Cosmography using gravitational waves

Surhud More

IUCAA

Course plan

- Basics of Cosmology and the Hubble tension
- Gravitational waves as standard sirens
- Mapping the expansion history of the Universe:
  - Bright sirens
  - Dark sirens
  - Utilizing the large scale clustering

Big questions

- How old is mankind?
- What are we made up of?
- How did we come into existence?
- What is the fate of mankind?

Flammarion's *L'atmosphère : météorologie populaire* (Paris: Hachette, 1888)

Age old questions in the domain of religion

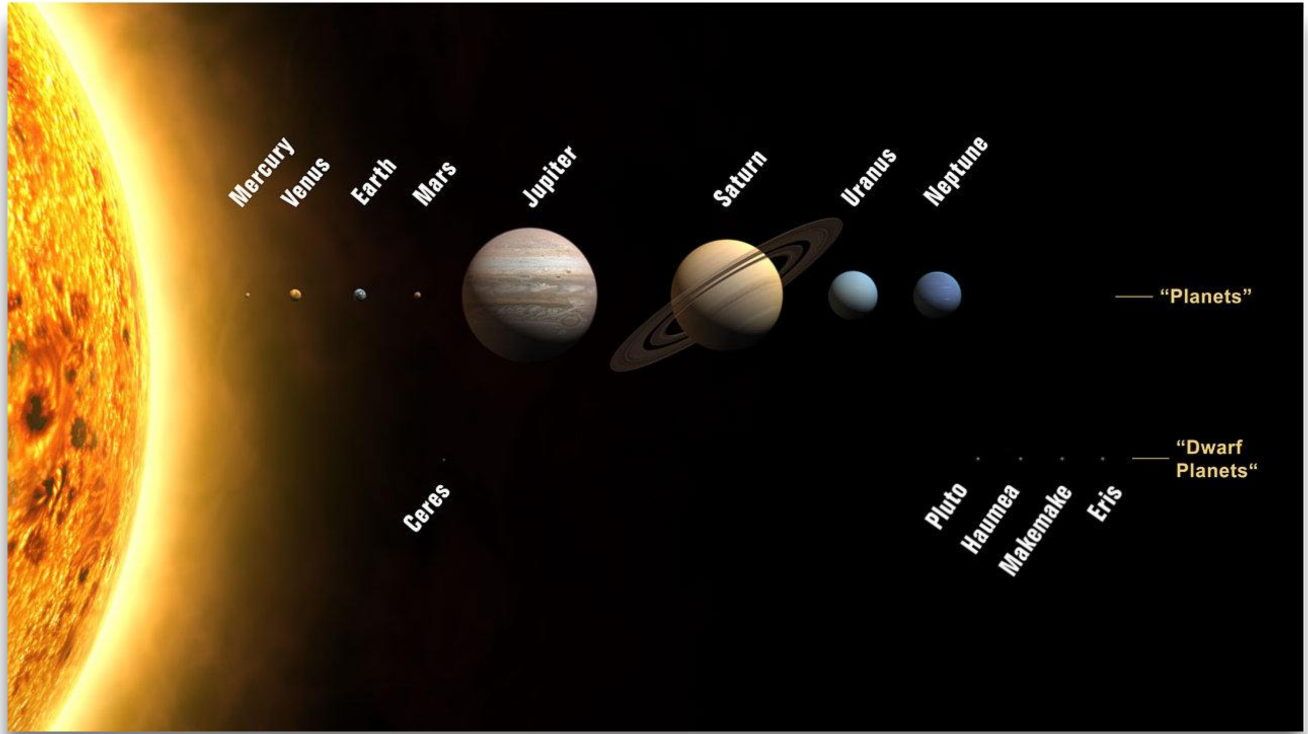
Big questions

- How old is the Universe?
- What is it made up of?
- How did the Universe come into existence?
- What is the fate of the Universe?

Now can be addressed scientifically

Flammarion's *L'atmosphère : météorologie populaire* (Paris: Hachette, 1888)

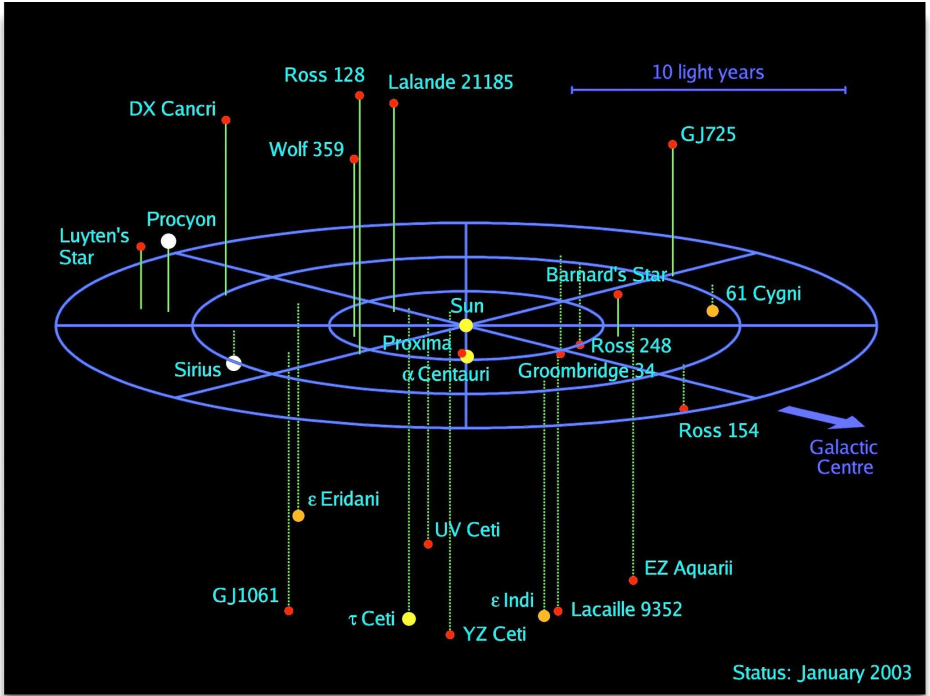
Observing the Universe



- Solar system: Light travel time of a few hours to a day!

https://en.wikipedia.org/wiki/Solar_System

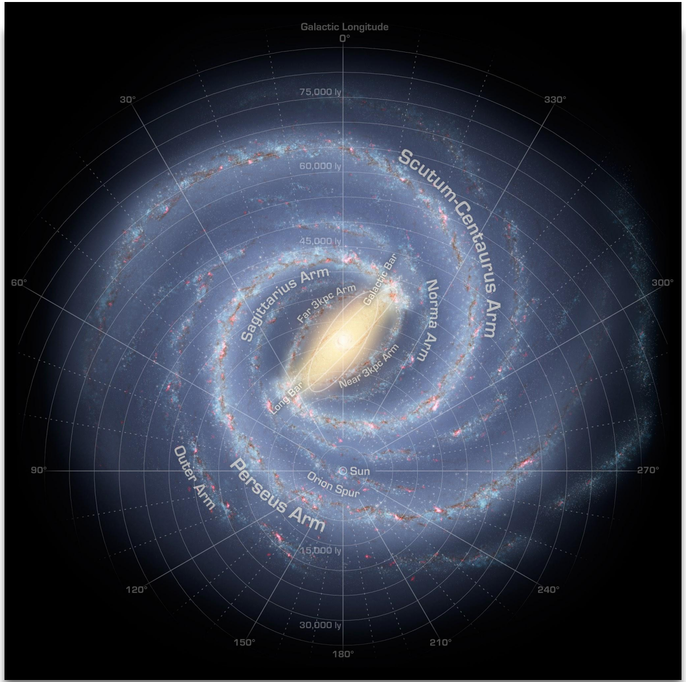
Observing the Universe



- Solar neighborhood: Light travel time of years (d ~ parsecs)

https://www.eso.org/public/images/eso0303c/

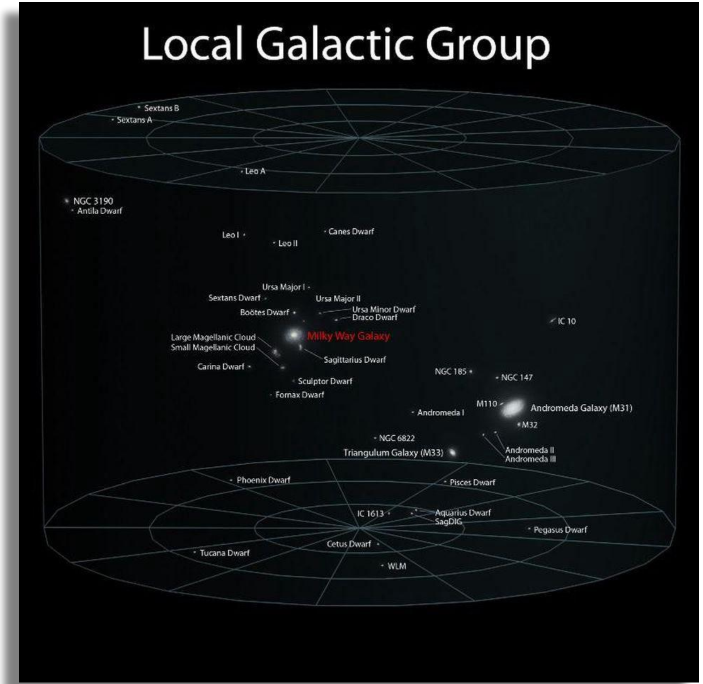
Observing the Universe



- The Milky way galaxy: Light travel time of 30000 years (d ~ 10 kpc)

https://solarsystem.nasa.gov/system/resources/detail_files/285_ssc2008-10b1.jpg

Observing the Universe



- The galactic neighborhood: Light travel time of millions of years (d ~ Mpc)

https://miro.medium.com/max/1920/0*82oP09fTzI1OV9hy.jpg

Observing the Universe

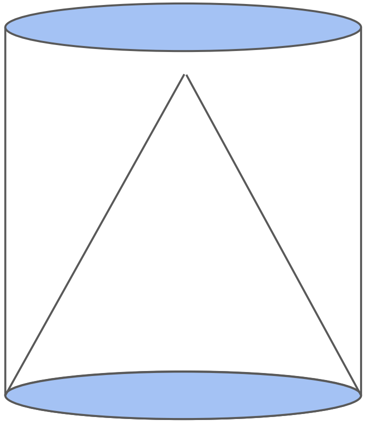
Observable Universe

- The Sloan digital sky survey: Billions of light years (d ~ Gpc)

https://www.youtube.com/watch?v=08LBltePDZw&t=6s

The cosmological principle

- **Homogeneity:**
  - Invariance under translations.
- **Isotropy:**
  - Invariance under rotations.

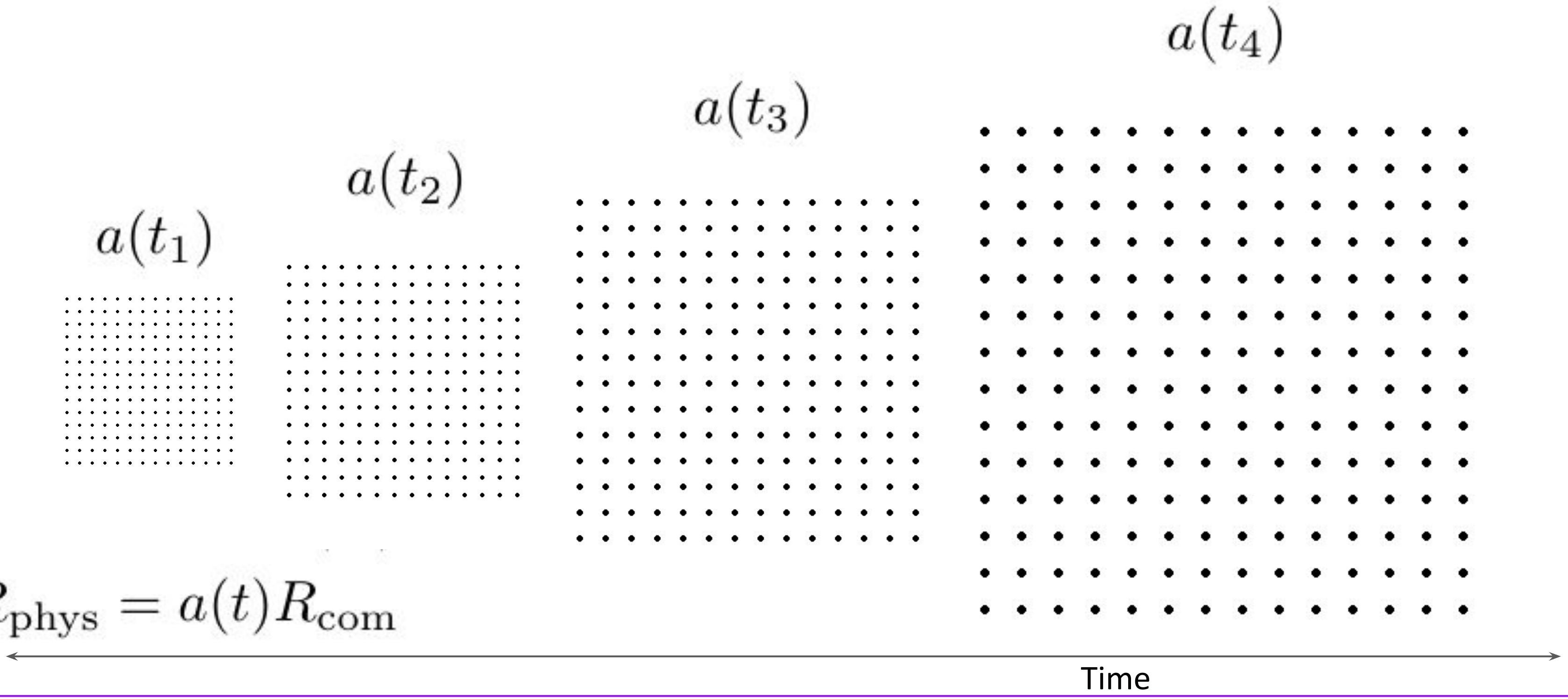


- **Our location in the Universe is not special and neither is any direction.**
- **The universe does change with time though!**

Observational evidence

- **Homogeneity:**
  - Variance in matter overdensity when smoothed on radius R
- **Isotropy:**
  - The cosmic microwave background radiation has the same temperature in any (non-contaminated) direction we observe it.
- **Time asymmetry:**
  - Galaxies evolve with time though

Fundamental observers



$$R_{\rm phys} = a(t) R_{\rm com}$$

Metric of spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

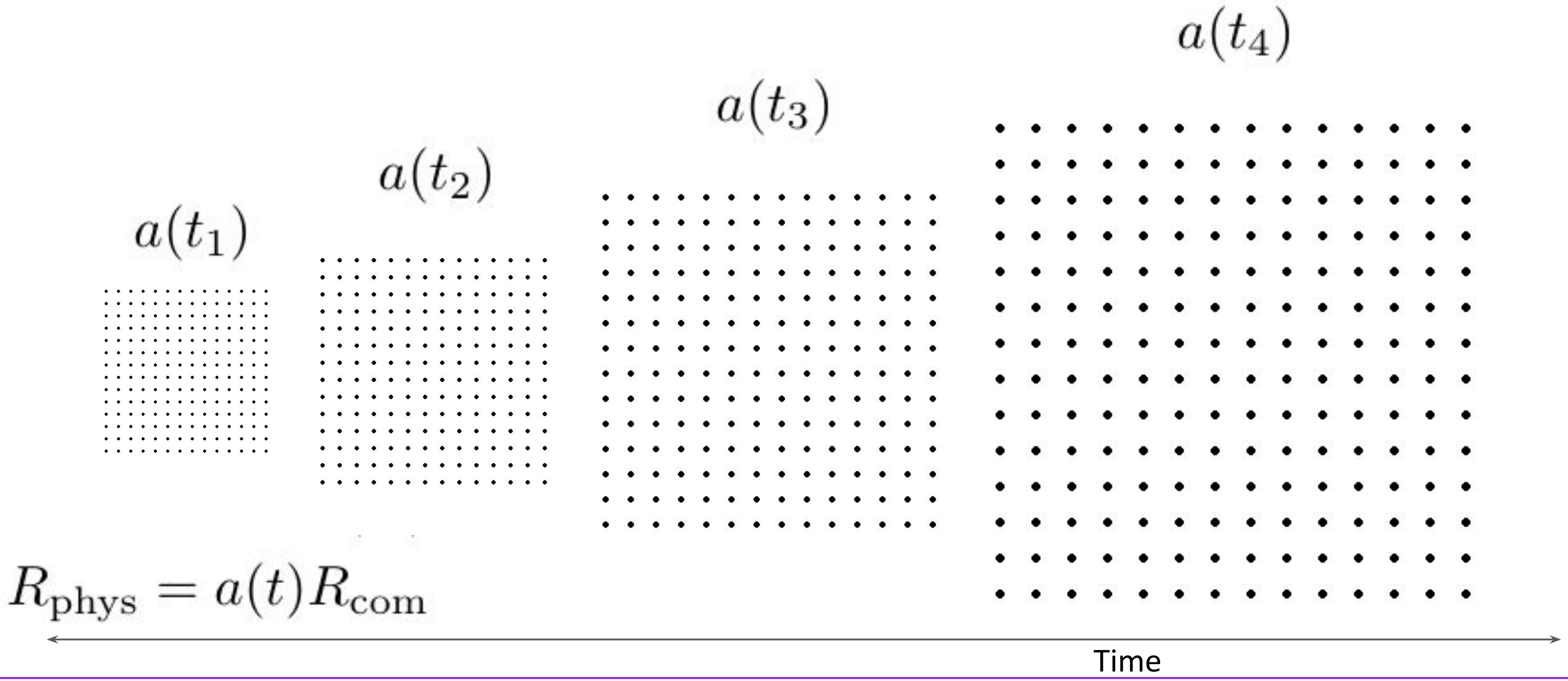
$$g_{00} = -c^2$$

$$g_{0i} = 0$$

$$ds^2 = -c^2 dt^2 + a^2(t) dl^2$$

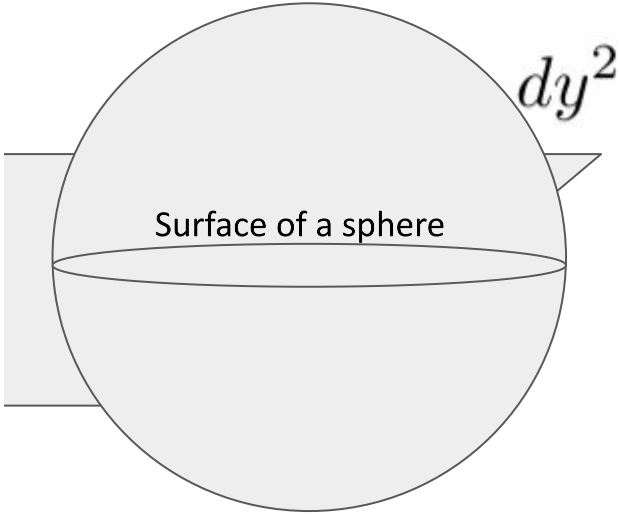
- 10 independent numbers in general describe the metric
- Time set to proper time for fundamental observers
- Isotropy implies no favored direction
- Foliate spacetime into time and homogeneous and isotropic space slices

Fundamental observers



Maximally symmetric surfaces

$$
\begin{aligned}
dl^2 &= dx^2 + dy^2 + dz^2 \\
R^2 &= x^2 + y^2 + z^2 \\
dl^2 &= R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2
\end{aligned}
$$



- **Easier to imagine two dimensional surfaces embedded in three dimensions.**

Curvature: https://www.feynmanlectures.caltech.edu/II_42.html

Maximally symmetric 3d metrics

$$R^2 = x^2 + y^2 + z^2 + w^2$$

$$dl^2 = dx^2 + dy^2 + dz^2$$
$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$
$$dl^2 = dx^2 + dy^2 + dz^2 - dw^2$$

4d space with a constraint equation

$$-R^2 = x^2 + y^2 + z^2 - w^2$$

Maximally symmetric 3d metrics

$$dl^2 = d\chi^2 + \chi^2[d\theta^2 + \sin^2\theta d\phi^2]$$

Flat space

$$dl^2 = d\chi^2 + R^2\sin^2(\chi/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

Positively curved space

$$dl^2 = d\chi^2 + R^2\sinh^2(\chi/R)[d\theta^2 + \sin^2\theta d\phi^2]$$

Negatively curved space

Friedmann Robertson Walker space

$$dl^2 = a^2(t)\left[d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

$$f(\chi) = \begin{cases} R\sin(\chi/R), & \text{Positive curvature} \\ \chi, & \text{Flat} \\ R\sinh(\chi/R), & \text{Negative curvature} \end{cases}$$

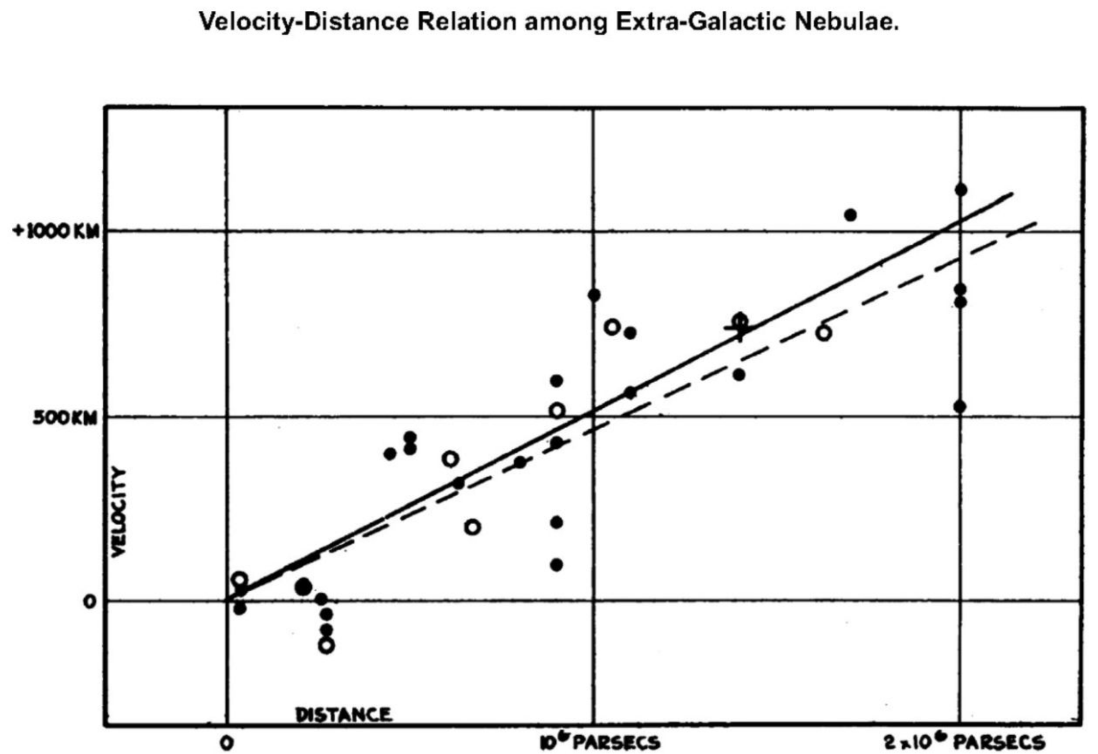
- **No equations of general relativity used yet, only homogeneity and isotropy!**

Expansion of the Universe

$$
\begin{aligned}
v = \frac{dR_{\rm phys}}{dt} &= \frac{d\left[a(t) R_{\rm com}\right]}{dt} \\
&= \frac{da}{dt} R_{\rm com} + a(t) \frac{dR_{\rm com}}{dt} \\
&= \frac{\dot{a}}{a} R_{\rm phys} + a(t) \frac{dR_{\rm com}}{dt}
\end{aligned}
$$

- **Velocities of galaxies consist of a cosmological recession velocity and peculiar velocity**

The expanding Universe



$$v = H_0 d$$

https://en.wikipedia.org/wiki/Edwin_Hubble

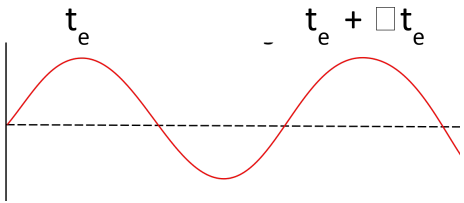
Redshift of photons in an expanding Universe

$$c\,\delta t = a(t)\,\delta\chi$$

$$\sum_{t_e}^{t_r} \frac{c\delta t}{a(t)} = \sum \delta\chi$$

$$\sum_{t_e+\delta t_e}^{t_r+\delta t_r} \frac{c\delta t}{a(t)} = \sum \delta\chi$$

Light moving radially towards us.



