pi^2$. What is the action?

$$
\begin{aligned}
0 &\leq (F_{\mu\nu} + \tilde{F}_{\mu\nu})^2 \\
0 &\leq F_{\mu\nu}^2 + 2F_{\mu\nu}\tilde{F}_{\mu\nu} + \tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} \\
2F_{\mu\nu}\tilde{F}_{\mu\nu} &\leq 2F_{\mu\nu}F_{\mu\nu}
\end{aligned}
$$

So $S = \int (1/4g^2) F_{\mu\nu}F_{\mu\nu} \geq 8\pi^2/g^2$

Instantons are suppressed by $e^{-8\pi^2/g^2}$ or more.

Instanton | Caloron



$$\int F\tilde{F} = 32\pi^2$$

$$\int F\tilde{F} = 0$$

But $\int d^3x\, K^0 = 1/2$

Application: Instantons in SU(3) QCD

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The most general Lagrangian for QCD is actually (Euclidean)

$$-\mathcal{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{i\Theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} + \sum_f \bar{\psi}_f \left(\not{D} + m_f \right) \psi_f$$

Θ plays same role as θ in toy model:
configs with instantons pick up phase $e^{iN_I\Theta}$, shifting free energy density

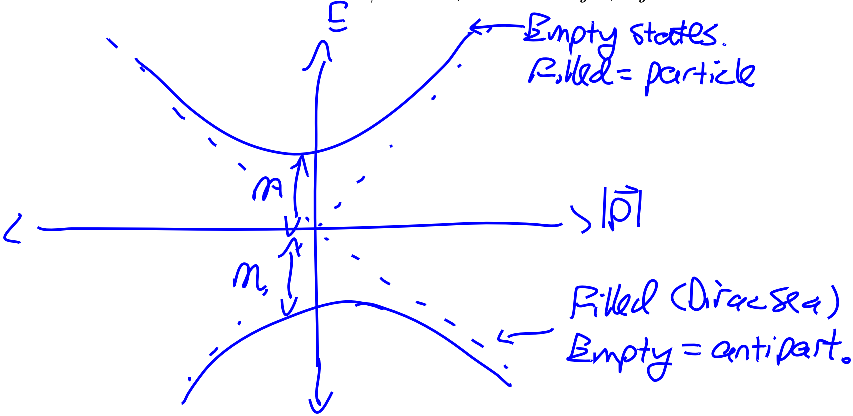
$$F(T, \Theta) = F(T, \Theta = 0) - \chi(T)(1 - \cos(\Theta))$$

Experiment: $|\Theta| < 10^{-10}$.
Axion makes Θ dynamical field. $F(T, \Theta)$ is effective potential, which shows why $\Theta = 0$ but also controls dynamics of axion field in astrophysics and cosmology. Another talk...

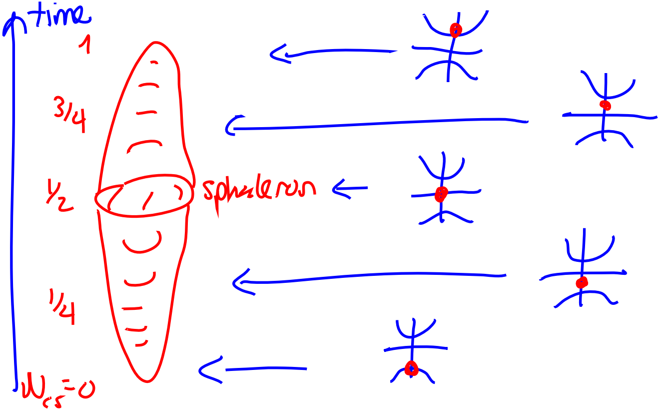
Application: Sphalerons in Standard Model

The SM has an SU(2) coupling to L-handed quarks Q,
and Higgs breaking the SU(2) and coupling Q to R-handed U, D

Fermionic Hamiltonian $\quad H_\psi = \bar{\psi}_i(\vec{\gamma}\cdot\vec{D} + y_{ij}h)\psi_j \quad$ gapped:



Consider a real-time sphaleron process...



Violation of B and L. Origin of matter-antimatter asymmetry??

Sphalerons in QCD

Equal left, right handed particles. But m_u, m_d small.
Sphaleron creates $u_L, d_L, \overline{u}_R, \overline{d}_R$.
Sphaleron processes create–and equilibrate–R, L imbalance
Possible role in, eg, chiral magnetic effect

Sphaleron rate in Electroweak

Use that $\alpha_w = g^2/4\pi = 1/30$ is small.
Sphalerons involve $\mathcal{O}(1/\alpha_w) \sim 30$ quanta;
Classical field approximation.
Allows *real-time* nonperturbative lattice treatment
Problem solved in 1990's

Situation different in QCD: even $\alpha_s \sim 1/10$ at $T \sim 100\,\mathrm{GeV}$ marginal for classical methods....

How do I compute Instantons in QCD?

- Low temperature $T < T_c$: chiral perturbation theory.
- Medium temperature $T_c < T < 10T_c$: lattice
- Very high temperature: pert. theory in instanton background

Lattice calculations are hard!

- Defining topology – and getting between topologies – on lattice
- Hi T – instantons become rare as T^{-7}.
  Need technique to overcome sampling problem
- Fermion almost-zero modes difficult to treat precisely

Modern techniques can overcome these problems .. another talk.

That's a secret!

or, current research

Summary

- Instantons – tunneling transitions between vacuum sectors
  Relevant for vacuum energy
- Sphalerons – real-time transitions between sectors
  Relevant for real-time dynamics
- These are *very different*, as toy model shows
- Nonabelian gauge theories have topology – Instanton, Sphaleron play roles.
- Sphalerons needed in EW theory for baryogenesis
  Maybe in QCD for chiral imbalance relaxation
- Instantons needed in QCD if there is an axion
- Calculation in SM is well controlled.
  Instantons in QCD getting under control
  Sphalerons in QCD: new techniques coming out!