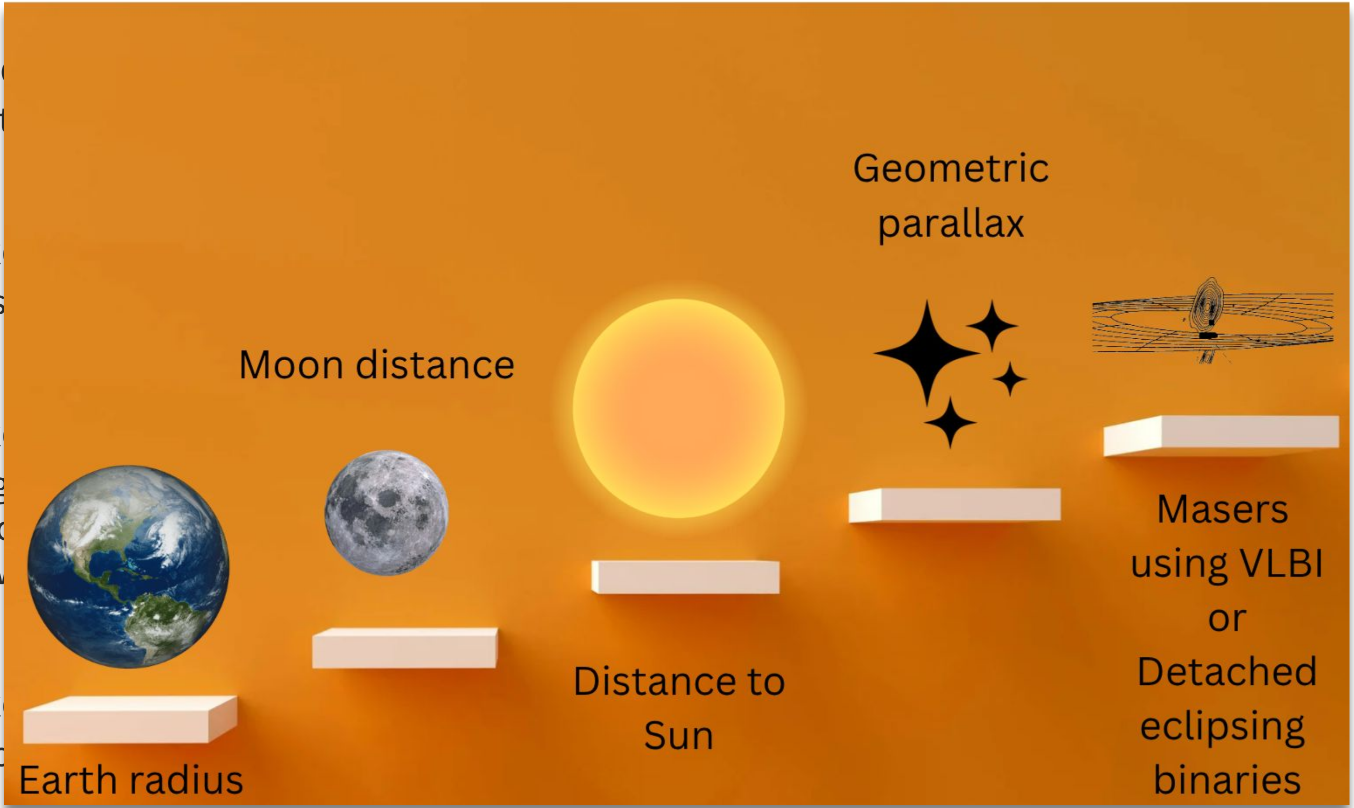
$$\frac{\lambda_r}{\lambda_e} = 1 + z = \frac{a(t_r)}{a(t_e)}$$

Distances in the Universe

- Angular distance:
  - Relation between angles (solid size) and physical size (physical area) of objects.
  - $$D_{\rm ang} = \left[\frac{A}{\Omega}\right]^{1/2} = \left[\frac{4\pi f^2(\chi)a^2(t)}{4\pi}\right]^{1/2} = \frac{f(\chi)}{1+z}$$
- Luminosity distance:
  - Relation between luminosity of an object and the flux received from it.
  - $$D_{\rm lum} = \left[\frac{L^{\rm bol}}{4\pi F^{\rm bol}}\right]^{1/2} = f(\chi)(1+z)$$

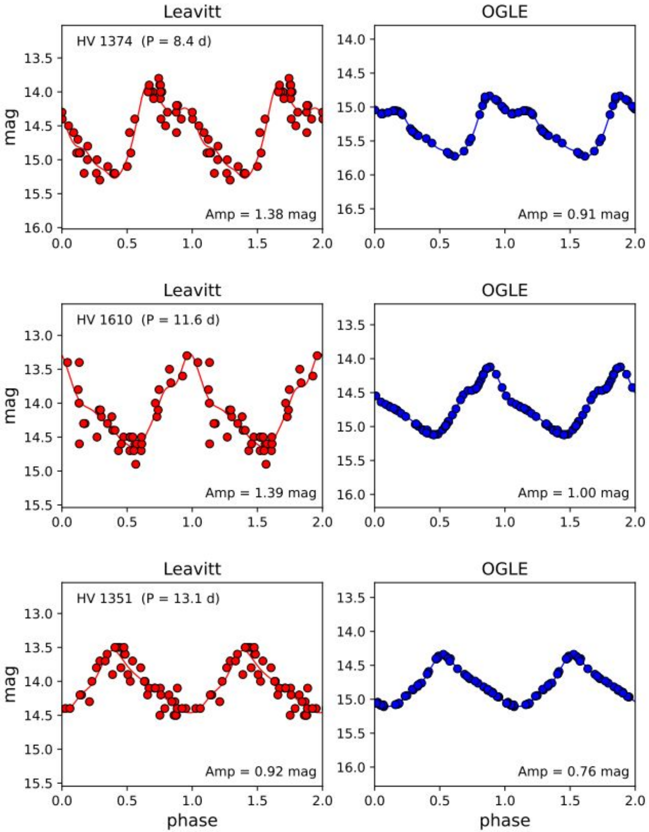
Cosmic Distance ladder

- Radius
  - Hist
- Distanc
  - Bas
- Distanc
  - Ang
  - Also
  - Now
- Distanc
  - Geo



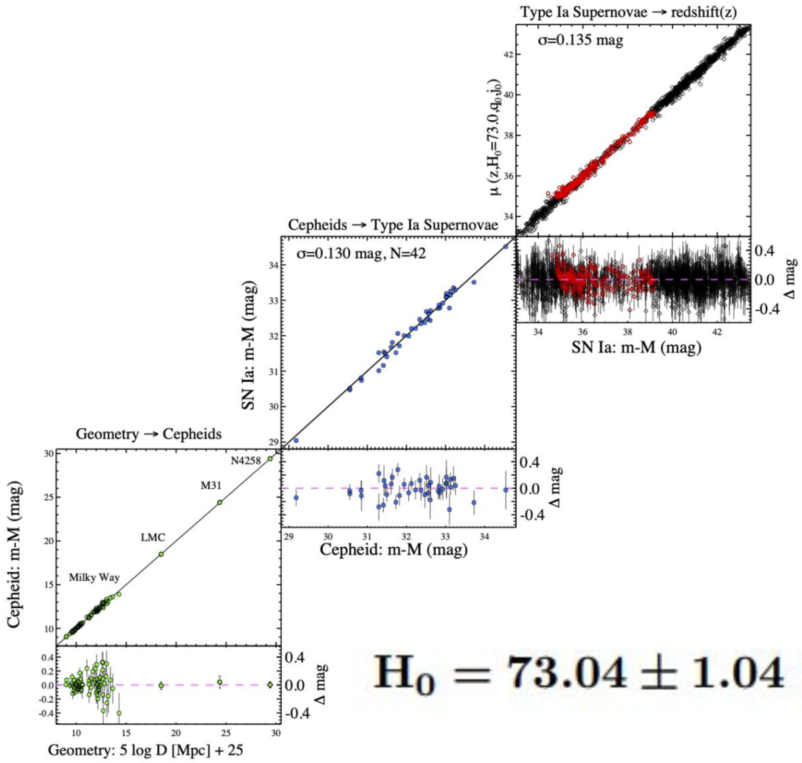
Standard candles: Cepheids

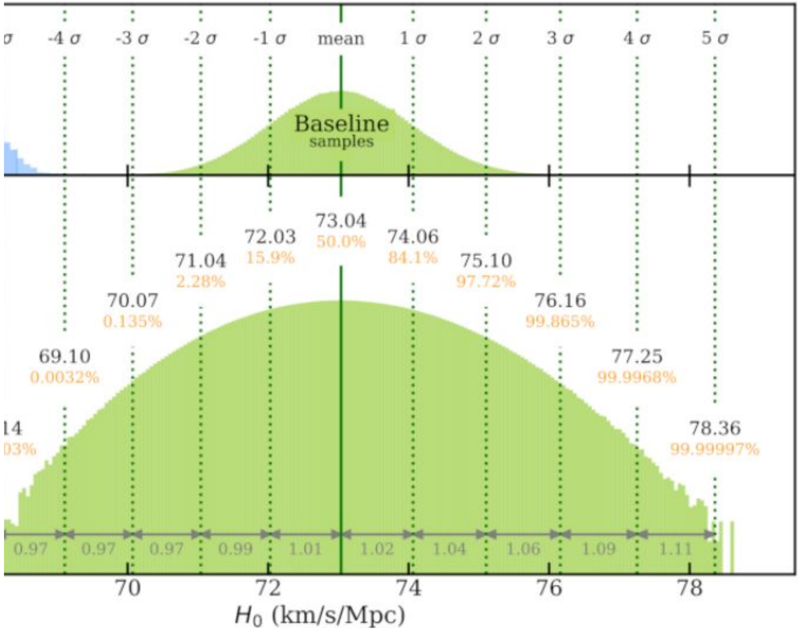
Henrietta Leavitt found the Period Luminosity relation for Cepheid variables



Breuval et al. 2025

Extragalactic distance ladder





$$H_0 = 73.04 \pm 1.04 \text{ km s}^{-1} \text{ Mpc}^{-1} \text{ (baseline with systematics)}$$

Riess et al. (2011)

General relativity

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Einstein's field equation relates the energy density content to the metric derivatives
  - Use the python package called EinsteinPy to define the FRW metric, and compute the Einstein tensor for the FRW metric.
- The right hand side requires us to specify the constituents of the Universe.
  - Use the Energy momentum tensor of an ideal fluid

$$T_{\mu\nu} = (\rho c^2 + p) u_\mu u_\nu + p c^2 g_{\mu\nu}$$

https://surhudmorg.github.io/Cosmology_March21/FRW_models.html

Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$

$$\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right] = \frac{8\pi G\rho}{3}$$

$$-\left[\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right] = 8\pi G\frac{p}{c^2}$$

- The 00 component and the ij th component of Einstein's equations give the Friedmann equations.
  - If all the constituents of matter have positive definite pressure, then acceleration is negative.

Friedmann equations

$$\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right] = \frac{8\pi G\rho}{3}$$

$$\frac{d\rho}{da} + \frac{3}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

- Specify the relationship between the pressure and the density (equation of state) and obtain density as a function of scale factor.
- The first equation then can be solved for scale factor as a function of time.

Behaviour of different constituents

$$p_{\rm mat} = 0$$
$$p_{\rm rad} = \frac{\rho_{\rm rad} c^2}{3}$$

$$\rho_{\rm mat} \propto a^{-3}$$
$$\rho_{\rm rad} \propto a^{-4}$$

Matter density falls down as the volume of the Universe.
Redshifting of photons causes radiation density to fall down faster.

Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\mathrm{m},0}a^{-3} + \Omega_{\mathrm{rad},0}a^{-4} + \Omega_{\mathrm{w},0}a^{-3(1+w)} + \Omega_k a^{-2}\right] = H_0^2 E^2(a)$$

- Specify the constituents of the Universe
- Compute a(t)
- Compute Chi(a)
- Compute age of the Universe, distances in the Universe

Cosmic microwave background

Arnos Penzias and Robert Wilson

- Accidental discovery in 1964 of an irreducible source of noise observed during transatlantic communication
- Impossible to eliminate, uniform in direction and time
- The temperature of this background corresponded to about ~ 3 K.

https://d1p0gxnqcu0lvz.cloudfront.net/images/86-300189001big_bang_confirmed.original.jpg

COBE: Cosmic Background Explorer

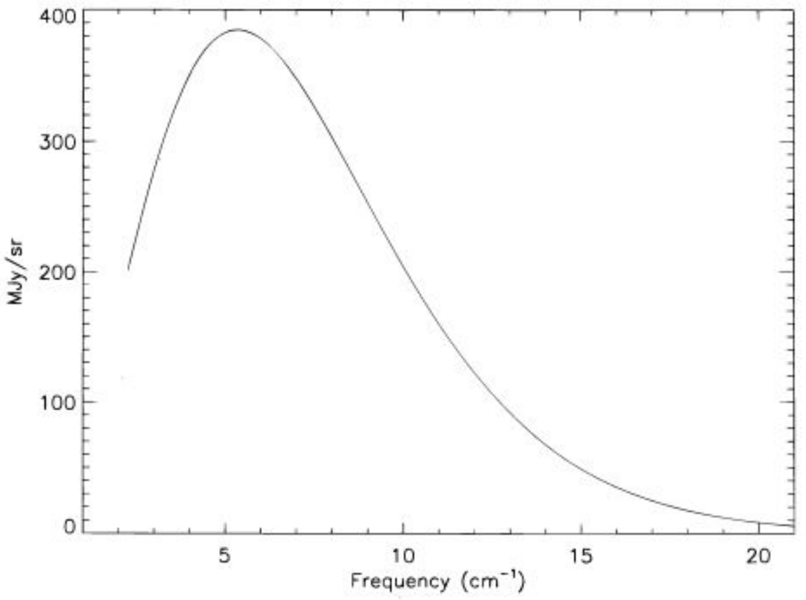


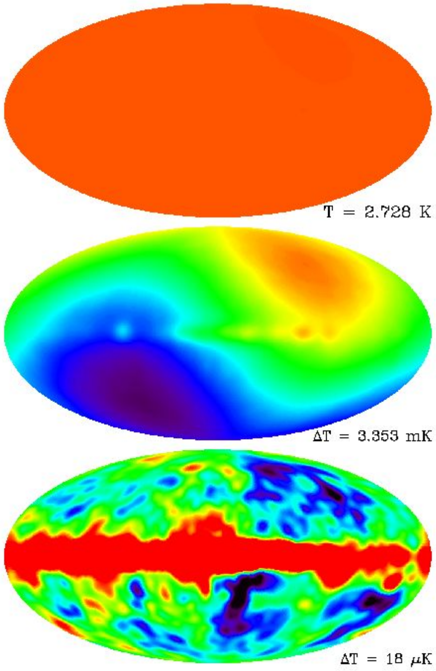
FIG. 4.—Uniform spectrum and fit to Planck blackbody (*T*). Uncertainties are a small fraction of the line thickness.

- One of the best naturally observed black body spectrum.
- Relic photon background from the Universe's hot and dense past, a direct observational proof of the thermal equilibrium in the early Universe.
- Discovery of inhomogeneities in the CMB.

https://science.nasa.gov/science-red/s3fs-public/thumbnails/image/COBEBIG.GIF

https://aether.lbl.gov/www/projects/cobe/COBE_Home/

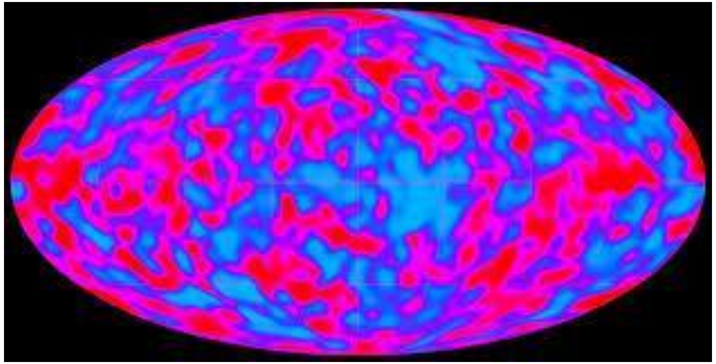
COBE Cosmic Microwave background



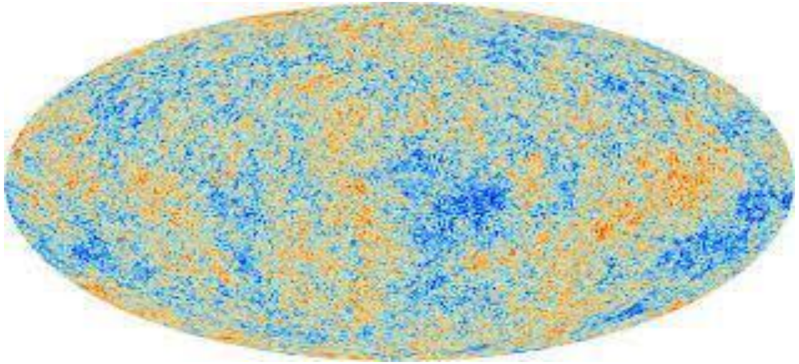
- Extremely isotropic emission, the first evidence of anisotropic behavior only at the milli Kelvin level.
- The dipole corresponds to our peculiar velocity with respect to the CMB.
- Fluctuations observed only at the level of 1 in 10^5.
- The seed fluctuations which result in the formation of structure in the Universe.
- Physics of sound wave propagation in the early Universe.

https://aether.lbl.gov/www/projects/cobe/COBE_Home/m_d_53s_1111.gif

Fluctuations in CMB

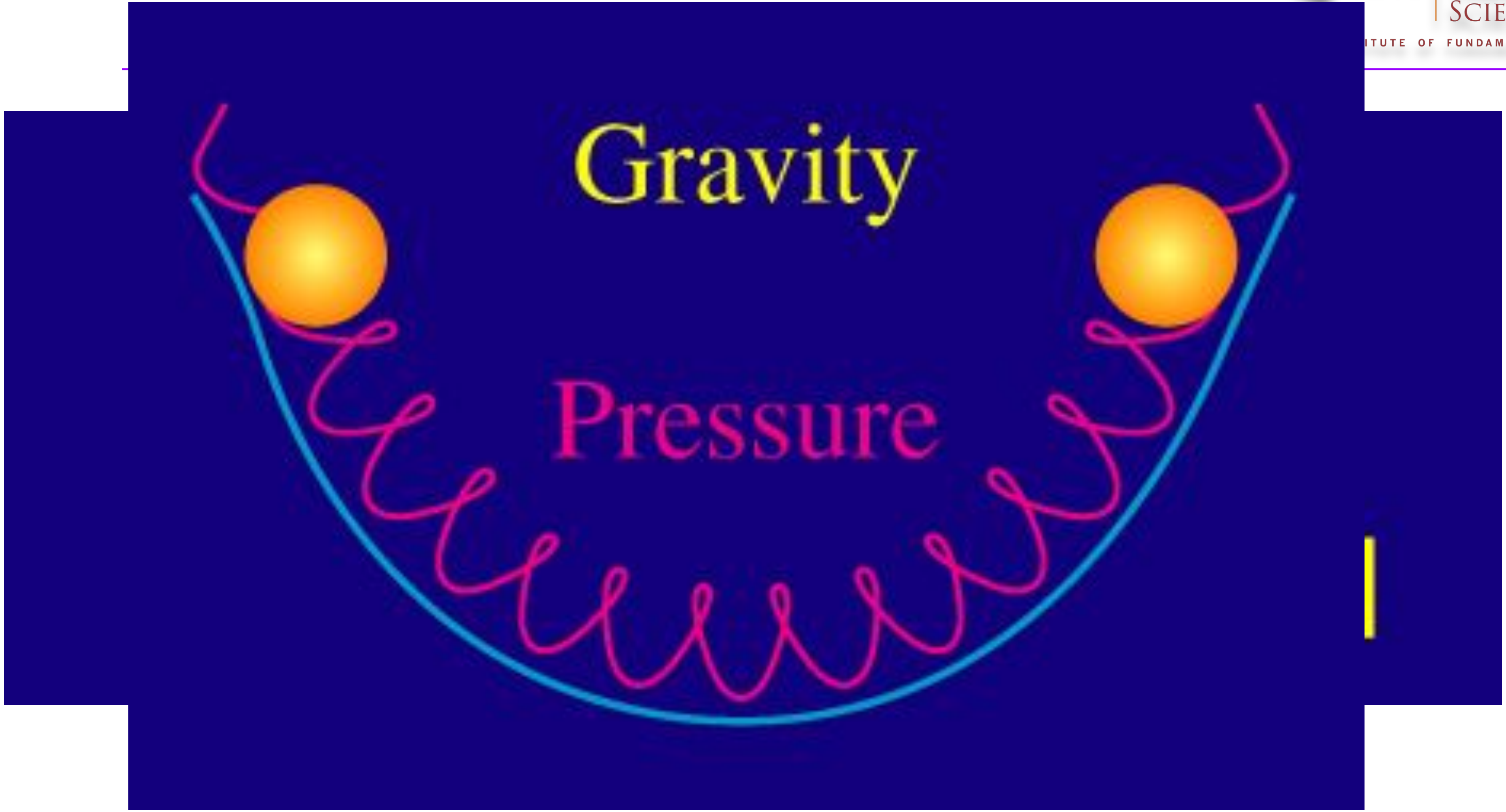


COBE collaboration 1990

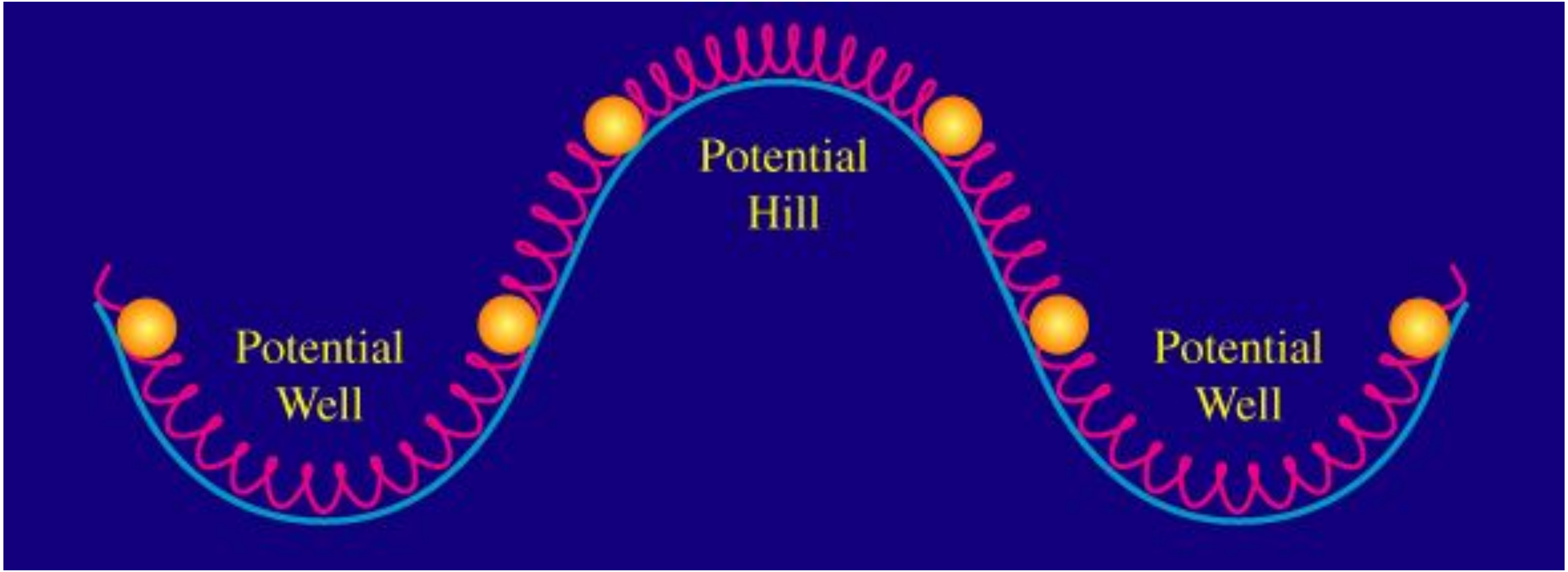


Planck collaboration 2015

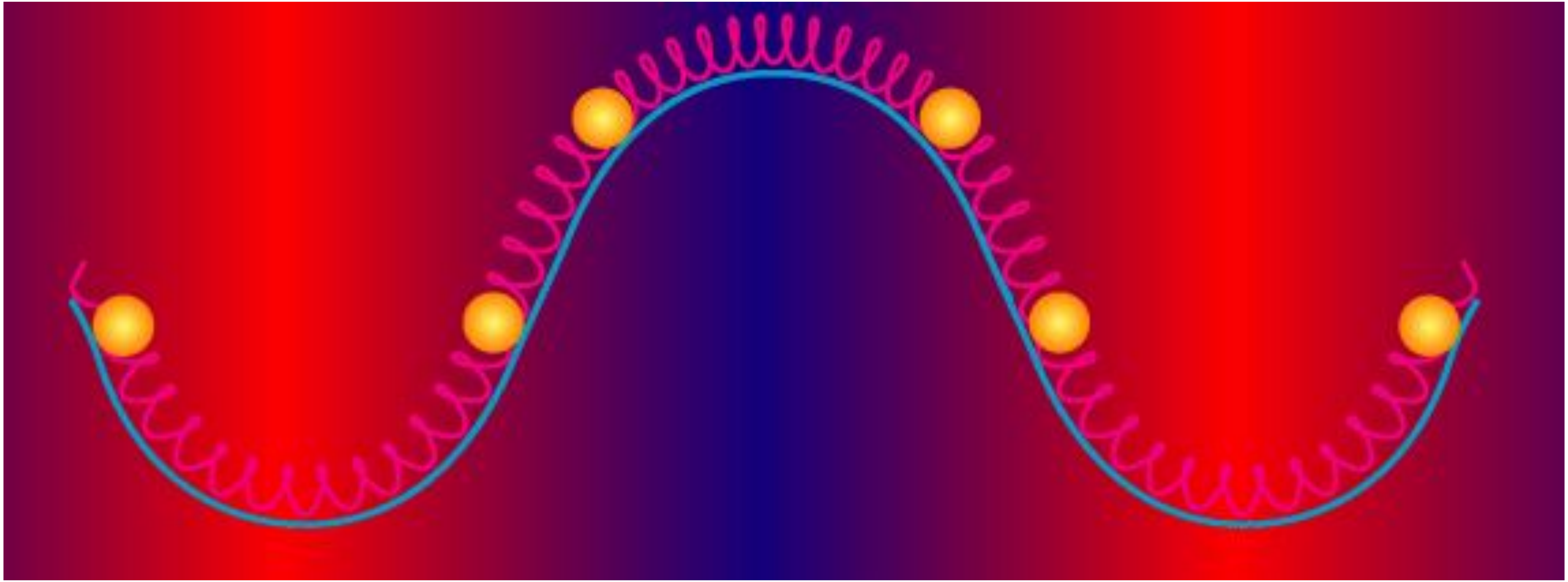
The origin of the inhomogeneity of the Universe



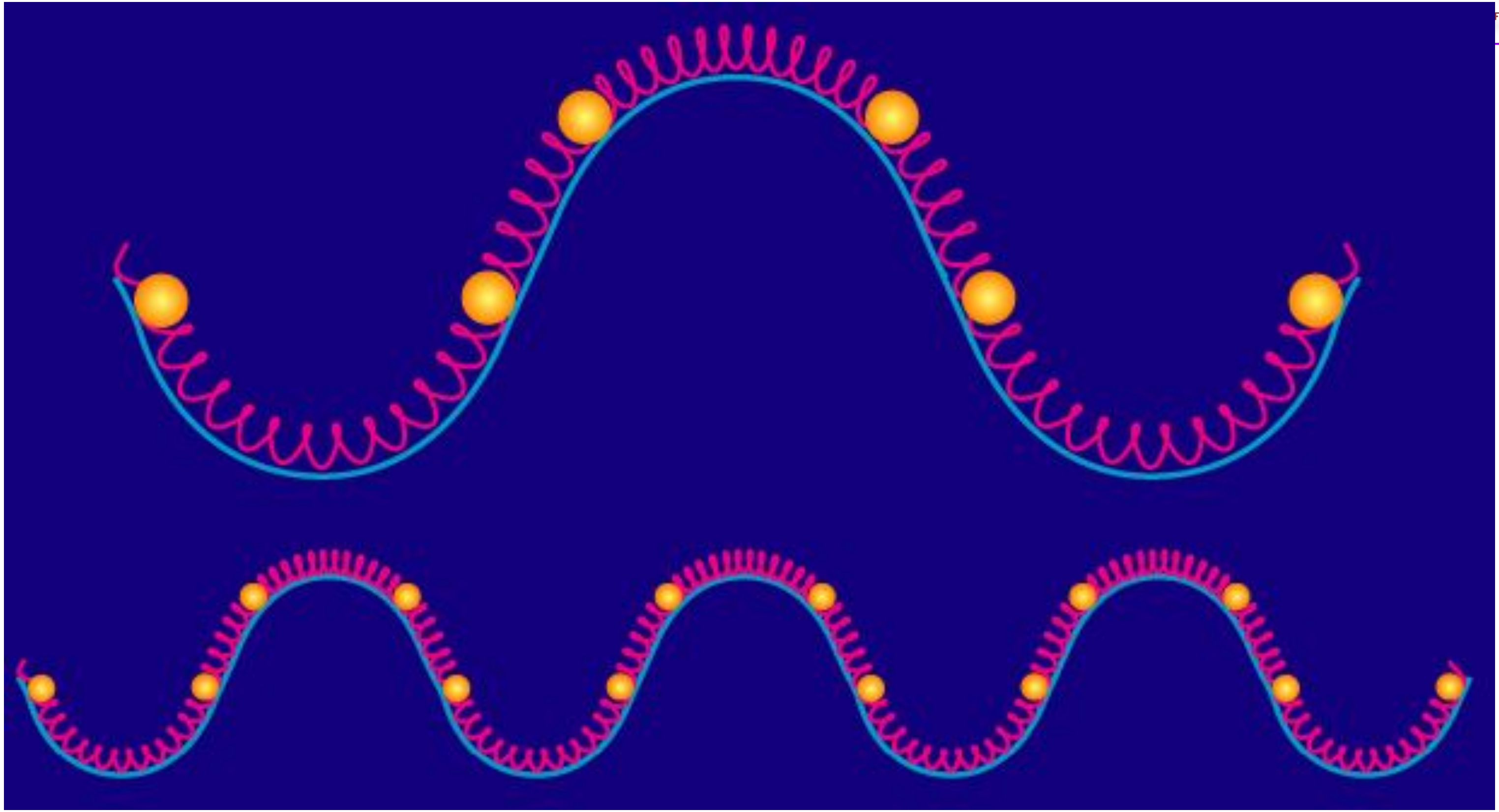
http://background.uchicago.edu/~whu/intermediate/basicoscil.gif http://background.uchicago.edu/~whu/intermediate/redshift3.gif



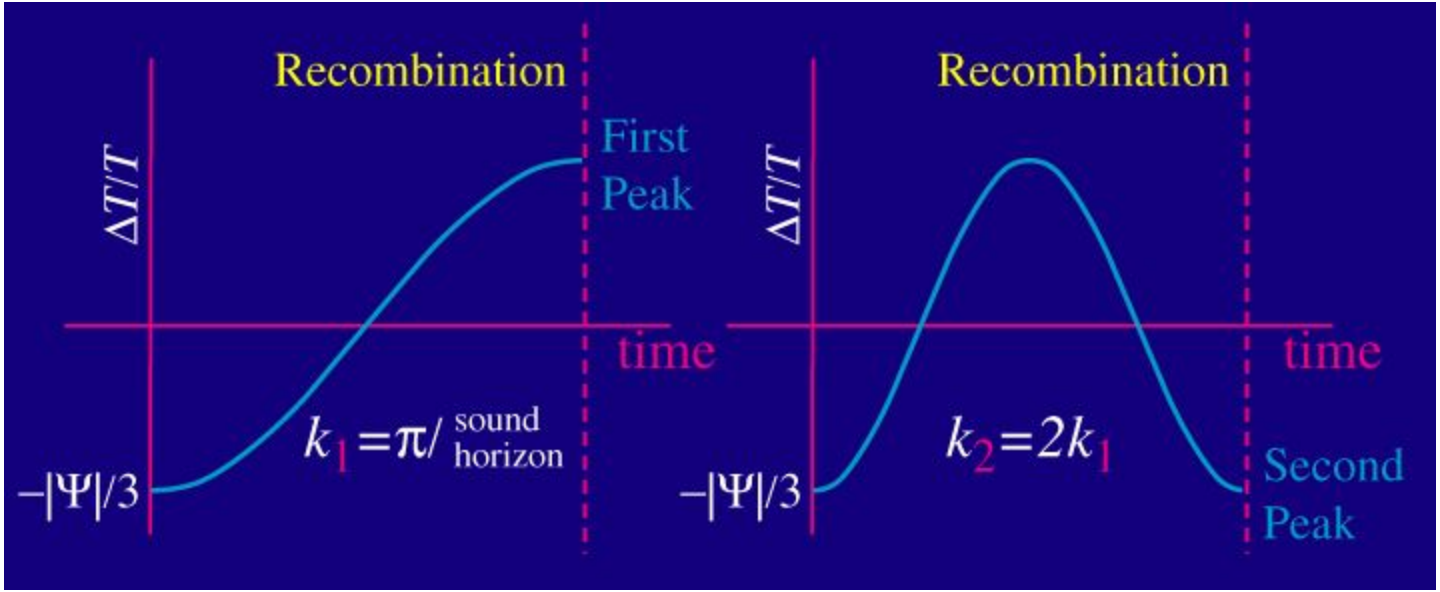
http://background.uchicago.edu/~whu/intermediate/plane.gif



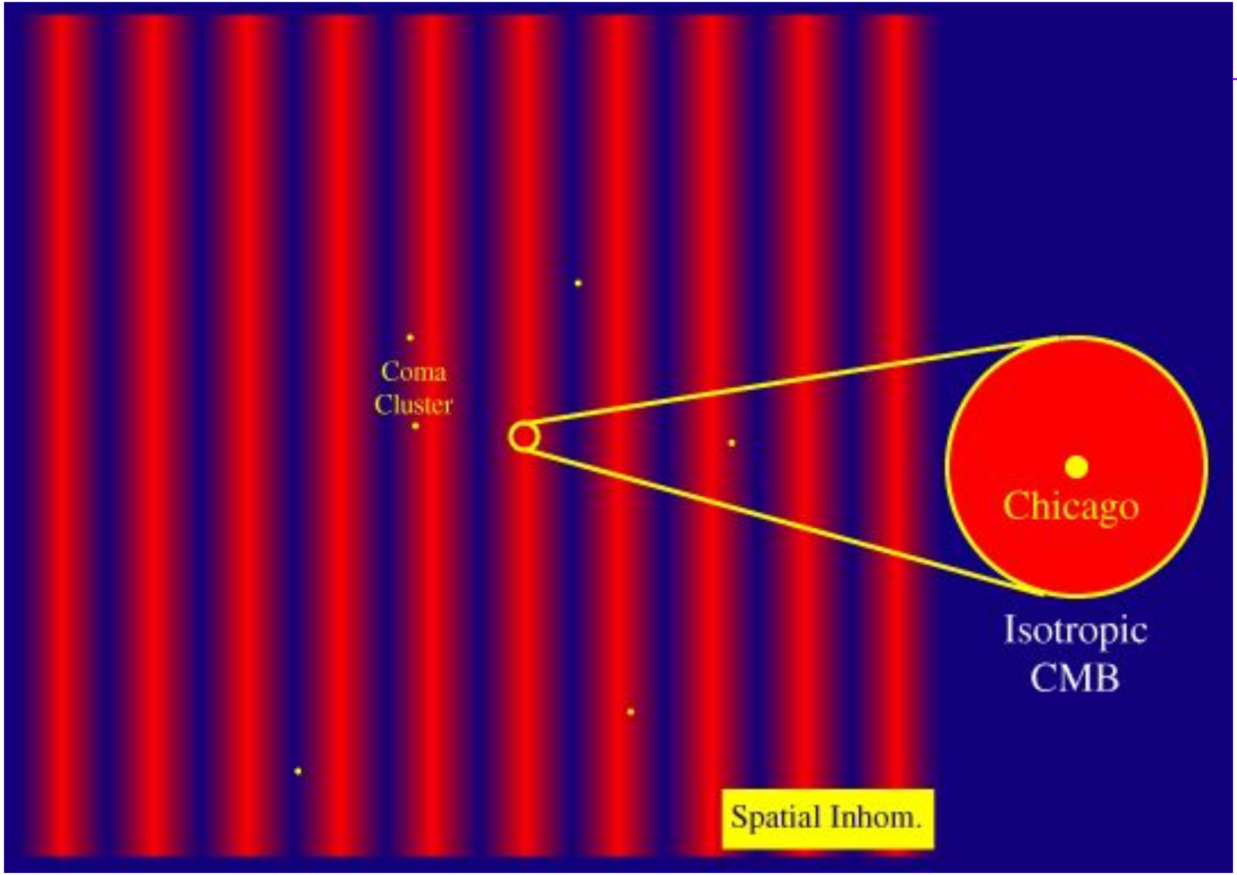
http://background.uchicago.edu/~whu/intermediate/hotcold.gif



http://background.uchicago.edu/~whu/intermediate/modes.gif



http://background.uchicago.edu/~whu/intermediate/extrema.gif



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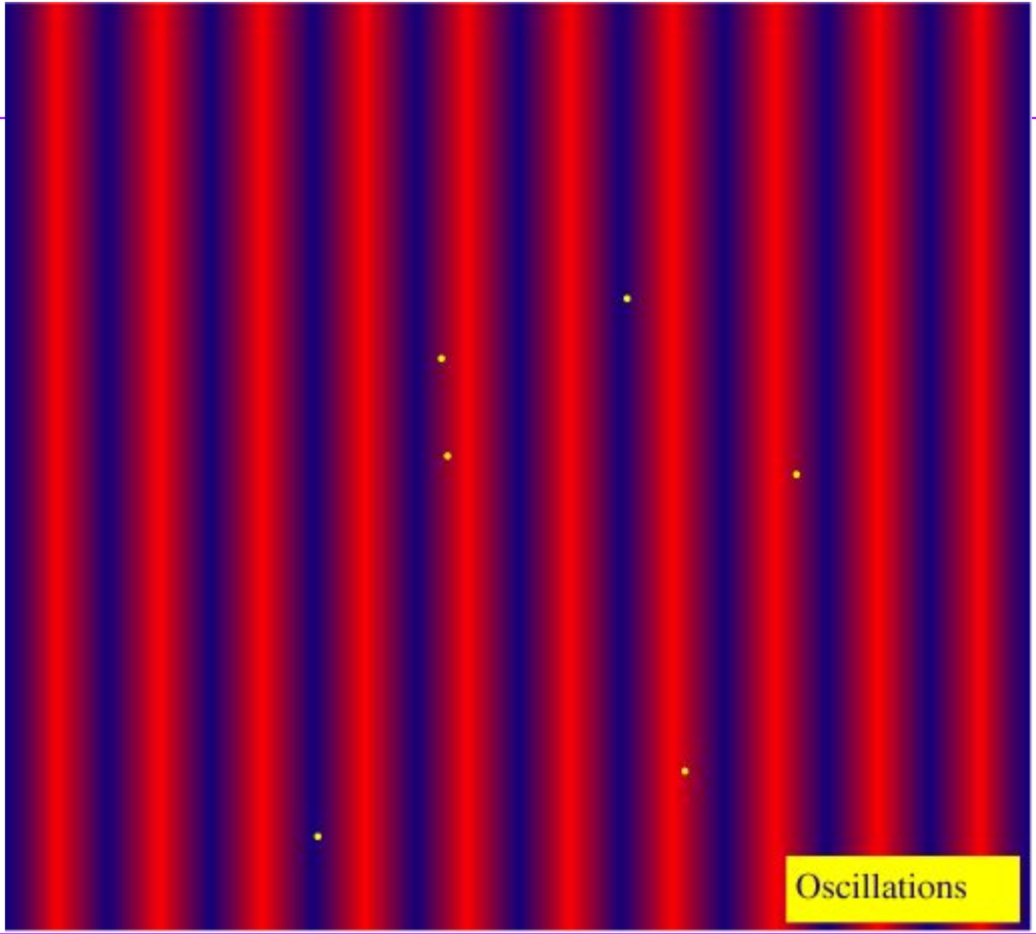
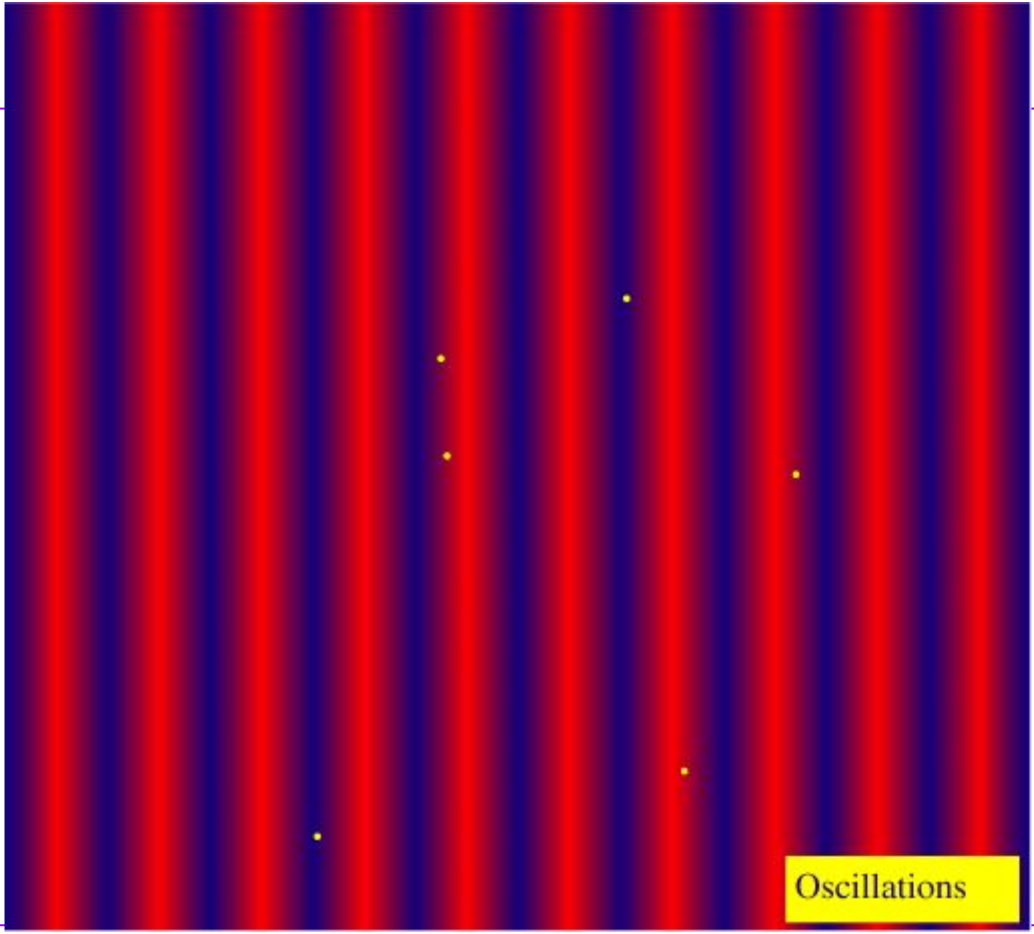
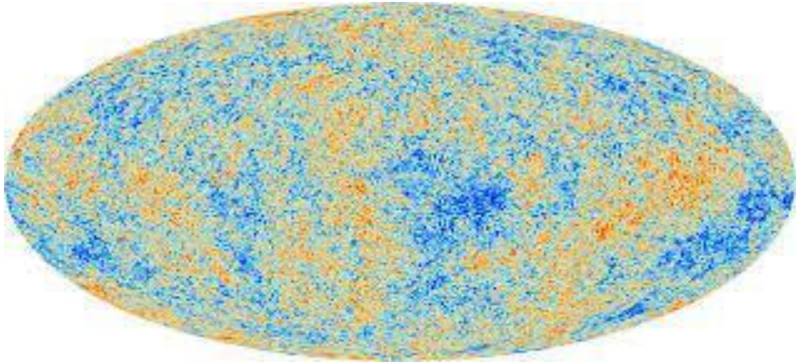


http://background.uchicago.edu/~whu/intermediate/brief1.gif



http://background.uchicago.edu/~whu/intermediate/brief.gif

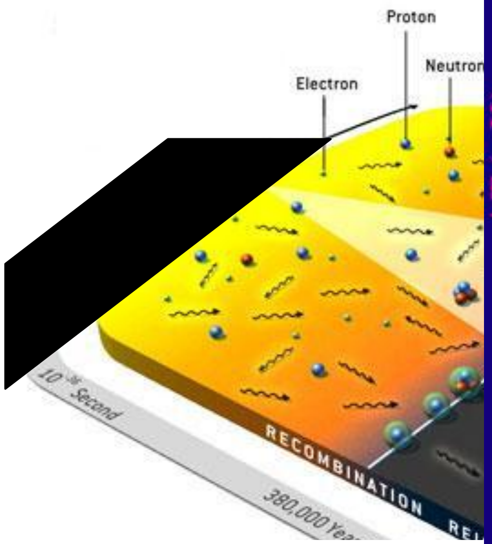
Cosmic microwave background

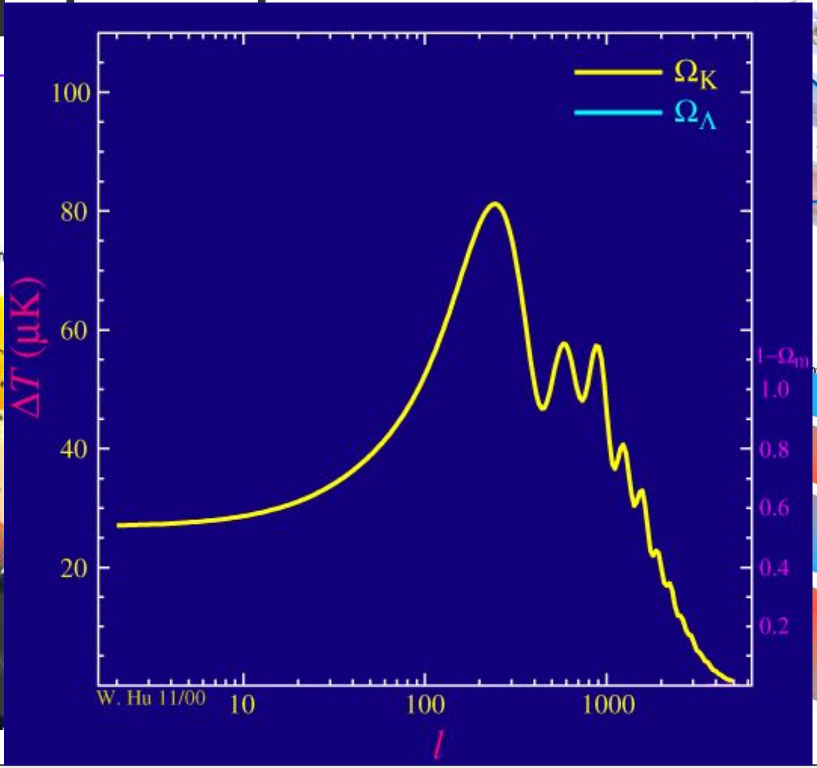


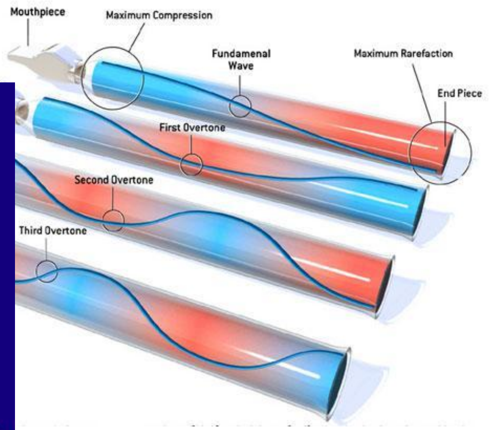
- The photons and baryons are coupled as a fluid.
- The speed of sound in such a medium is approximately equal to $c/\sqrt{3}$ and the sound horizon is given by $c_s t_{rec}$.
- Modes with wavenumbers corresponding to integral multiples of π/(sound horizon) give rise to fluctuation enhancements.

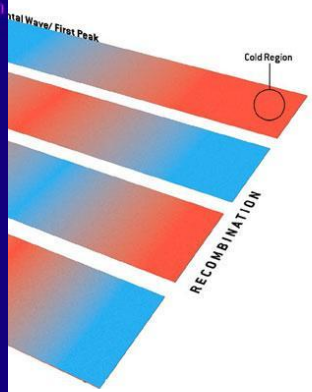
Hu & White (2004)

Primordi

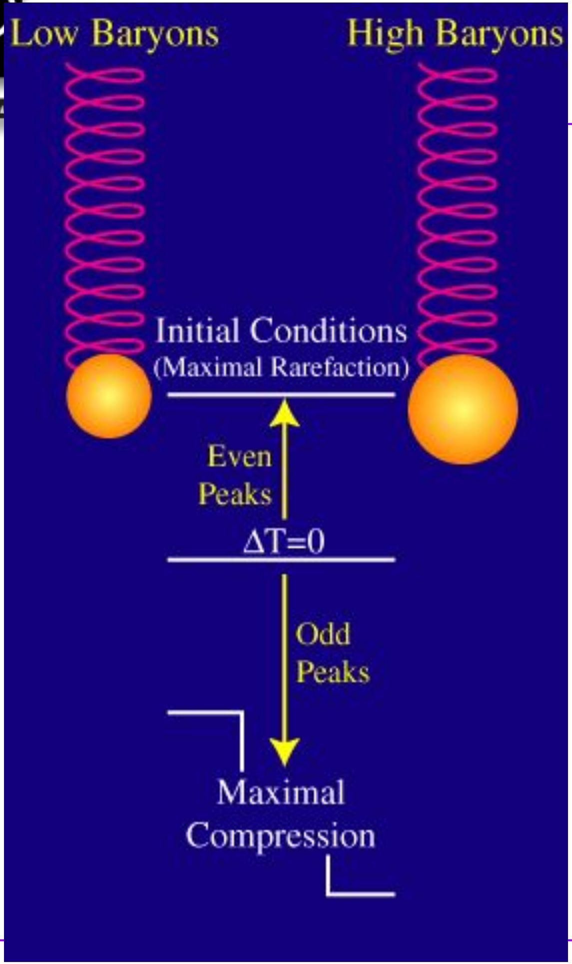


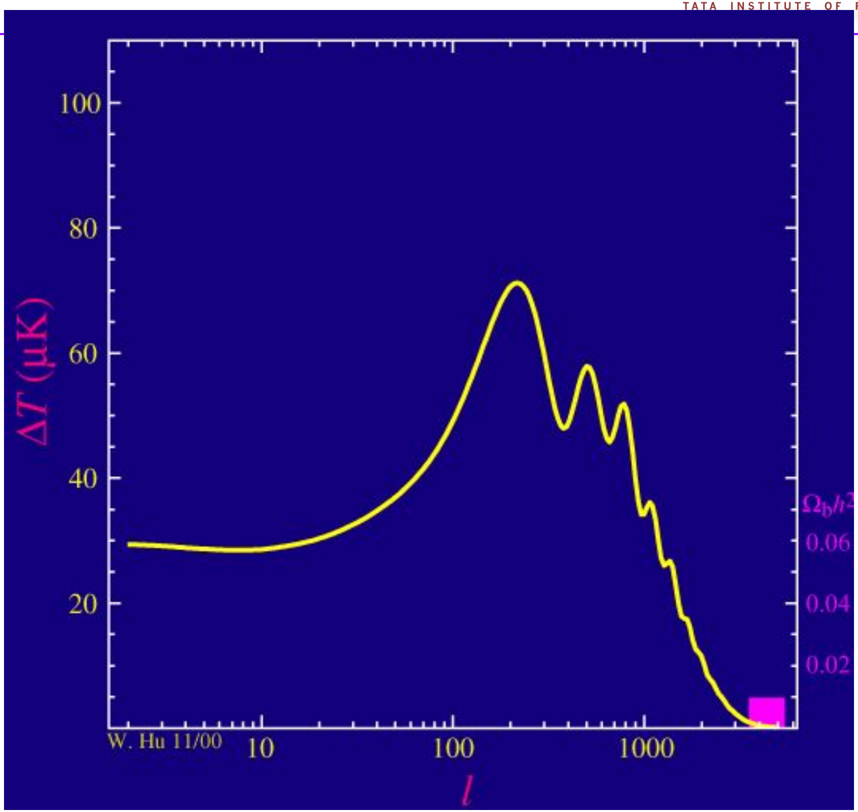




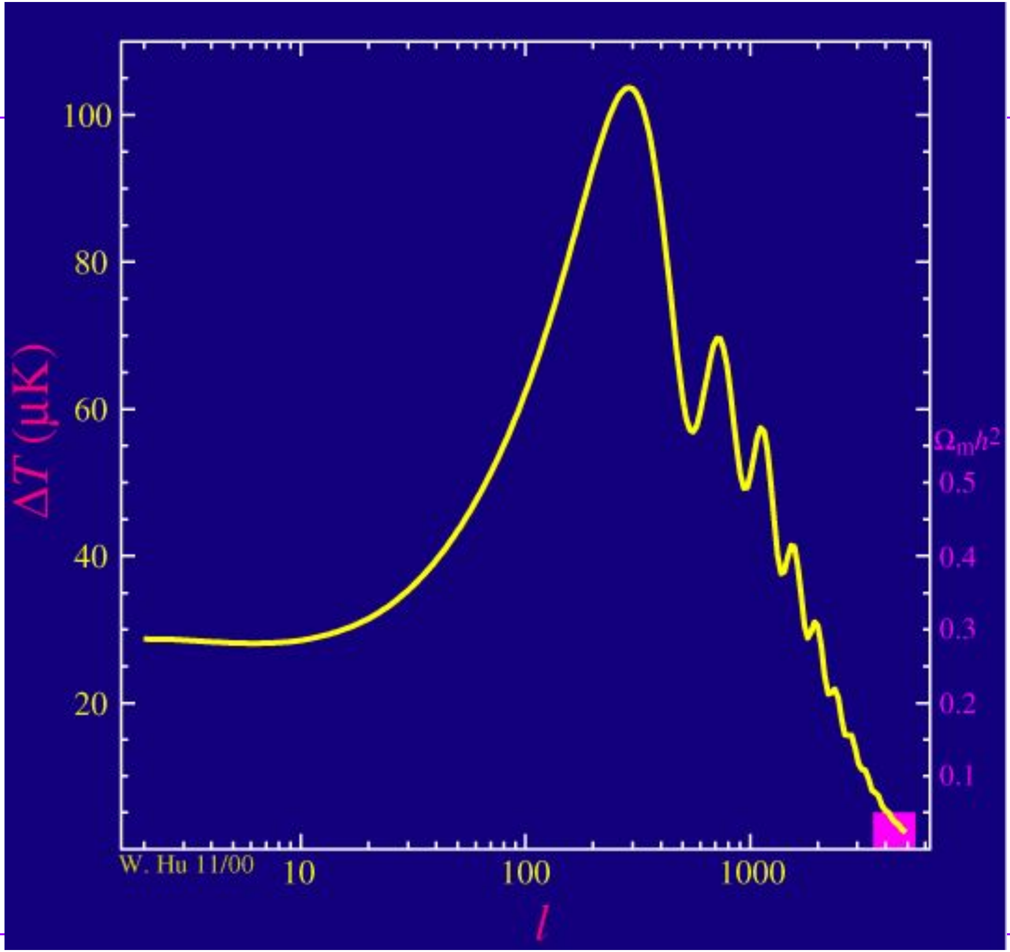


Typical size of CMB spots: a standardizable ruler

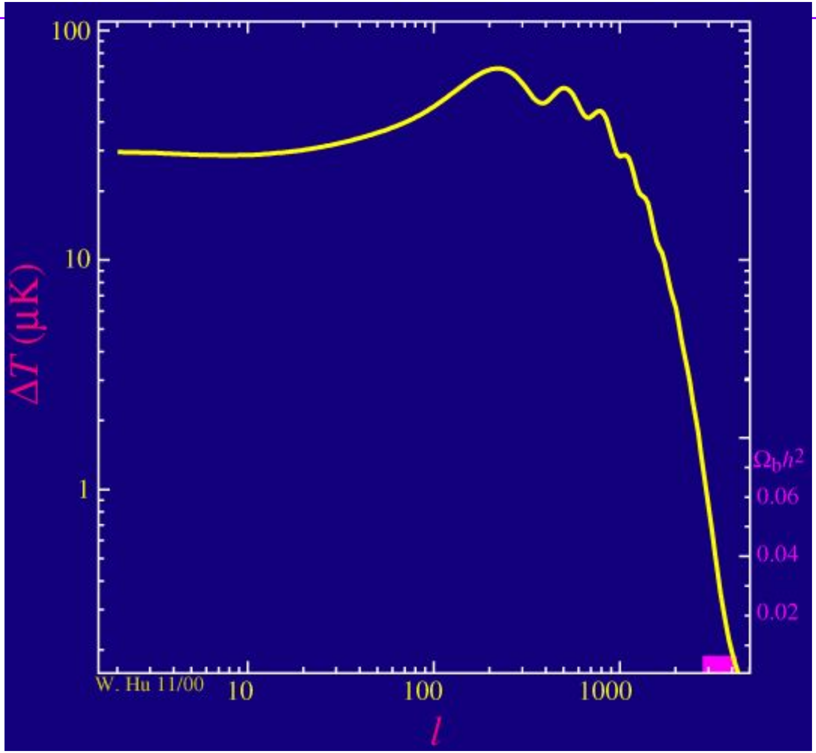




http://background.uchicago.edu/~whu/intermediate/baryonspring.gif



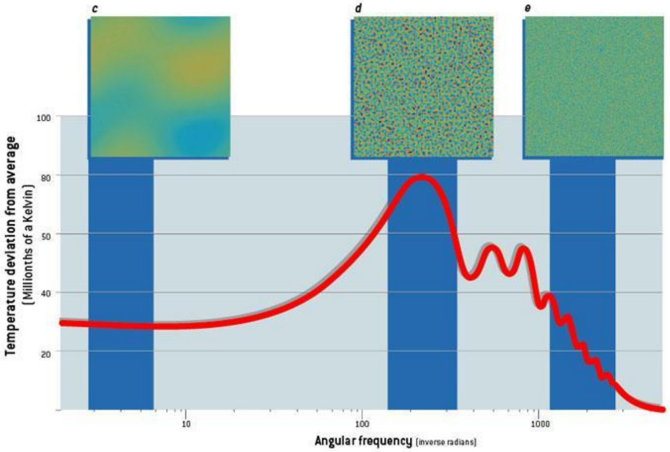
http://background.uchicago.edu/~whu/intermediate/clmatter.gif



http://background.uchicago.edu/~whu/intermediate/clbaryon.gif

Hu & White (2004)

The power spectrum

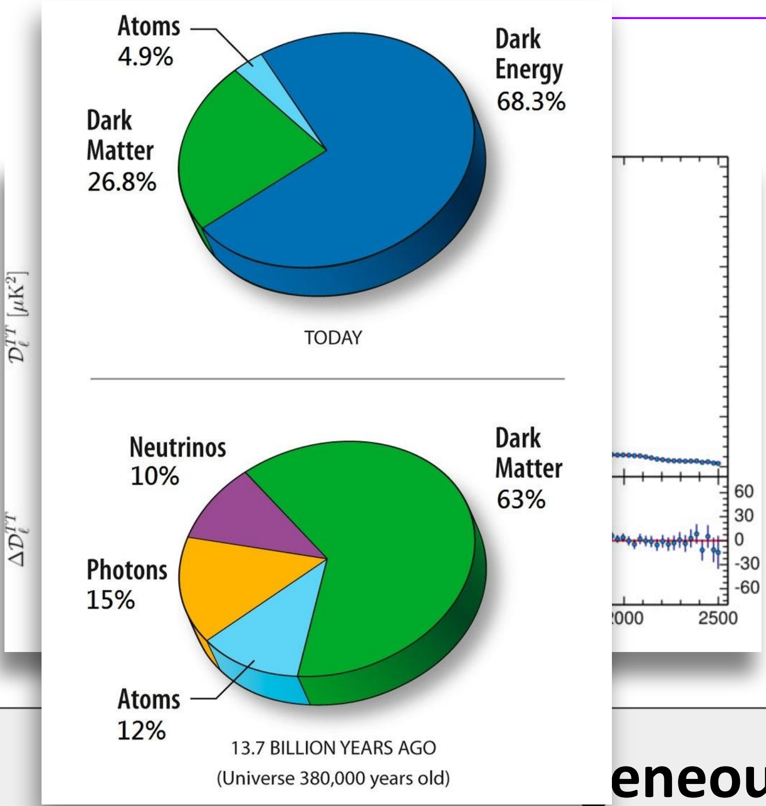


- The location of the first peak informs us about the geometry of the Universe
- The relative heights of the peaks about dark matter and baryon densities

Geometry and the constituents in the early Universe

Planck collaboration

The cosmological model



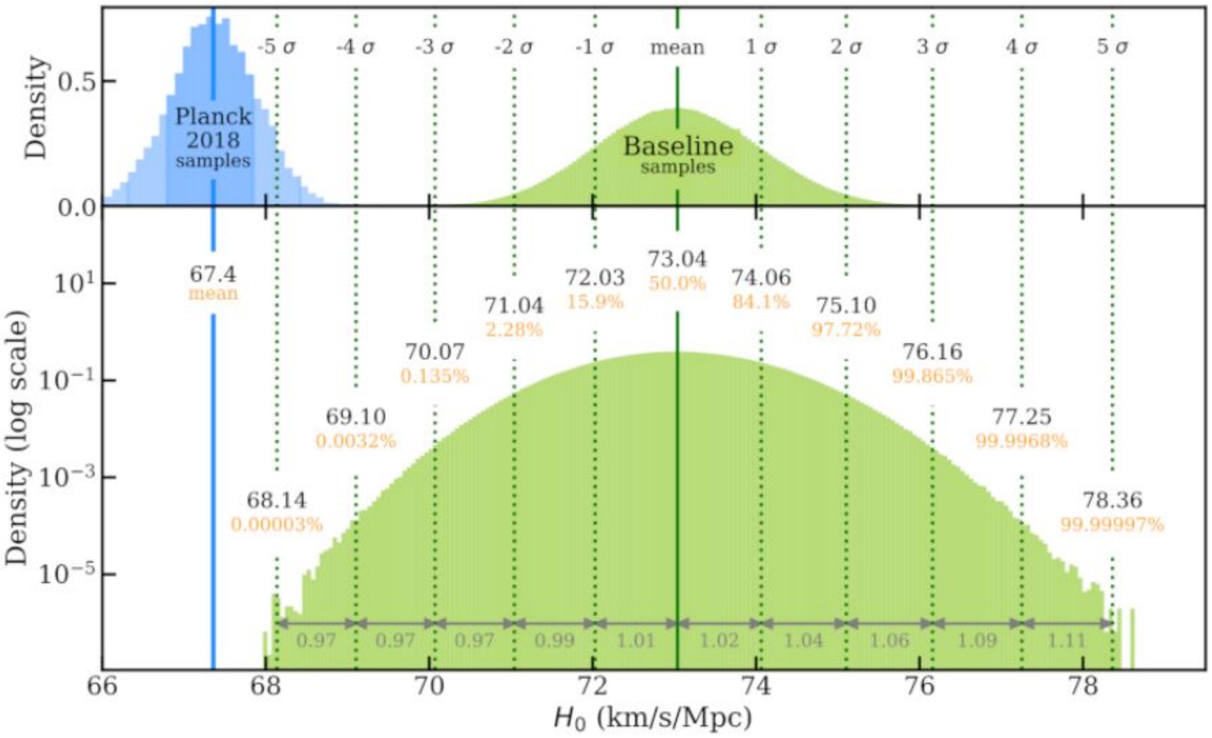
- The Universe is flat and has critical density
- Baryons make up about 15 percent of the matter sector, rest is dark matter.

Planck collaboration 2015

eneous Universe

Planck collaboration

Hubble tension



Riess et al. (SH0ES collaboration)

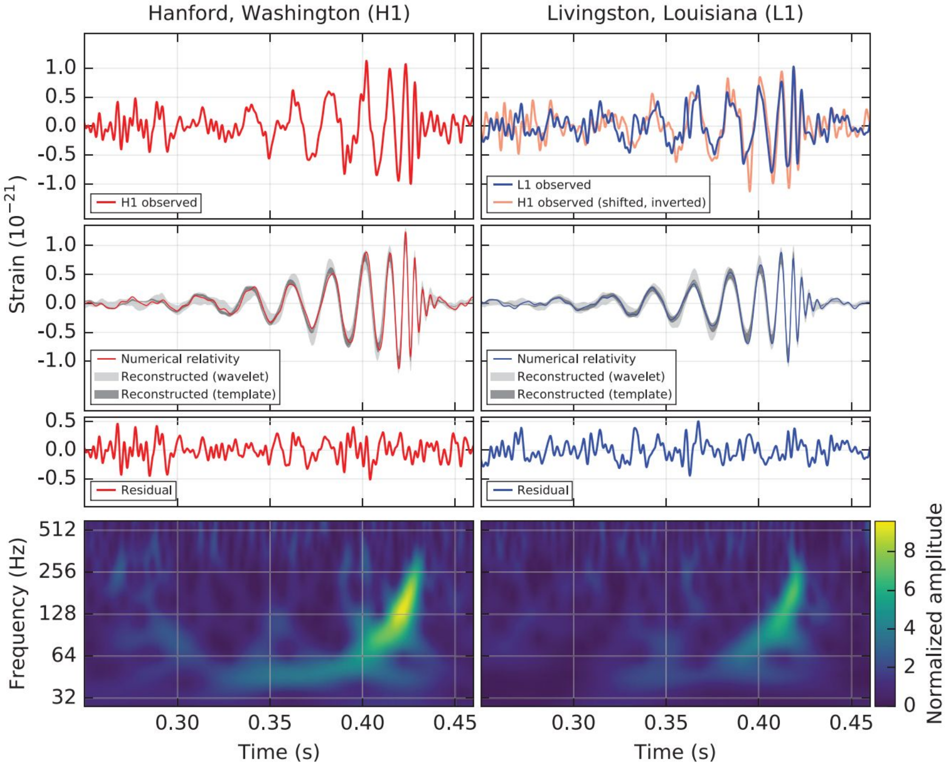
What could have gone wrong?

- The Universe is not flat LCDM?
- Some exotic early dark energy?

Systematics:

- Standard candles? Calibrations?

Gravitational waves



- Gravitational waves were directly detected for the first time by the LIGO Virgo collaboration in 2015 - GW150914.
- A loud nearby event where two black holes of mass ~ 30 solar masses merged.
- Event detected in both the Hanford and Livingston interferometers
- Opened a new era for Gravitational wave astronomy

Abbott et al. 2016

Understanding Gravitational waves

As the compact objects spiral inwards, the orbital time period decreases (frequency increases), leads to generation of gravitational waves, and loss of further energy.

Decaying orbits

For two bodies of masses M and m, separated from their center of mass by R and r, respectively, we have:

$$\omega^2 = \frac{G(M+m)}{(R+r)^3}$$

The sum of kinetic plus potential energy is given by:

$$E_{\rm tot} = -\frac{1}{2}\frac{G^{2/3}Mm}{(M+m)^{1/3}}\omega^{2/3}.$$

The power radiated is proportional to the square of the moment of inertia, and is related to G, c, and the angular frequency itself. Dimensional analysis will show that:

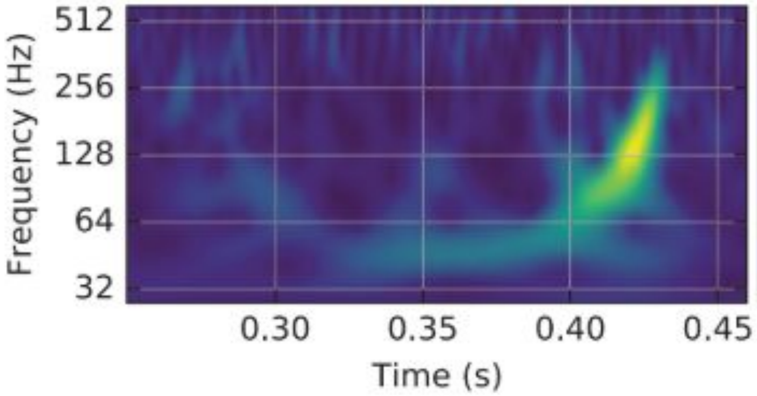
$$\mathcal{I} = \frac{mM}{m+M}(r+R)^2 \qquad P_{\rm rad} = \alpha\frac{G\mathcal{I}^2\omega^6}{c^5}. \qquad -\frac{dE_{\rm tot}}{dt} = \frac{1}{3}G^{2/3}\frac{Mm}{(M+m)^{1/3}}\omega^{-1/3}\frac{d\omega}{dt}$$

See e.g. Mathur et al. 2016

Chirp mass

$$\mathcal{M} = \frac{(mM)^{3/5}}{(m+M)^{1/5}} = \frac{c^3}{G}\left(\frac{1}{3\alpha}\omega^{-11/3}\frac{d\omega}{dt}\right)^{3/5}$$

$$2\pi f_{\rm gw} = \omega_{\rm gw} = 2\omega$$



Frequency starts to diverge after a finite amount of time.

The observed signal yields the chirp mass.

See e.g. Mathur et al. 2016

Effect of redshift

$$\mathcal{M} = \frac{(mM)^{3/5}}{(m+M)^{1/5}} = \frac{c^3}{G}\left(\frac{1}{3\alpha}\omega^{-11/3}\frac{d\omega}{dt}\right)^{3/5}$$

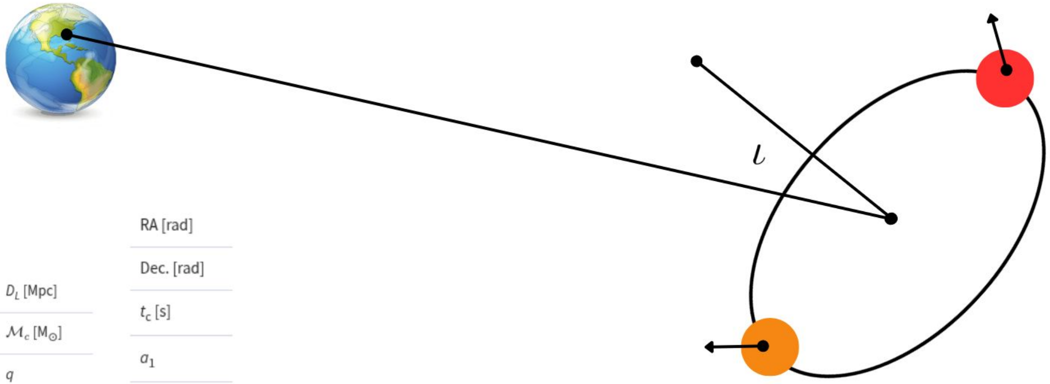
$$\omega_{\mathrm{d}} = \omega/(1+z)$$
$$t_{\mathrm{d}} = t(1+z)$$

$$\mathcal{M}_{\mathrm{d}} = \mathcal{M}(1+z)$$

Replacing the above, one gets the same equality between the detector frame quantities as well. Thus the observed signal gives a degenerate combination of these quantities.

See e.g. Mathur et al. 2016

GW merger parameters



D_L [Mpc]
$\mathcal{M}_c$ [$M_\odot$]
q
ι [rad]
ϕ_c [rad]
ψ [rad]

RA [rad]
Dec. [rad]
t_c [s]
a_1
a_2
θ_1
θ_2
$\phi_{1,2}$
$\phi_{J,L}$

Luminosity distance D_L, chirp mass Mc, Mass ratio q, inclination ι, Orbital phase at coalescence ϕ_c, Polarization phase ψ, right ascension RA, declination Dec., time at coalescence t_c, spin magnitude on the heavier BH a_1, spin magnitude on the lighter BH a_2, spin tilt angle on the heavier black hole θ_1, spin tilt angle on the lighter black hole θ_2, the angle between the two spin vectors $\phi_{1,2}$, and the angle between the orbital and total angular momentum $\phi_{J,L}$.

GW signal amplitude

$$\tilde{h}_+(f) = A(f,\mathcal{M})\left[\frac{1+\cos^2\iota}{2}\right]\exp\left(i\Psi(f,\mathcal{M})\right)$$

$$\tilde{h}_\times(f) = A(f,\mathcal{M})[\cos\iota]\exp\left(\frac{\pi}{2}+i\Psi(f,\mathcal{M})\right)$$

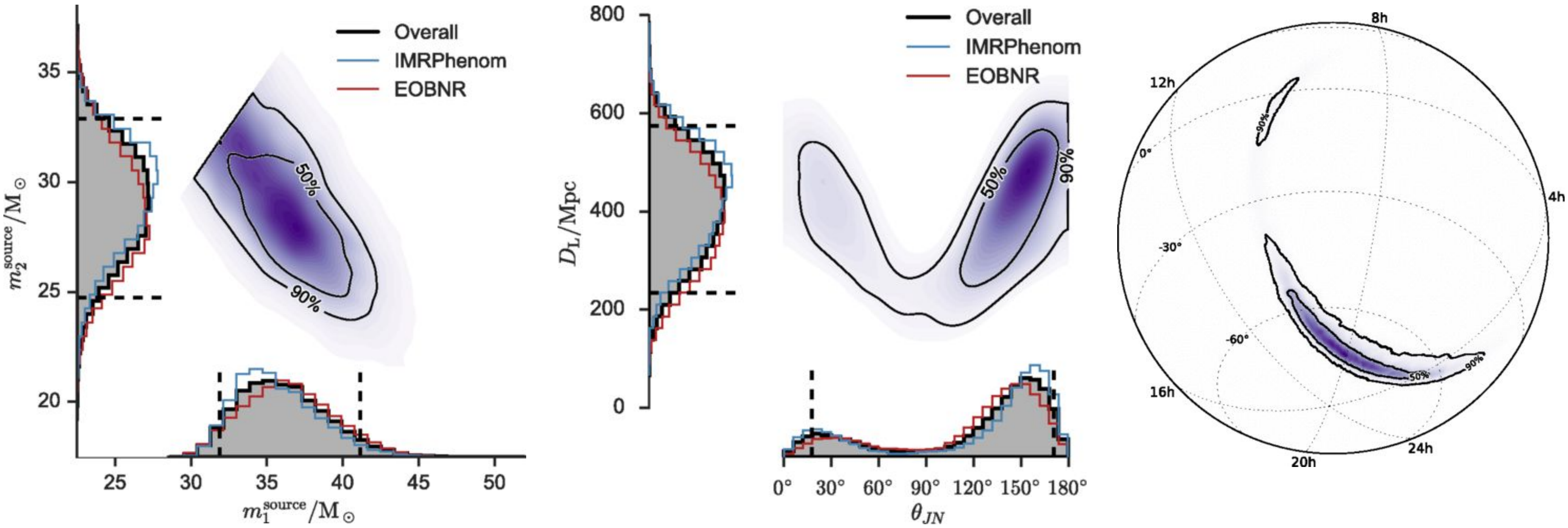
$$\Psi(f,\mathcal{M}) = 2\pi f t_m - \frac{\pi}{4} - \phi_c + \frac{3}{128}\left(\frac{\pi G\mathcal{M}}{c^3}\right)^{-5/3}\frac{1}{f^{5/3}}$$

$$A(f,\mathcal{M}) = \frac{1}{d}\frac{5}{24\pi^{4/3}}\frac{(G\mathcal{M})^{5/6}}{c^{3/2}}\frac{1}{f^{7/6}}.$$

- Two polarizations of GWs
- Amplitudes inversely proportional to the distance, and related to the chirp mass.
- Degeneracy with the inclination angle.
- A global network can better constrain each of the polarizations and help reduce the degeneracy.

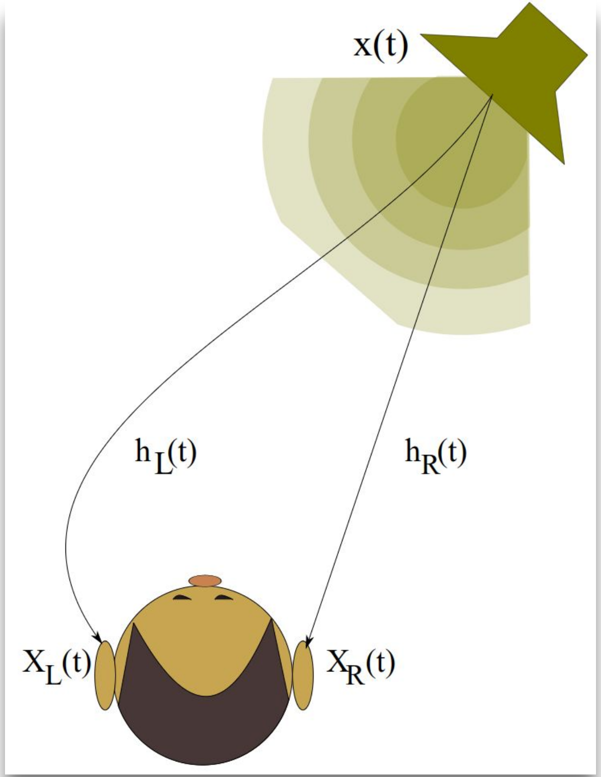
See e.g., Maggiore, Gravitational Waves, 2008

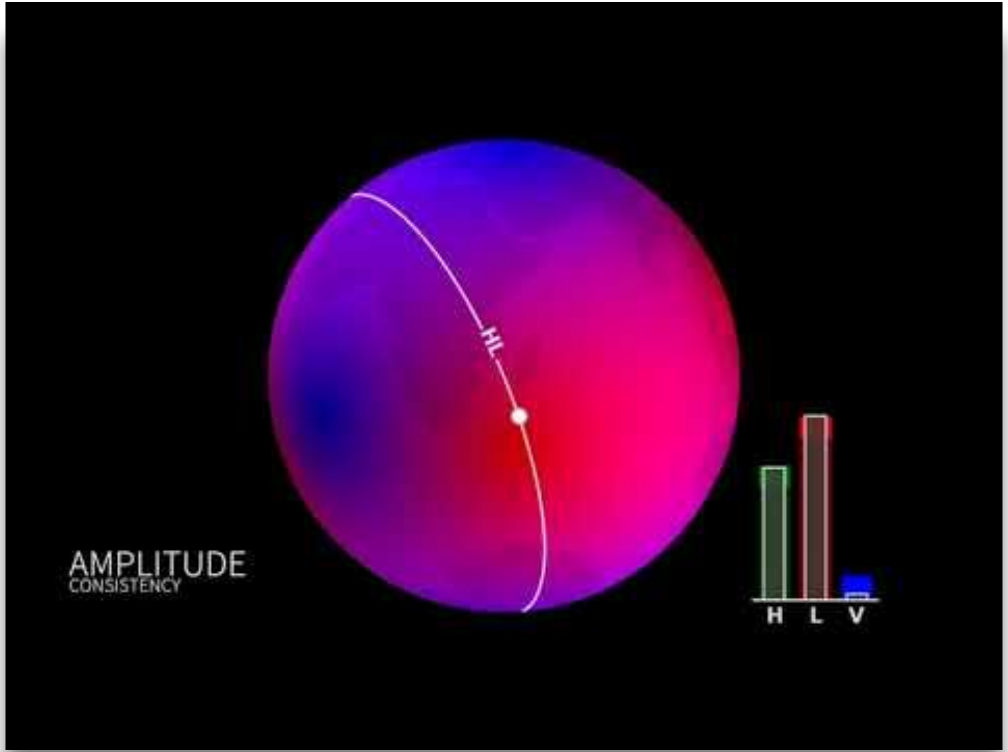
GW150914



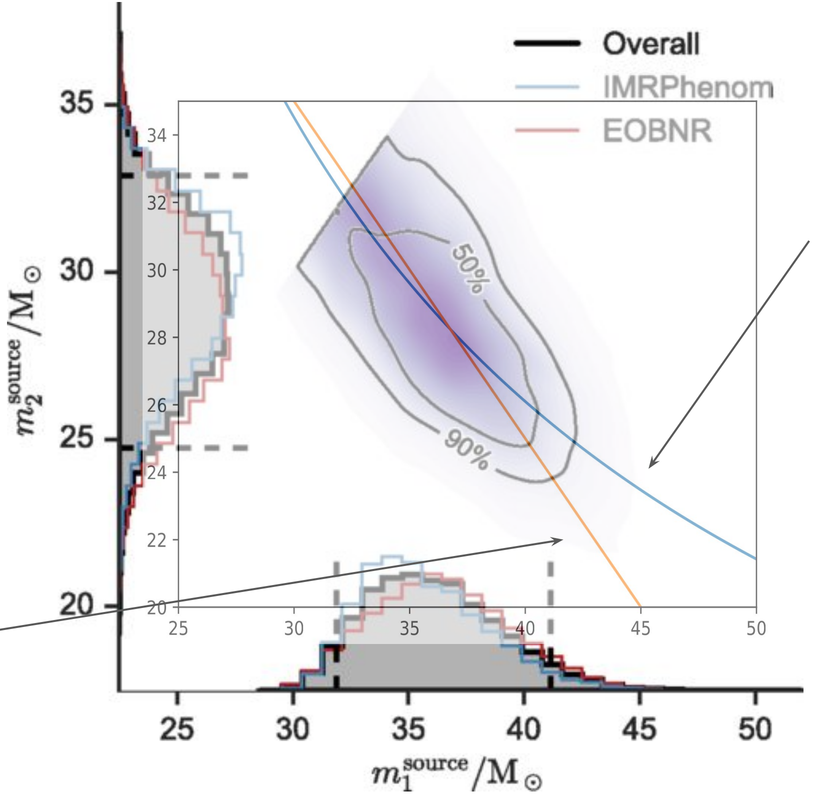
Chirp mass from the chirp, Dl from the amplitude, source mass assumes the redshift from fiducial cosmology

Localization of GW events



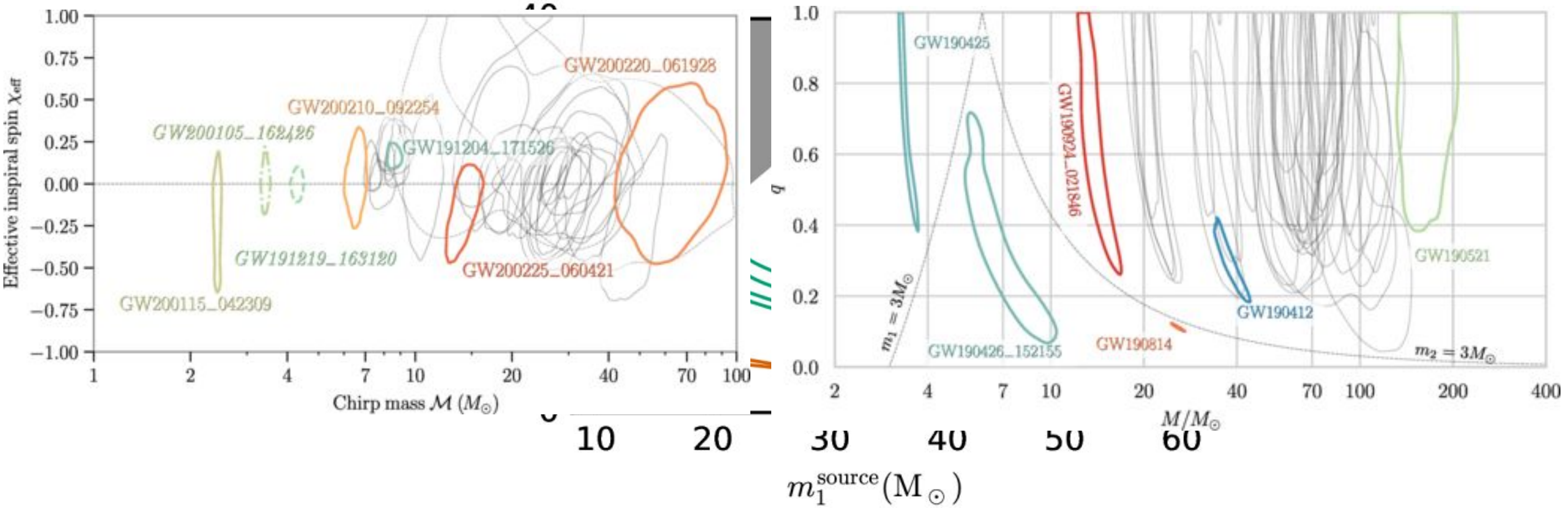


GW150914



Chirp mass degeneracy

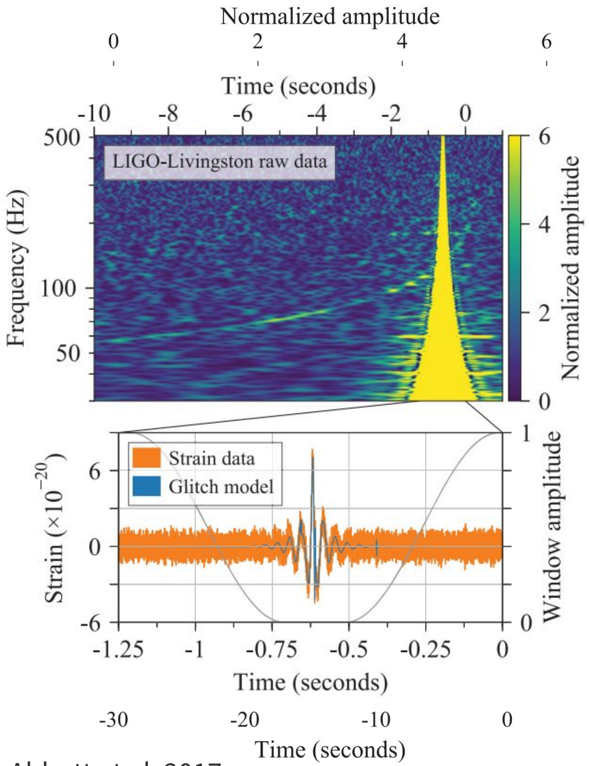
Total mass degeneracy

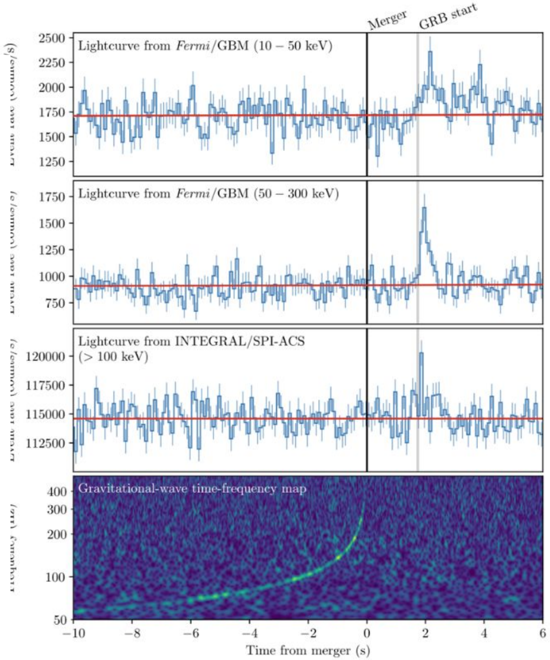

Effective inspiral spin $\chi_{\rm eff}$
Chirp mass $\mathcal{M}$ ($M_\odot$)
GW200220_061928
GW200210_092254
GW200105_162426
GW191204_171526
GW191219_163120
GW200225_060421
GW200115_042309
GW190425
GW190924_021846
GW190521
GW190412
GW190426_152155
GW190814
$m_1 = 3M_\odot$
$m_2 = 3M_\odot$
q
$M/M_\odot$
$m_1^{\rm source}(M_\odot)$

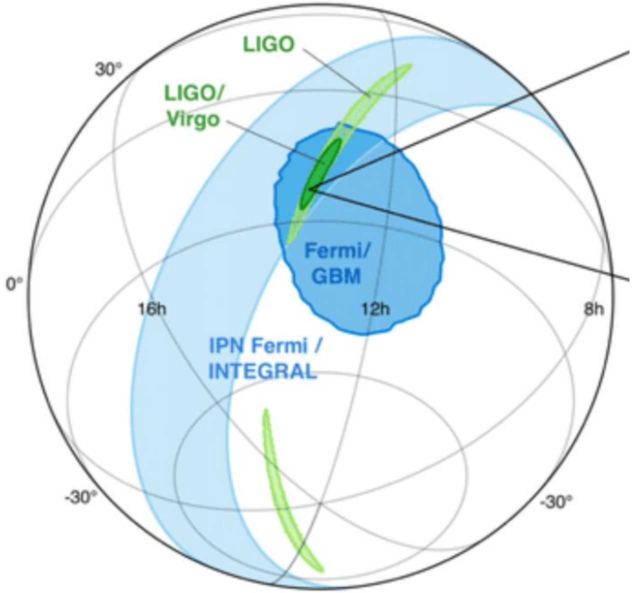
Bayesian analysis

- Bayes theorem:
  - $P(A, B) = P(A|B)\, P(B) = P(B|A)\, P(A)$
- Marginalization:
  - $P(A) = \int P(A, C)\, d C = \int P(A|C)\, P(C)\, dC$
- Any of these equations you can add information to the right hand side of all probabilities, for example:
  - $P(A|B, C)\, P(B|C) = P(B|A, C)\, P(A|\, C)$
  - $P(A|I) = \int P(A, C|I)\, d C = \int P(A|C, I)\, P(C|I)\, dC$
- These will be quite useful manipulations and should be kept as cheat sheet with us

Bright siren: GW170817



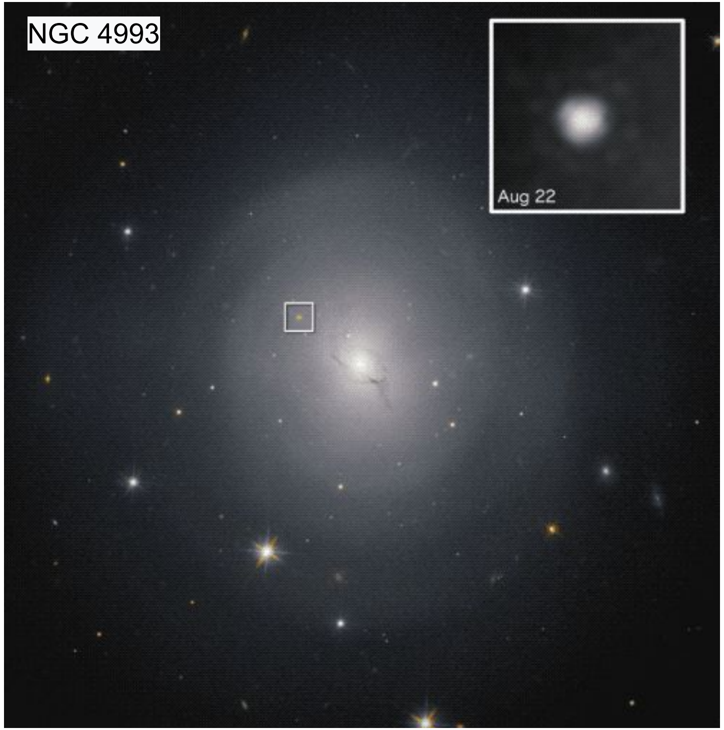




GW events localized with an electromagnetic counterpart

Abbott et al. 2017

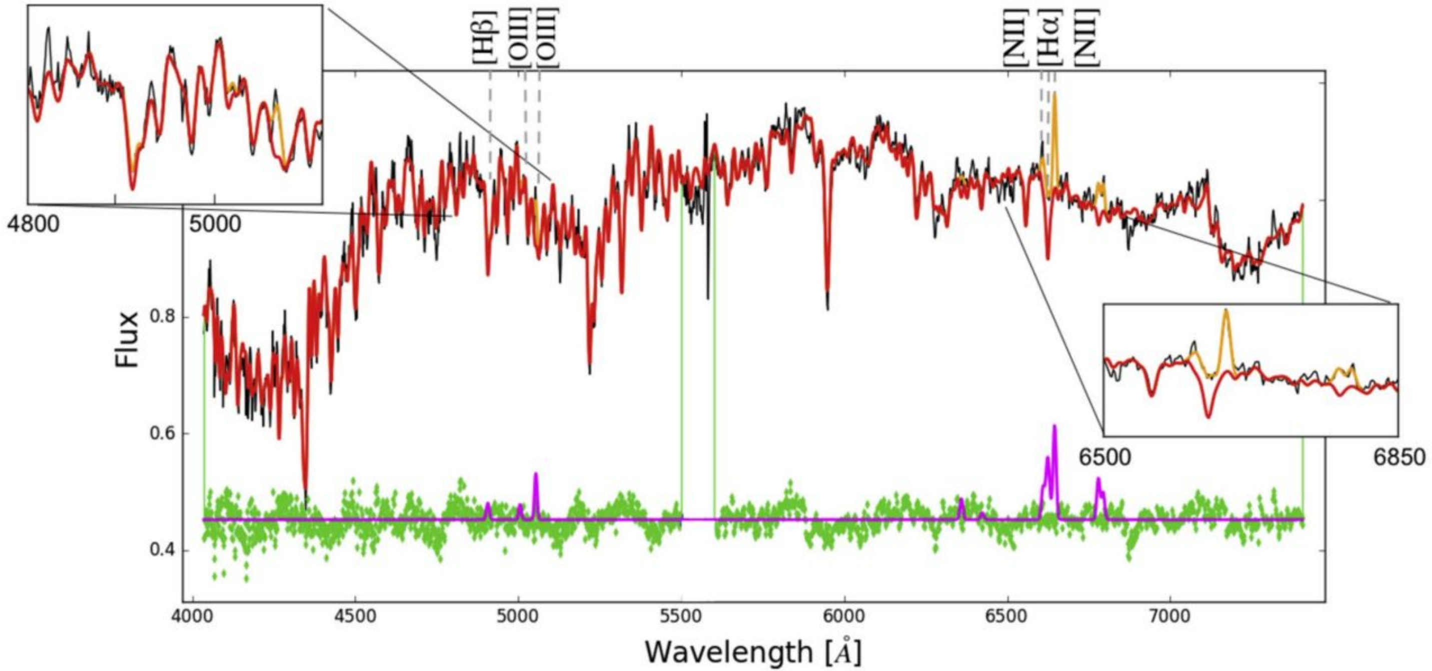
Bright siren: EM follow up



NASA/ESA Hubble space telescope

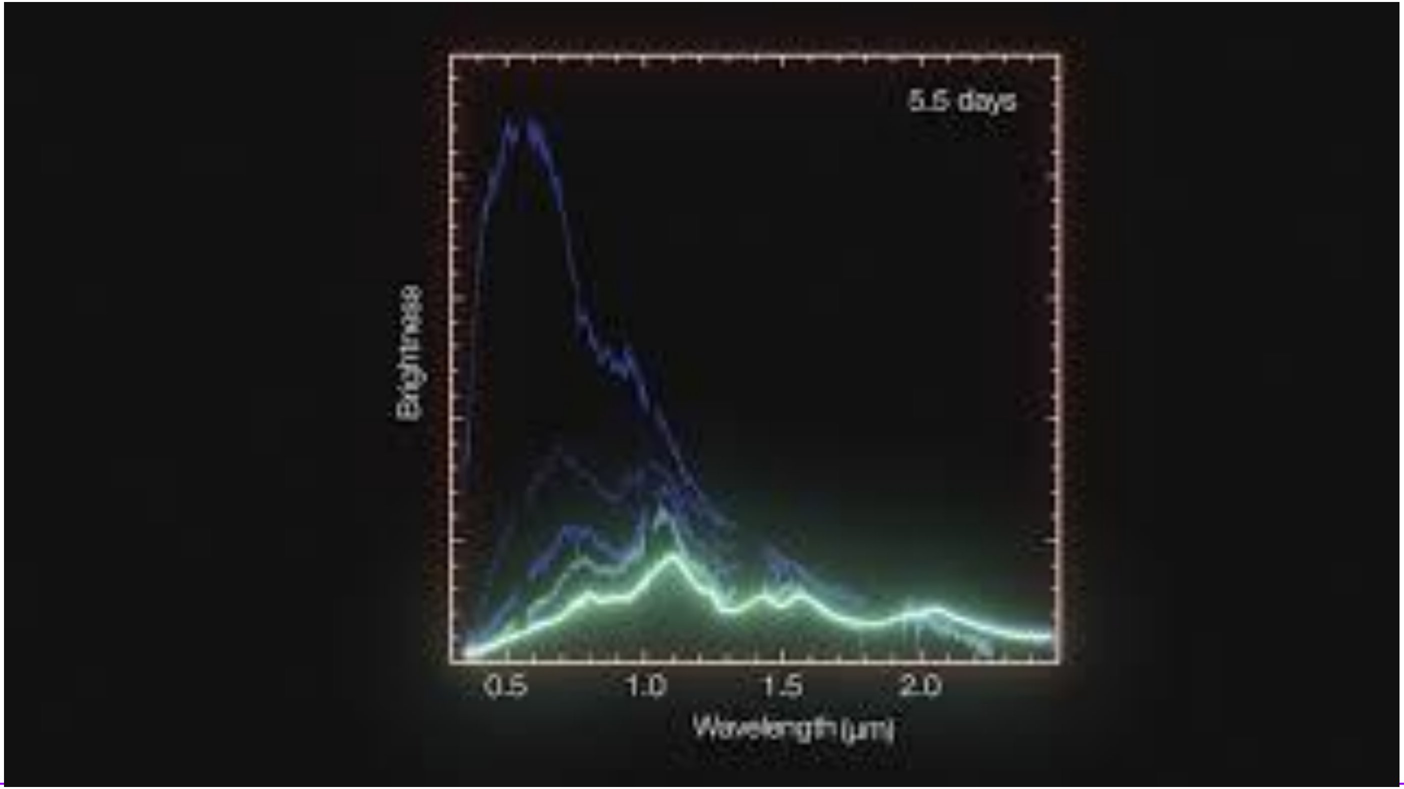
- Typical localizations are 100s of sq degrees.
- Optical telescopes have order 1 sq degree field of view
- Optimum plan to target galaxies to look for transients.
  - Weight galaxies by their luminosity to decide targeting priorities
  - But some fields set earlier and some later due to rotation of the Earth
- GW170817 progenitor discovered after 11 hours of search

NGC 4993: Redshift from spectrum



Palmese et al. (2017) 6dF spectrum

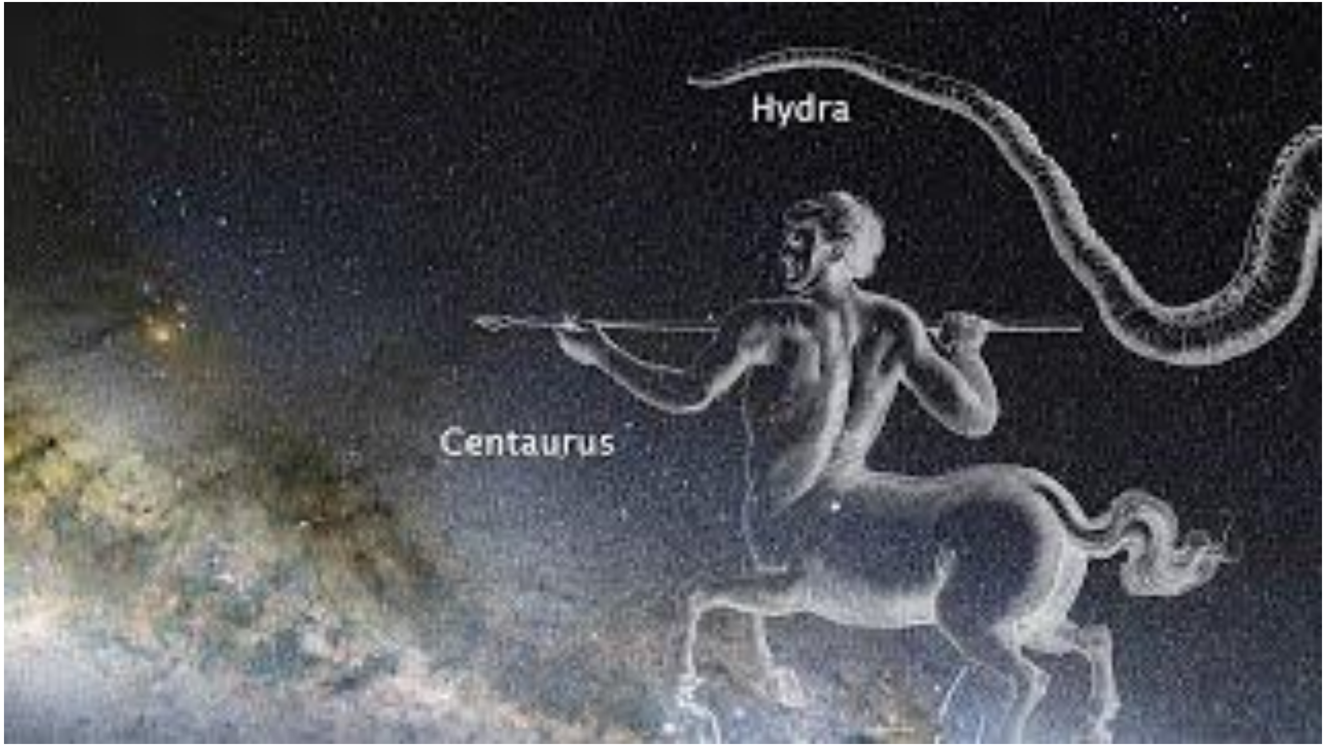
Spectrum of the transient



Hubble constant from nearby bright sirens

- Nearby Universe the Hubble law reduces to $v_{cosmo} = H_0 d$
- But peculiar velocities can make it difficult to discern the cosmological recession velocity from the peculiar velocity
- Remember that the peculiar velocity here is not the velocity of GW event, but the galaxy itself.
- The frequency shift in the GW waveform comes from:
  - Cosmological recession velocity of the host galaxy.
  - Peculiar velocity due to the galaxies neighbours.
  - Velocity of the GW event with respect to the host galaxy.

NGC 4993



NASA/ESA Hubble and DSS

NGC 4993 part of galaxy group with velocity

$$v_{\rm r} = 3327 \pm 72\,{\rm km\,s^{-1}}$$

Peculiar velocity:

$$v_{\rm p} = 310 \pm 150\,{\rm km\,s^{-1}}$$

Likelihood: GW170817 observables

$$p(x_{\mathrm{GW}}, v_r, \langle v_p \rangle \mid d, \cos\iota, v_p, H_0) =$$
$$p(x_{\mathrm{GW}} \mid d, \cos\iota)\, p(v_r \mid d, v_p, H_0)\, p(\langle v_p \rangle \mid v_p)$$

$$p\left(v_r \mid d, v_p, H_0\right) = N\left[v_p + H_0 d, \sigma^2_{v_r}\right](v_r)$$

$$p\left(\langle v_p \rangle \mid v_p\right) = N\left[v_p, \sigma^2_{v_p}\right]\left(\langle v_p \rangle\right)$$

Formalism allows to account for uncertainties in all of the measurements

Abbott et al. 2017

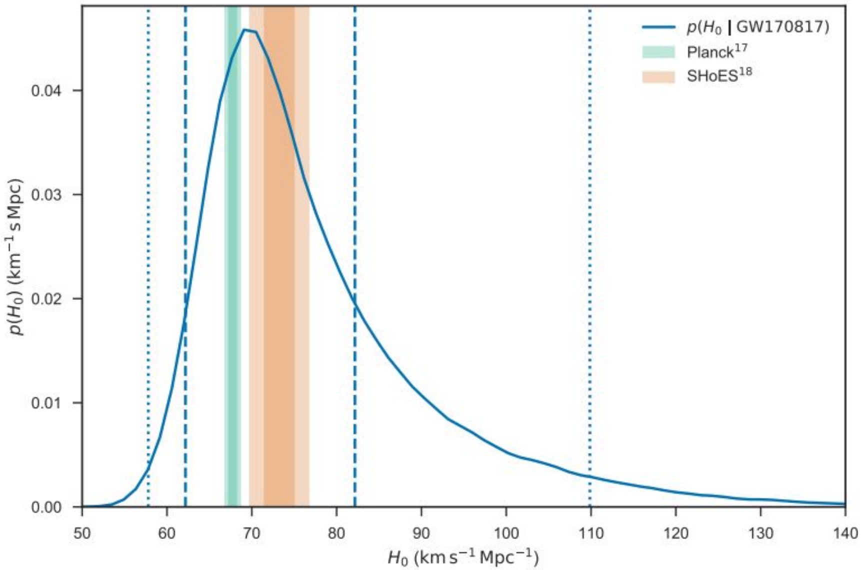
Posterior for the Hubble constant

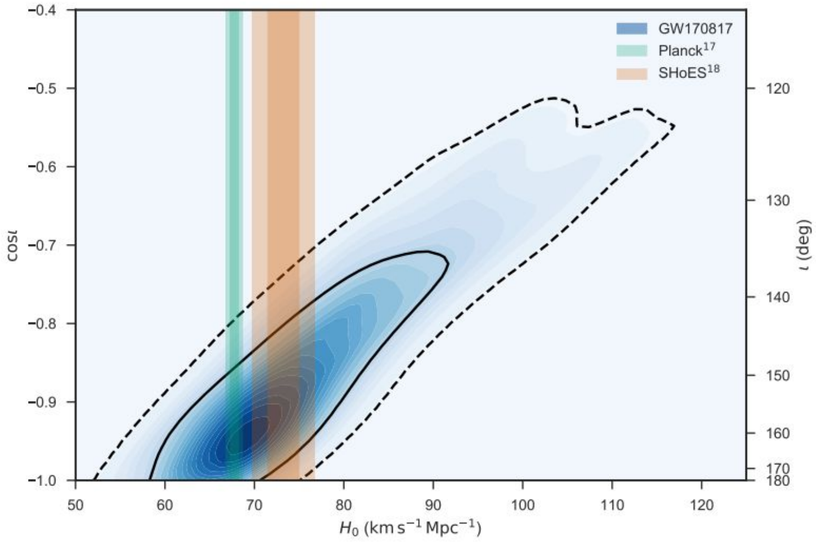
$$p(H_0, d, \cos\iota, v_p \mid x_{\rm GW}, v_r, \langle v_p \rangle)$$
$$\propto \frac{p(H_0)}{\mathcal{N}_{\rm s}(H_0)} p(x_{\rm GW} \mid d, \cos\iota)\, p(v_r \mid d, v_p, H_0)$$
$$\times\, p(\langle v_p \rangle \mid v_p)\, p(d)\, p(v_p)\, p(\cos\iota),$$

Also need to account for selection effects, not all GW events will be observable

Abbott et al. 2017

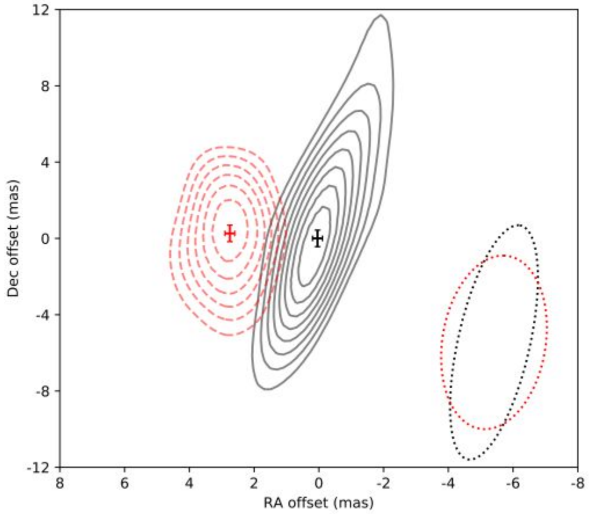
Hubble constant constraint

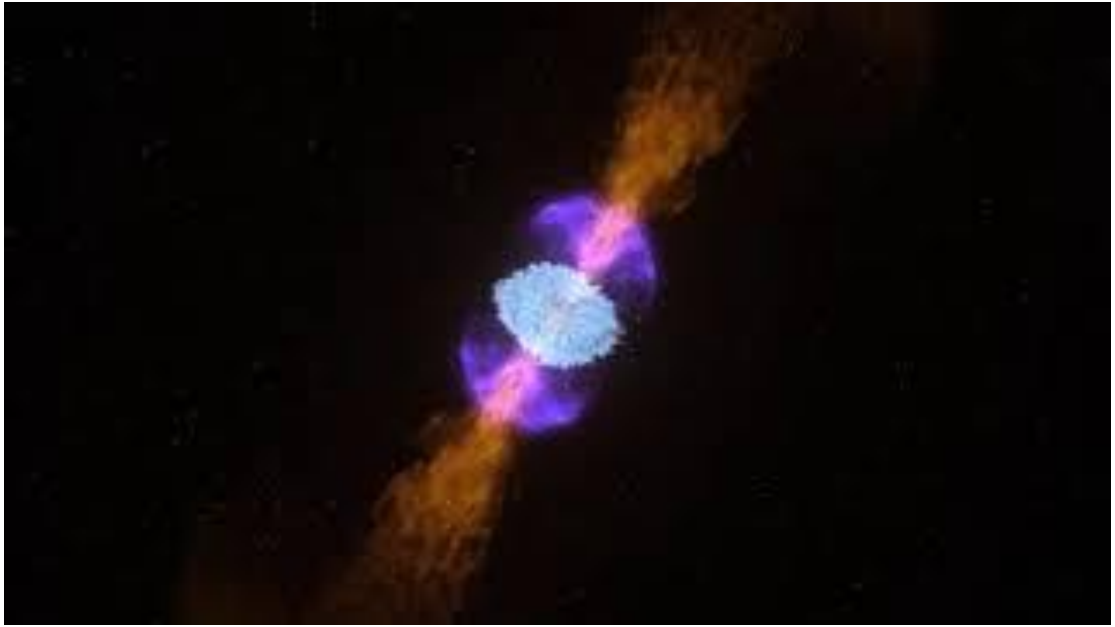




Abbott et al. 2017

Superluminal motion of the jet

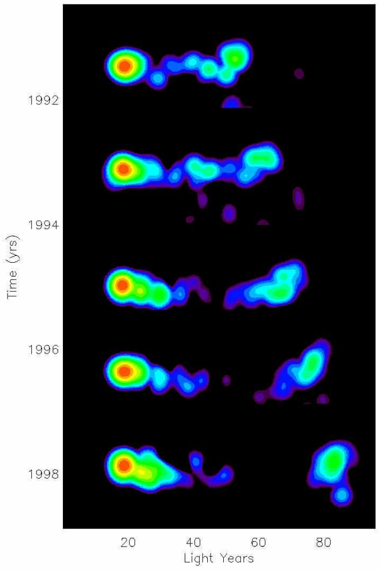


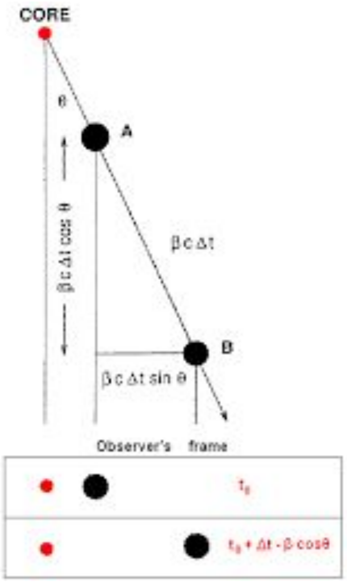


Very long baseline interferometry can be used to detect the superluminal motion

Mooley et al. 2018

Apparent superluminal motion

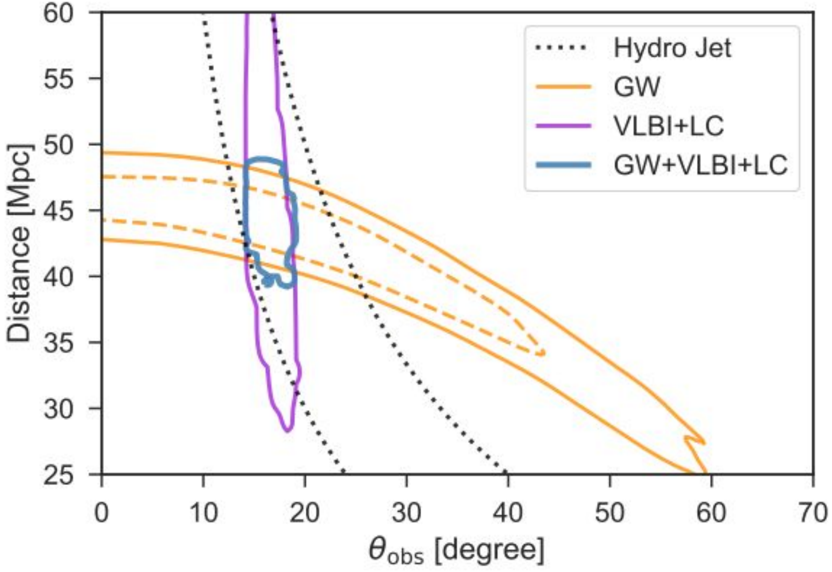


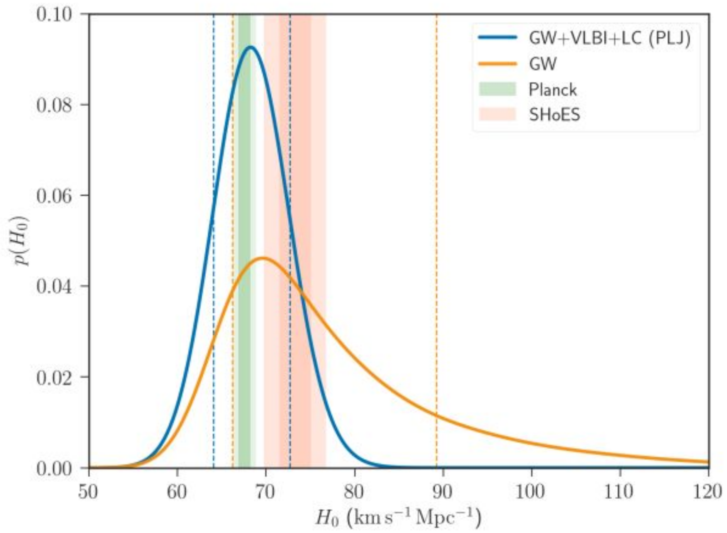


- The jet is chasing the radiation
- So the blob emitting the radiation may have already travelled closer to us and has to cover smaller distance.
- Apparent motion can seem superluminal for a relativistic jet.

Credits : NRAO/MPG

Combination of GW + VLBI data

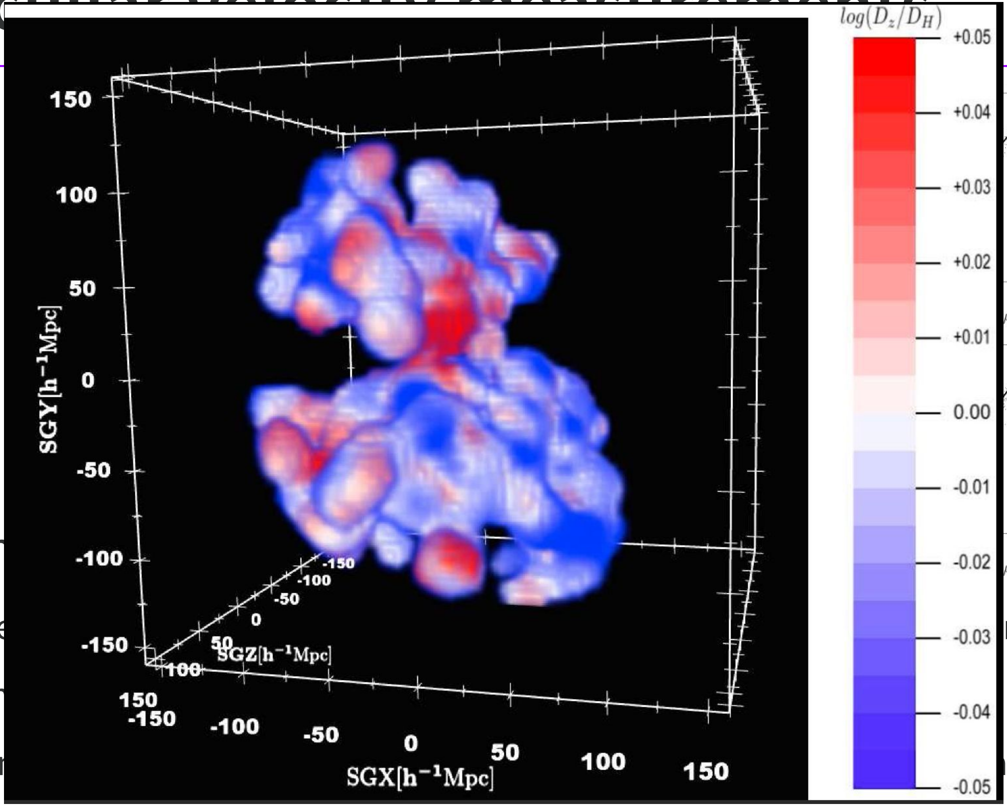


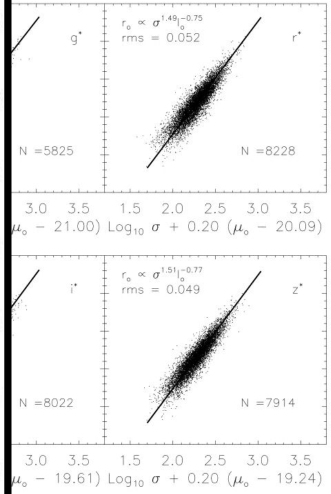


Measurements breaks degeneracy between distance and inclination

Hotokezaka et al. 2018

Peculiar velocity measurements

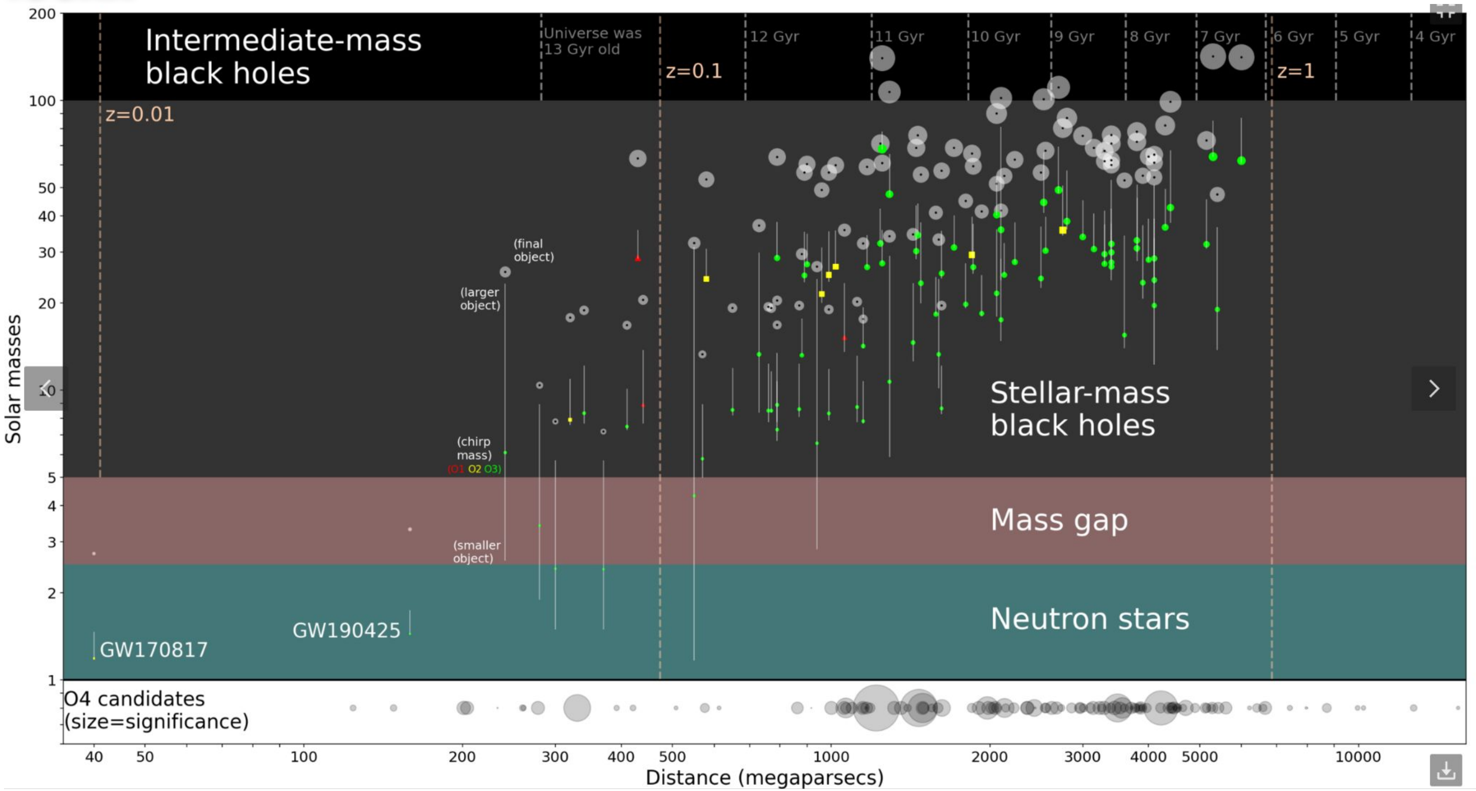




- Galaxy scalin
  - Tully Fishe … residuals in the fundamen
  - Fundamen … nd velocity dispersion. Residuals again correlated with peculiar velocities

Courteau and Rix (1998) Bernardi et al. 2003 Springob et al. 2014

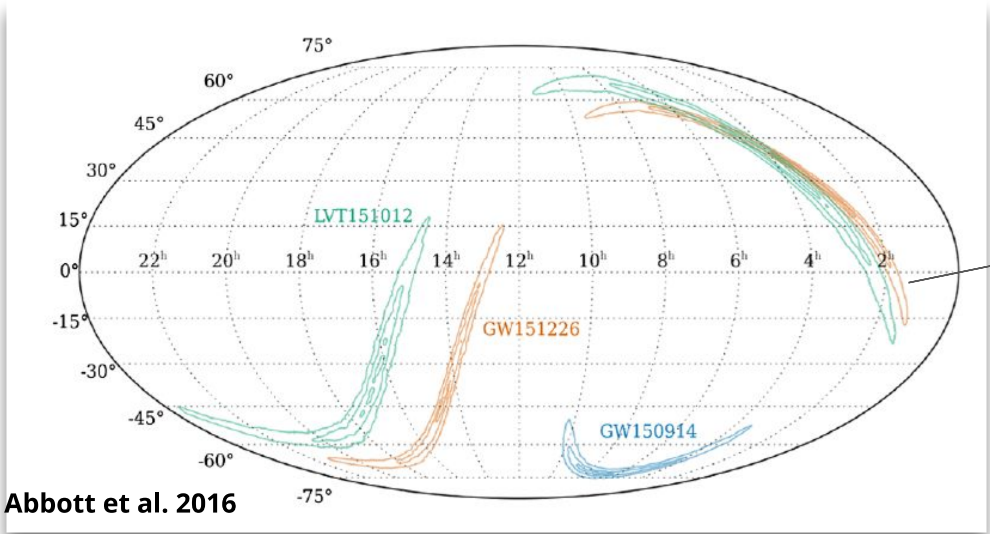
LVK detections in O3b, O4 candidates



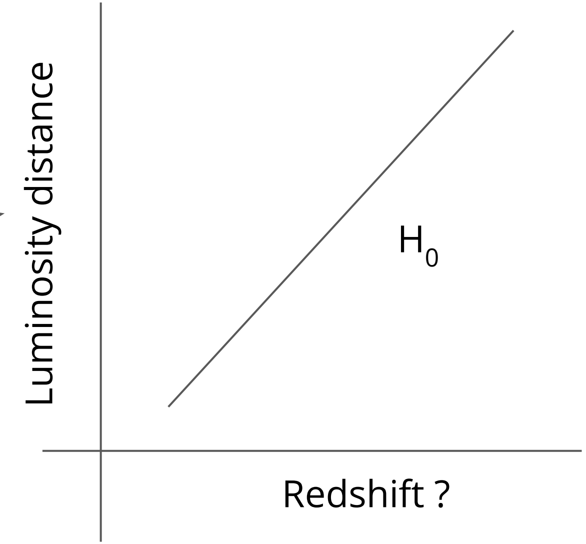
Many candidates, but only a single EM counterpart

Wikipedia (Exoplanetaryscience)

Hubble constant from BBH events



Abbott et al. 2016



Merging binary black holes give luminosity distance

Redshift information not readily available as no electromagnetic counterparts

Consider galaxies (with known redshifts) in the localization region as potential hosts.

Schutz 1986

Hubble constant from GW+galaxies

If we accept that H_0 is less than some H_{max} (say 120 km s^{-1} Mpc^{-1}), then the error box can be surveyed for bright galaxies (up to ~15 mag) with velocities below $H_{max}r$, where we consider at first only events with $r<100$ Mpc. Existing surveys (see ref. 8) show that galaxies with velocities <12,000 km s^{-1} cluster strongly, with ~1 cluster per square degree, and that bright galacies are good tracers of these clusters. In our error box, therefore, the source ought to be in one of ~36 clusters. Each cluster redshift gives a candidate value for H_0. As r is known

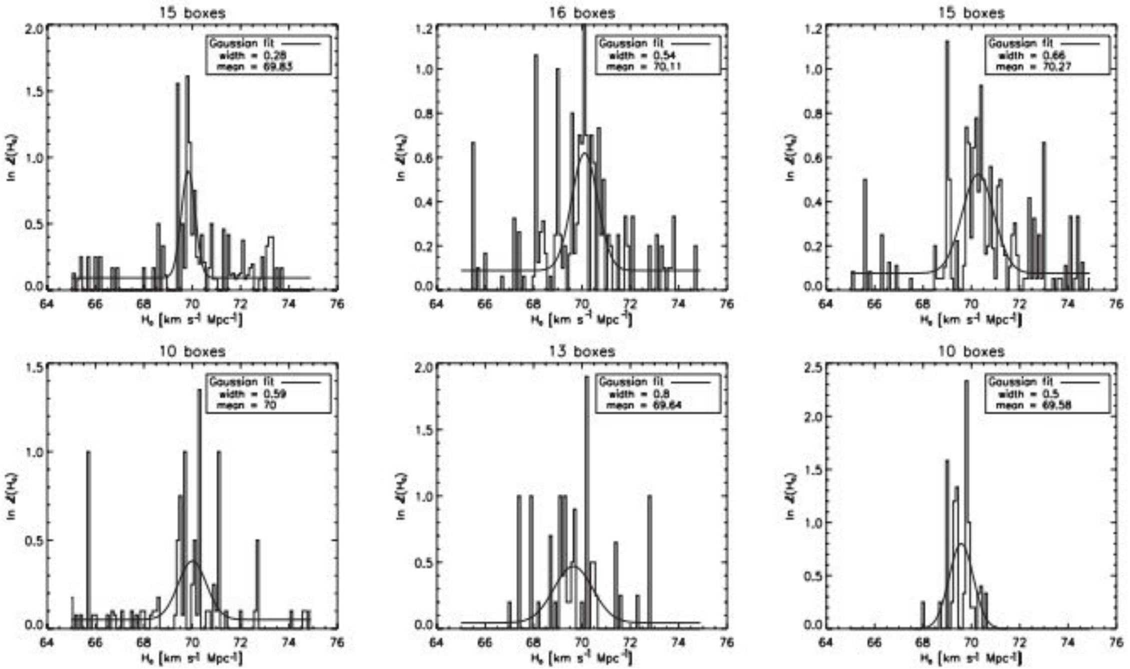
Schutz 1986

Schutz method in simulations

CHELSEA L. MACLEOD AND CRAIG J. HOGAN PHYSICAL REVIEW D 77, 043512 (2008)

Single interferometer

Two interferometers

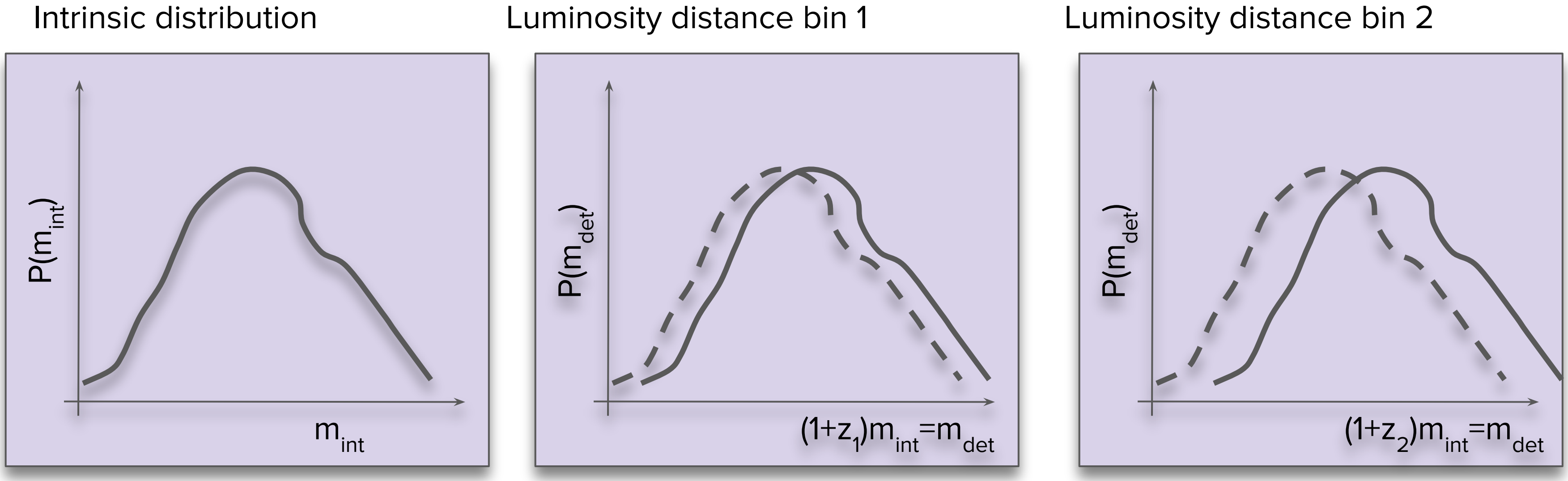


Statistical host identification

- Ignores the clustering of galaxies
- If we assume that the potential host of the GW event is in the galaxy catalog, then the method of Schutz et al. 1986 will still work.
- Use every galaxy in the localization as a potential host, and use its redshift to obtain Hubble constant.
- There is always one galaxy with the correct Hubble constant, rest are randomly distributed.
- As you get more and more events the correct Hubble constant will coherently arise from the random distribution of the Hubble constants inferred from other unrelated galaxies.
- But needs to deal with incompleteness in real galaxy catalogs

Taylor+2012, Taylor & Gair 2012, Farr+2019, Ezquiaga & Holz 2021, Mastrogiovanni et al. 2021

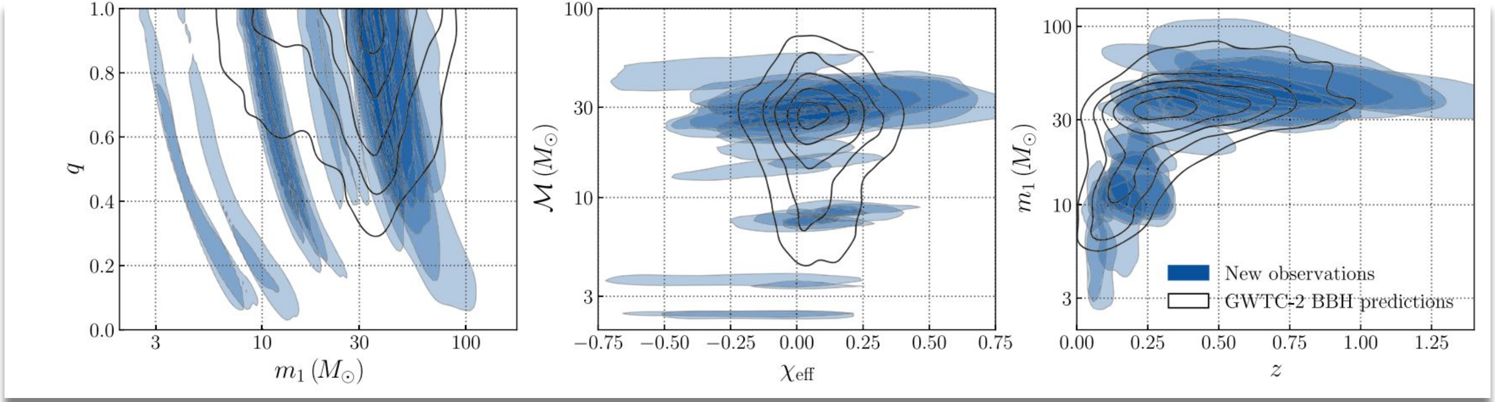
Spectral sirens



Gravitational wave events measure the detector frame mass

Relies on the universality of the mass distribution of BBH merger events

Measuring mass distributions



- Masses measured by gravitational wave events have errors, and are degenerate with a number of other parameters.
- Simplest would be to make histograms, but clearly that cannot work
- Histograms are still better than distributing masses based on their errors
- Need to carry out full Bayesian analysis.

LVK Collaboration, 2023

Bayesian analysis

$$\lambda = \{\Omega, \lambda', N\}$$

Cosmological parameters,
Rate parameters, Shape parameters

$$p(\vec{\lambda}|\{\vec{d}_i\}) = \frac{p(\{\vec{d}_i\}|\vec{\lambda})\pi(\vec{\lambda})}{p(\{\vec{d}_i\})}$$

Bayes' theorem

$$\frac{\mathrm{d}N}{\mathrm{d}\vec{\theta}}(\vec{\lambda}) = N p_{\mathrm{pop}}(\vec{\theta}|\vec{\lambda}')$$

Normalization and shape

Ignore cosmology and normalization for the moment

Mandel et al. 2018

No measurement errors

$$p(\{\vec{\theta}_i\}|\vec{\lambda}') = \prod_{i=1}^{N_{\rm obs}} \frac{p_{\rm pop}(\vec{\theta}_i|\vec{\lambda}')}{\int d\vec{\theta}\; p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}$$

Parameters of individual events drawn from sample properties

But due to selection effects, we only observe events which have been detected.

$$p(\{\vec{\theta}_i\}|\vec{\lambda}') = \prod_{i=1}^{N_{\rm obs}} \frac{p_{\rm pop}(\vec{\theta}_i|\vec{\lambda}')p_{\rm det}(\vec{\theta}_i)}{\int d(\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')p_{\rm det}(\vec{\theta})} = \prod_{i=1}^{N_{\rm obs}} \frac{p_{\rm pop}(\vec{\theta}_i|\vec{\lambda}')}{\int d(\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')p_{\rm det}(\vec{\theta})}$$

Mandel et al. 2018

Noisy data vector

$$p_{\rm det}(\vec{\theta}) = \int_{\vec{d}>\text{threshold}} p(\vec{d}|\vec{\theta})\mathrm{d}\vec{d} = \int \mathbf{I}(\vec{d})p(\vec{d}|\vec{\theta})\mathrm{d}\vec{d}$$

Think about d as a noisy data vector from GW events, and we only see events with some SNR threshold.

$$p(\vec{d}|\vec{\lambda}') = \frac{\int \mathrm{d}\vec{\theta}\, p(\vec{d}|\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}{\alpha(\vec{\lambda}')}$$

$$p(\{\vec{d}_i\}|\vec{\lambda}') = \prod_{i=1}^{N_{\rm obs}} \frac{\int \mathrm{d}\vec{\theta}\, p(\vec{d}_i|\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}{\int \mathrm{d}\vec{\theta}\, p_{\rm det}(\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}$$

$$\begin{aligned}\alpha(\vec{\lambda}') &\equiv \int_{\vec{d}>\text{threshold}} \mathrm{d}\vec{d} \int \mathrm{d}\vec{\theta}\, p(\vec{d}|\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}') \\ &= \int \mathrm{d}\vec{\theta}\left[\int_{\vec{d}>\text{threshold}} \mathrm{d}\vec{d}\, p(\vec{d}|\vec{\theta})\right] p_{\rm pop}(\vec{\theta}|\vec{\lambda}') \\ &\equiv \int \mathrm{d}\vec{\theta}\, p_{\rm det}(\vec{\theta})p_{\rm pop}(\vec{\theta}|\vec{\lambda}')\,.\end{aligned}$$

Mandel et al. 2018

Gravitational wave case

Prior to our meta analysis, parameter estimation runs are already performed as soon as an event is detected.

$$p(\vec{d}_i|\vec{\theta}_i) = \frac{p(\vec{\theta}_i|\vec{d}_i)\,p(\vec{d}_i)}{\pi(\vec{\theta})}$$

Remove priors which were already imposed to get the likelihood itself

$$p(\{\vec{d}_i\}|\vec{\lambda}') = \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} p_{\rm pop}({}^j\vec{\theta}_i|\vec{\lambda}')\frac{p(\vec{d}_i)}{\pi(\vec{\theta})}}{\int d\vec{\theta}\, p_{\rm det}(\vec{\theta})\,p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}$$

Can reuse existing “S” PE samples and perform Monte-Carlo integral

Mandel et al. 2018

Posterior distribution on shape

$$p(\vec{\lambda}'|\{\vec{d}_i\}) = \frac{\pi(\vec{\lambda}')}{p(\{\vec{d}_i\})} \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} \frac{p_{\rm pop}({}^j\vec{\theta}_i|\vec{\lambda}')}{\pi(\vec{\theta})} p(\vec{d}_i)}{\int d\vec{\theta}\, p_{\rm det}(\vec{\theta})\, p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}$$

$$= \pi(\vec{\lambda}') \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} \frac{p_{\rm pop}({}^j\vec{\theta}_i|\vec{\lambda}')}{\pi(\vec{\theta})}}{\int d\vec{\theta}\, p_{\rm det}(\vec{\theta})\, p_{\rm pop}(\vec{\theta}|\vec{\lambda}')} .$$

Explicit dependence of priors on the hyperparameters, and PE runs can be seen here

Mandel et al. 2018

What about the number N?

$$P(N_{\rm obs}) = \exp^{-N_{\rm det}} N_{\rm det}^{N_{\rm obs}}$$

Poisson likelihood

$$N_{\rm det}\left(\vec{\lambda}\right) \equiv \int_{\vec{d}>\rm threshold} \mathrm{d}\vec{d}\, \mathrm{d}\vec{\theta}\; p(\vec{d}|\vec{\theta}) \frac{\mathrm{d}N}{\mathrm{d}\vec{\theta}}(\vec{\lambda}) = N\alpha(\vec{\lambda}')$$

Average expectation

$$p(\vec{\lambda}', N|\{\vec{d}_i\}) = \pi(\vec{\lambda}')\pi(N) \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} \frac{p_{\rm pop}({}^j\vec{\theta}_i|\vec{\lambda}')}{\pi(\vec{\theta})}}{\int \mathrm{d}\vec{\theta}\, p_{\rm det}(\vec{\theta}) p_{\rm pop}(\vec{\theta}|\vec{\lambda}')} \times e^{-N_{\rm det}} (N_{\rm det})^{N_{\rm obs}} .$$

Final expression for hyperparameters of CBC population given data!

Mandel et al. 2018

CBC hyperparameter posterior

Prior information

Likelihood

$$p(\vec{\lambda}', N|\{\vec{d}_i\}) = \pi(\vec{\lambda}')\pi(N) \prod_{i=1}^{N_{\rm obs}} \frac{\frac{1}{S_i}\sum_{j=1}^{S_i} \frac{p_{\rm pop}({}^j\vec{\theta}_i|\vec{\lambda}')}{\pi(\vec{\theta})}}{\int {\rm d}\vec{\theta}\, p_{\rm det}(\vec{\theta})\, p_{\rm pop}(\vec{\theta}|\vec{\lambda}')}$$

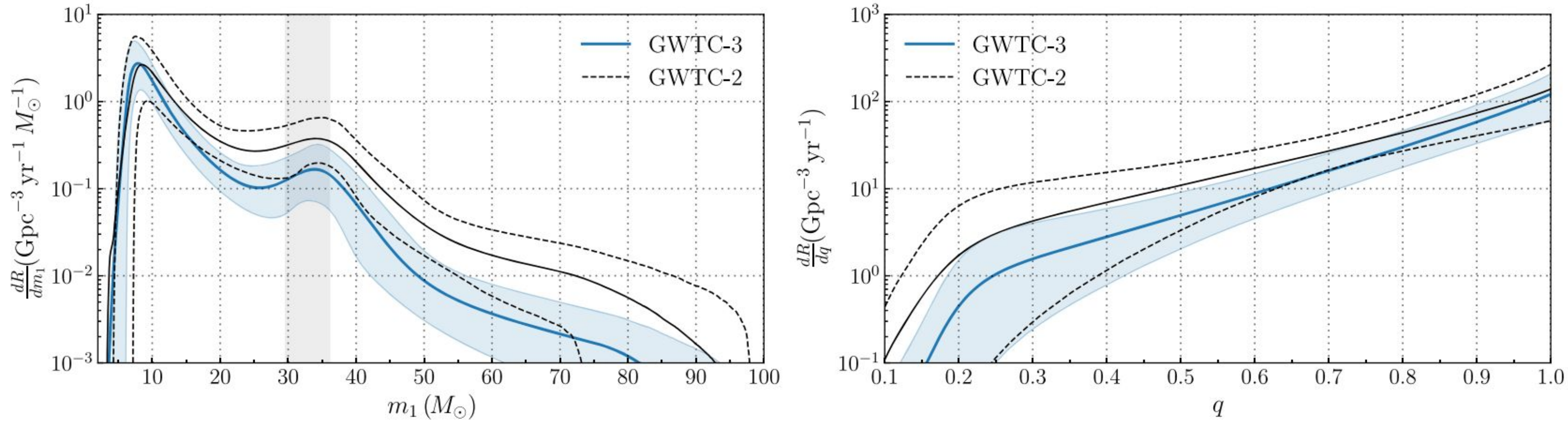
$$\times e^{-N_{\rm det}} (N_{\rm det})^{N_{\rm obs}} \,.$$

Prior information

Selection effect

Mandel et al. 2018

Rate posterior given data



This is for fixed cosmology though!

LVK Collaboration, 2023

GWcosmo: Joint spectral siren and galaxy catalog method

Prior

Population sampling

$$p(\Lambda|\{x_{\rm GW}\},\{D_{\rm GW}\},I) \propto p(\Lambda|I)p(N_{\rm det}|\Lambda,I)\prod_i^{N_{\rm det}} \frac{\int p(x_{{\rm GW}i}|\theta,\Lambda,I)p(\theta|\Lambda,I)d\theta}{\int p(D_{{\rm GW}i}|\theta,\Lambda,I)p(\theta|\Lambda,I)d\theta},$$

Poisson distribution

$p_{\rm det}(\vec{\theta})$

Gray et al. 2023

GWcosmo: Joint spectral siren and galaxy catalog method

This introduces sky pixel dependent information

$$\left[\int \sum_{j}^{N_{\rm pix}} p(x_{\rm GW}{}_i|\Omega_j,\theta,\Lambda,I)p(\theta|\Omega_j,\Lambda,I)p(\Omega_j|I)d\theta\right]$$

$$p(\Lambda|\{x_{\rm GW}\},\{D_{\rm GW}\},I) \propto p(\Lambda|I)p(N_{\rm det}|\Lambda,I)\prod_{i}^{N_{\rm det}} \frac{\int p(x_{\rm GW}{}_i|\theta,\Lambda,I)p(\theta|\Lambda,I)d\theta}{\int p(D_{\rm GW}{}_i|\theta,\Lambda,I)p(\theta|\Lambda,I)d\theta},$$

Gray et al. 2023

Statistical host identification

$$\left[\iint \sum_{j}^{N_{\rm pix}} p(x_{\rm GW}{}_i|\Omega_j, z, \theta', \Lambda, I)p(\theta'|\Omega_j, \Lambda, I)p(z|\Omega_j, \Lambda, I)d\theta' dz\right]$$

$$p(\Lambda|\{x_{\rm GW}\}, \{D_{\rm GW}\}, I) \propto p(\Lambda|I)p(N_{\rm det}|\Lambda, I) \prod_{i}^{N_{\rm det}} \frac{\int p(x_{\rm GW}{}_i|\theta, \Lambda, I)p(\theta|\Lambda, I)d\theta}{\int p(D_{\rm GW}{}_i|\theta, \Lambda, I)p(\theta|\Lambda, I)d\theta},$$

The line-of-sight redshift prior can be used to introduce information from galaxy catalogs.

Gray et al. 2023

Redshift prior

$$p(z|\Omega_i,\Lambda,s,I) = \iint \sum_{g=G,\bar{G}} p(z,M,m,g|\Omega_i,\Lambda,s,I)\, dM dm,$$
$$= p(G|\Omega_i,\Lambda,s,I) \iint p(z,M,m|G,\Omega_i,\Lambda,s,I)\, dM dm$$
$$+ p(\bar{G}|\Omega_i,\Lambda,s,I) \iint p(z,M,m|\bar{G},\Omega_i,\Lambda,s,I)\, dM dm,$$

In catalog

Out of catalog

- The redshift of the GW event could be related to galaxies with apparent magnitude m, absolute magnitude M, that are either in the catalog or not in the catalog
- The symbol s denotes the presence of a GW event, which was hidden inside

Gray et al. 2023

In catalog term

$$p(z, M, m|G, \Omega_i, \Lambda, s, I)$$
$$= \frac{p(z, M, m|G, \Omega_i, \Lambda, I)p(s|z, M, m, G, \Omega_i, \Lambda, I)}{p(s|G, \Omega_i, \Lambda, I)},$$
$$= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} p(M|z, m, G, \Omega_i, \Lambda, I)p(z, m|G, \Omega_i, \Lambda, I)p(s|z, M, \Lambda, I),$$
$$= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} \delta(M - M(z, m, \Lambda))p(z, m|G, \Omega_i, I)p(s|z, M, \Lambda, I).$$

Conditional probability
P(A, B) = P(A|B)P(B)

Connecting apparent and absolute magnitude via cosmological parameters

Gray et al. 2023

In catalog term

$$\iint p(z, M, m | G, \Omega_i, \Lambda, s, I)\, dM\, dm$$

$$= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} \int p(z, m|G, \Omega_i, I) p(s|z, M(z, m, \Lambda), \Lambda, I)\, dm.$$

$$= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} \frac{1}{N_{\mathrm{gal}}(\Omega_i)} \sum_{k}^{N_{\mathrm{gal}}(\Omega_i)} p(z|\hat{z}_k) p(s|z, M(z, \hat{m}_k, \Lambda), \Lambda, I)$$

Allows incorporation of redshift uncertainty of galaxies in the analysis

Gray et al. 2023

Out of catalog term

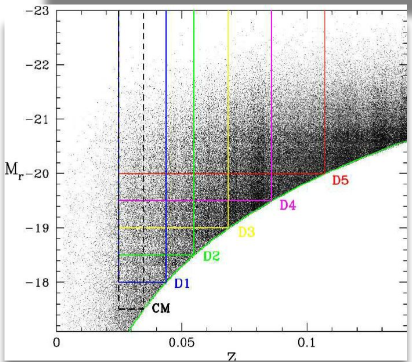
$$
\begin{aligned}
p(z, M, m|\bar{G}, \Omega_i, \Lambda, s, I) \\
&= \frac{p(z, M, m|\bar{G}, \Omega_i, \Lambda, I)p(s|z, M, m, \bar{G}, \Omega_i, \Lambda, I)}{p(s|\bar{G}, \Omega_i, \Lambda, I)}, \\
&= \frac{1}{p(s|\bar{G}, \Omega_i, \Lambda, I)} p(z, M, m|\bar{G}, \Omega_i, \Lambda, I)p(s|z, M, \Lambda, I), \\
&= \frac{1}{p(s|\bar{G}, \Omega_i, \Lambda, I)} \frac{p(\bar{G}|z, M, m, \Omega_i, \Lambda, I)p(z, M, m|\Omega_i, \Lambda, I)}{p(\bar{G}|\Omega_i, \Lambda, I)} p(s|z, M, \Lambda, I)
\end{aligned}
$$

Again similar mathematics as before for the in catalog term until here

Gray et al. 2023

Out of catalog term

$$p(\bar{G}|z, M, m, \Omega_i, H_0, I) = \Theta[m - m_{\rm th}(\Omega_i)]\Theta[z_{\rm cut} - z] + \Theta[z - z_{\rm cut}]$$



Galaxies fainter than magnitude threshold but within redshift range

Galaxies outside redshift range

$$p(z, M, m|\Lambda, I) = \delta(m - m(z, M, \Lambda))p(z, M|\Lambda, I).$$

Purely cosmology dependent term

Gray et al. 2023

Out of catalog term

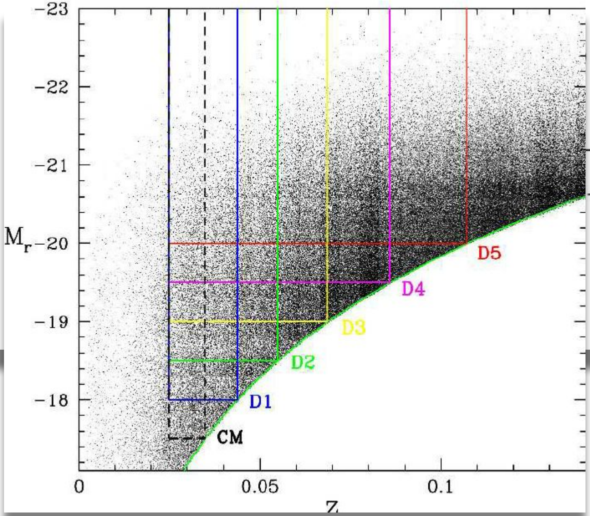
$$\iint p(z,M,m|\bar{G},\Omega_i,\Lambda,s,I)\,dM\,dm$$

$$= \frac{1}{p(s|\bar{G},\Omega_i,\Lambda,I)p(\bar{G}|\Omega_i,\Lambda,I)} \int \left(\Theta[m(z,M,\Lambda) - m_{\rm th}(\Omega_i)]\Theta[z_{\rm cut}-z] + \Theta[z - z_{\rm cut}]\right)$$

$$\ldots I)\,dM,$$

$$\ldots I)} \left[\Theta[z_{\rm cut}-z] \int_{M(z,m_{\rm th}(\Omega_i),\Lambda)}^{M_{\rm max}(H_0)} p(z,M|\Lambda,I)p(s|z,M,\Lambda,I)\,dM \right.$$

$$\left. + \Theta[z - z_{\rm cut}] \int_{M_{\rm min}(H_0)}^{M_{\rm max}(H_0)} p(z,M|\Lambda,I)p(s|z,M,\Lambda,I)\,dM \right]$$



Gray et al. 2023

Redshift prior

$$p(z|\Omega_i, \Lambda, s, I) = \iint \sum_{g=G,\bar{G}} p(z, M, m, g|\Omega_i, \Lambda, s, I)\, dM dm,$$

$$= p(G|\Omega_i, \Lambda, s, I) \iint p(z, M, m|G, \Omega_i, \Lambda, s, I)\, dM dm$$

$$+ p(\bar{G}|\Omega_i, \Lambda, s, I) \iint p(z, M, m|\bar{G}, \Omega_i, \Lambda, s, I)\, dM dm,$$

In catalog

Out of catalog

- The redshift of the GW event could be related to galaxies with apparent magnitude m, absolute magnitude M, that are either in the catalog or not in the catalog
- The symbol s denotes the presence of a GW event, which was hidden inside

Gray et al. 2023

Line of sight redshift prior

$$
\begin{aligned}
p(z|\Omega_i, \Lambda, s, I) \\
&= \frac{p(G|\Omega_i, \Lambda, s, I)}{p(s|G, \Omega_i, \Lambda, I)} f(\text{IN CAT}) + \frac{p(\bar{G}|\Omega_i, \Lambda, s, I)}{p(s|\bar{G}, \Omega_i, \Lambda, I) p(\bar{G}|\Omega_i, \Lambda, I)} f(\text{OUT OF CAT}) \\
&= \frac{p(s|G, \Omega_i, \Lambda, I) p(G|\Omega_i, \Lambda, I)}{p(s|G, \Omega_i, \Lambda, I) p(s|\Omega_i, \Lambda, I)} f(\text{IN CAT}) \\
&\quad + \frac{p(s|\bar{G}, \Omega_i, \Lambda, I) p(\bar{G}|\Omega_i, \Lambda, I)}{p(s|\Omega_i, \Lambda, I) p(s|\bar{G}, \Omega_i, \Lambda, I) p(\bar{G}|\Omega_i, \Lambda, I)} f(\text{OUT OF CAT}), \\
&= \frac{1}{p(s|\Omega_i, \Lambda, I)} \left[p(G|\Omega_i, \Lambda, I) f(\text{IN CAT}) + f(\text{OUT OF CAT}) \right].
\end{aligned}
$$

$$
p(G|\Omega_i, \Lambda, I) = \int_0^{z_{\text{cut}}} \int_{M_{\text{min}}(H_0)}^{M(z, m_{\text{th}}(\Omega_i), \Lambda)} p(z, M|\Lambda, I) \, dM dz
$$

Gray et al. 2023

Line of sight redshift prior

Galaxies from a catalog

$$p(z|\Omega_i,\Lambda,s,I)=\frac{1}{p(s|\Omega_i,\Lambda,I)}\Bigg[p(G|\Omega_i,\Lambda,I)\frac{1}{N_{\rm gal}(\Omega_i)}\sum_k^{N_{\rm gal}(\Omega_i)} p(z|\hat{z}_k)p(s|z,M(z,\hat{m}_k,\Lambda),\Lambda,I)$$
$$+\Bigg(\Theta[z_{\rm cut}-z]\int_{M(z,m_{\rm th}(\Omega_i),\Lambda)}^{M_{\rm max}(H_0)} p(z,M|\Lambda,I)p(s|z,M,\Lambda,I)\,dM$$
$$+\Theta[z-z_{\rm cut}]\int_{M_{\rm min}(H_0)}^{M_{\rm max}(H_0)} p(z,M|\Lambda,I)p(s|z,M,\Lambda,I)\,dM\Bigg)\Bigg].$$

Requires prior information about galaxies not in the catalog that could potentially be hosts of GW events

Gray et al. 2023

Separability of LOS redshift prior

$$\Lambda = \{\Lambda_{\rm cosmo}, \Lambda_{\rm mass}, \Lambda_{\rm rate}\}$$

Separate parameters: cosmological, or correspond to normalization or shape of mass distribution

$$\begin{aligned} P(s|z, M, \Lambda, I) &\equiv P(s|z, M, \Lambda_{\rm rate}, I) \\ &\equiv P(s|z, \Lambda_{\rm rate}, I) P(s|M, I) \end{aligned}$$

$$p(s|M, I) \propto \begin{cases} L(M) & \text{if using luminosity weighting} \\ \text{const} & \text{if assuming uniform weighting.} \end{cases}$$

Gray et al. 2023

LOS redshift prior (precomputed)

$$
\begin{aligned}
p(z|\Omega_i, \Lambda, s, I) \\
= \frac{p(s|z, \Lambda_{\rm rate}, I)}{p(s|\Omega_i, \Lambda, I)} \Bigg[p(G|\Omega_i, \Lambda, I) \frac{1}{N_{\rm gal}(\Omega_i)} \sum_k^{N_{\rm gal}(\Omega_i)} p(z|\hat{z}_k) p(s|M(z, \hat{m}_k, \Lambda_{\rm cosmo}), I) \\
+ p(z|\Lambda_{\rm cosmo}, I) \Bigg(\Theta[z_{\rm cut} - z] \int_{M(z, m_{\rm th}(\Omega_i), \Lambda_{\rm cosmo})}^{M_{\rm max}(H_0)} p(M|z, \Lambda_{\rm cosmo}, I) p(s|M, I)\, dM \\
+ \Theta[z - z_{\rm cut}] \int_{M_{\rm min}(H_0)}^{M_{\rm max}(H_0)} p(M|z, \Lambda_{\rm cosmo}, I) p(s|M, I)\, dM \Bigg) \Bigg].
\end{aligned}
$$

This LOS prior can be pre-computed to save runtime. This is the approach used in GWcosmo.

Gray et al. 2023

Calculation of selection effects

$$\frac{N_{\rm exp}}{N}(\Lambda) = \iiint p(D_{\rm GW}|m_1^s, m_2^s, z, \Lambda, I)p(m_1^s, m_2^s, z|\Lambda, I){\rm d}m_1^s{\rm d}m_2^s{\rm d}z,$$
$$\approx \frac{1}{N_{\rm sim}} \sum_{i=1}^{N_{\rm det}} \frac{p(m_{1,i}^s, m_{2,i}^s, z_i|\Lambda, I)}{\pi_{\rm inj}^s(m_{1,i}^s, m_{2,i}^s, z_i|\Lambda, I)},$$

Inject large population of GW event with a wide coverage in parameter space, these injections allow one to compute the probability of detection of events with a given set of parameters

Gray et al. 2023

Parameter inference: priors

POWER LAW

Parameter	Description	Prior
$M_{\rm min}^{\rm BH}$	Minimum mass of the PL comp… hole mass distribution.	
$M_{\rm max}^{\rm BH}$	Maximum mass of the PL com… hole mass distribution.	
α	Spectral index for the PL of th… distribution.	
μ_g	Mean of the Gaussian compone… mass distribution.	
σ_g	Width of the Gaussian component in the primary mass distribution.	$\mathcal{U}(0.4\mathrm{M}_\odot, 10.0\mathrm{M}_\odot)$
λ_g	Fraction of the model in the Gaussian compone…	
δ_m	Range of mass tapering on the lower end of the… mass distribution.	
β	Spectral index for the PL of the secondary mas… distribution.	
$M_{\rm min}^{\rm NS}$	Minimum mass of the uniform neutron star mass distribution.	$\mathcal{U}(1.0\mathrm{M}_\odot, 1.5\mathrm{M}_\odot)$
$M_{\rm max}^{\rm NS}$	Maximum mass of the uniform neutron star mass distribution.	$\mathcal{U}(2.0\mathrm{M}_\odot, 5.0\mathrm{M}_\odot)$

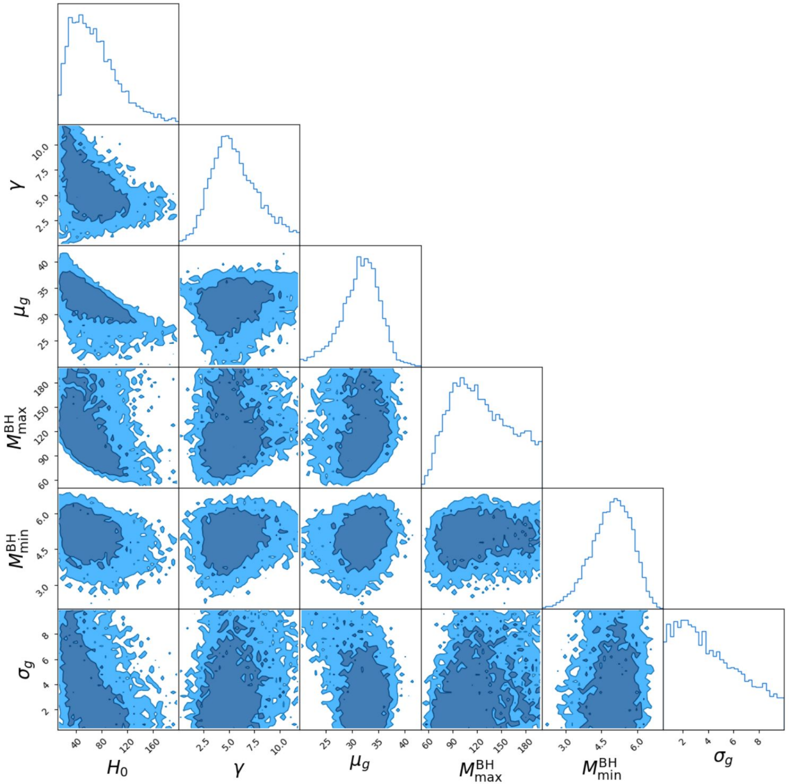
MERGER RATE SHAPE PARAMETERS

Parameter	Description	Prior
R_0	Local merger rate.	$1/R_0$ (implicit)
γ	Power-law index describing the merger rate at low redshift.	$\mathcal{U}(0, 12.0)$
κ	Power-law index describing the merger rate at high redshift.	$\mathcal{U}(0, 6.0)$
z_p	The redshift where the slope of the merger rate changes.	$\mathcal{U}(0, 4.0)$

$$R(z) = R_0(1+z)^\gamma \frac{1+(1+z_p)^{-(\gamma+\kappa)}}{1+\left(\frac{1+z}{1+z_p}\right)^{\gamma+\kappa}}.$$

Gray et al. 2023

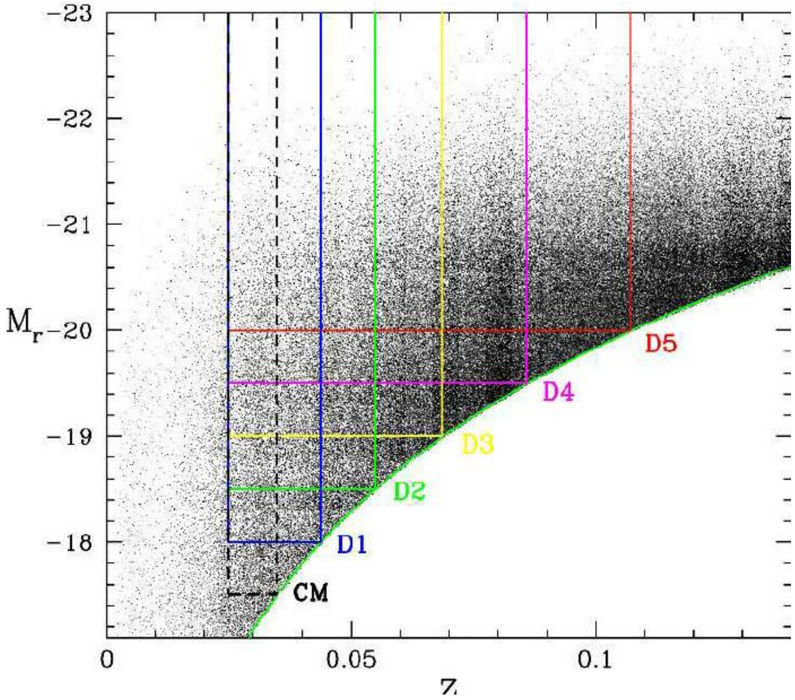
Posterior of parameters of interest



- Note degeneracy between the Hubble constant and features in mass distribution space
  - the mean of the Gaussian
  - Maximum mass of black hole
- Earlier analysis used fixed mass model parameters, so they can yield better Hubble constant measurements, although the posterior will be prior dominated.

Gray et al. 2023

Galaxy surveys: flux limit

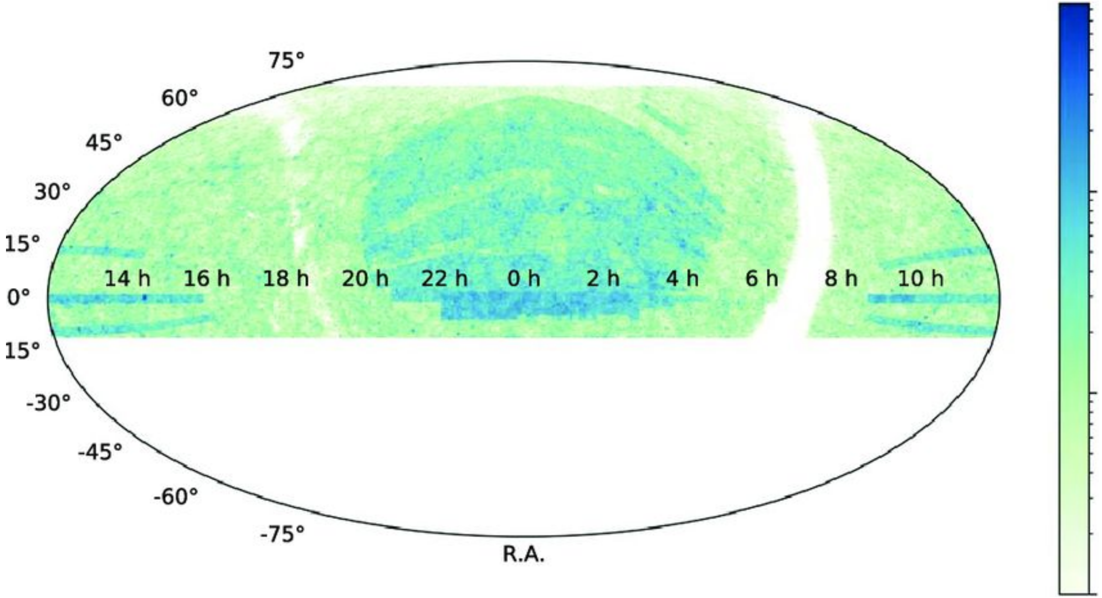


SDSS: Spectroscopic galaxies

- Large imaging or spectroscopic Galaxy surveys have flux limits
- Brighter galaxies can be seen out to higher redshifts
- Survey inhomogeneities can be severe
- The host galaxy of the GW event may actually not be inside of the catalog.
- In this case, the statistical host identification method can run in to problems if it does not account for such missing galaxies.

Choi et al. 2006, Zehavi et al. 2012

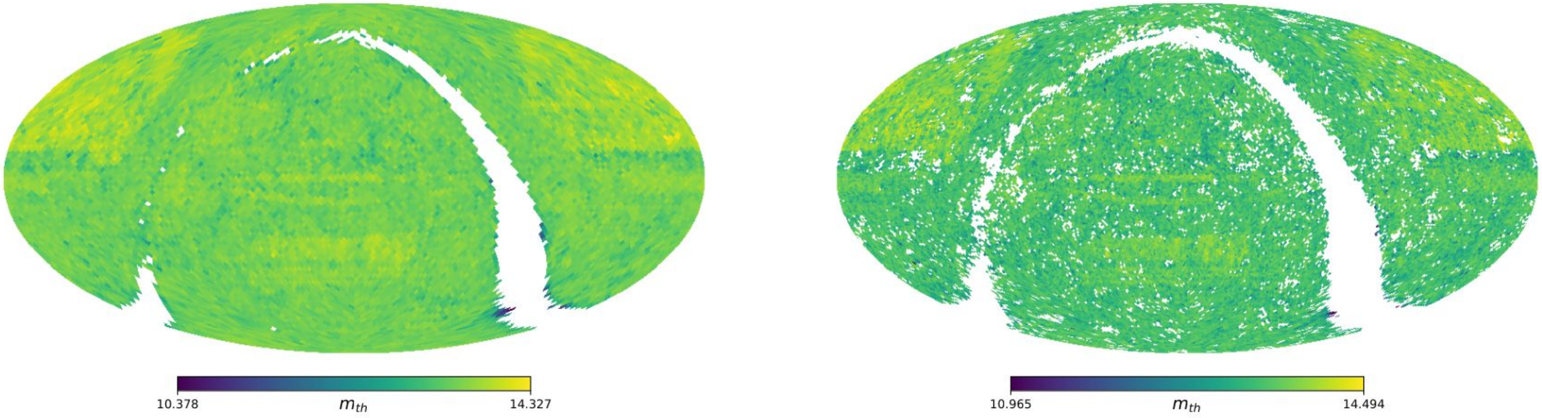
Galaxy surveys: flux limit



- Surveys are also non-homogeneous, can contain holes, or galaxies could be compiled from heterogeneous surveys with varying magnitude thresholds.

Bustamente et al. 2023, Dalya et al. 2018

Galaxy catalogs



(a) m_{th} map for $n_{\rm map} = 32$. The sky is divided into 12,288 pixels, each covering an area of 3.36 deg^2.

(b) m_{th} map for $n_{\rm map} = 64$. The sky is divided into 49,152 pixels, each covering an area of 0.84 deg^2.

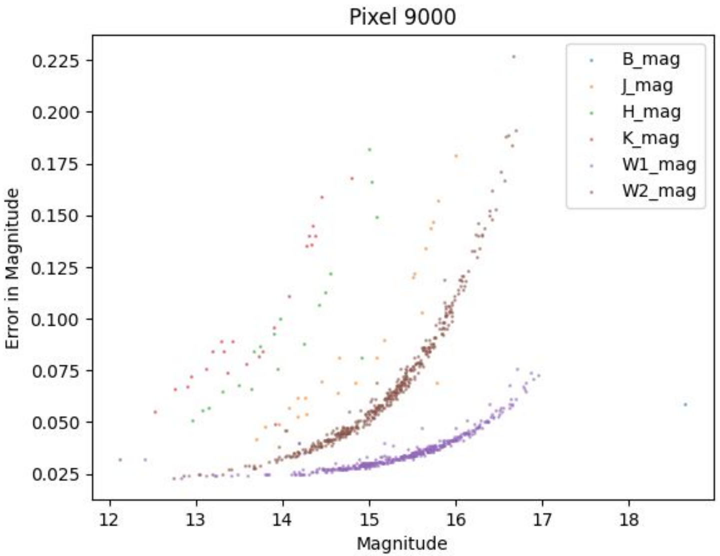
Shows the magnitude threshold in various regions of sky at different resolutions

Dalya et al. 2021

Computing magnitude threshold

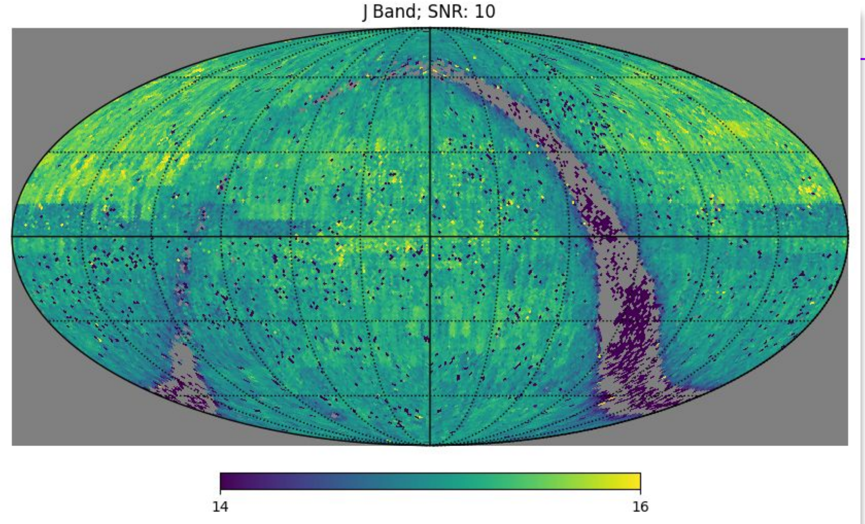
$$m = -2.5 \log F + m_0$$

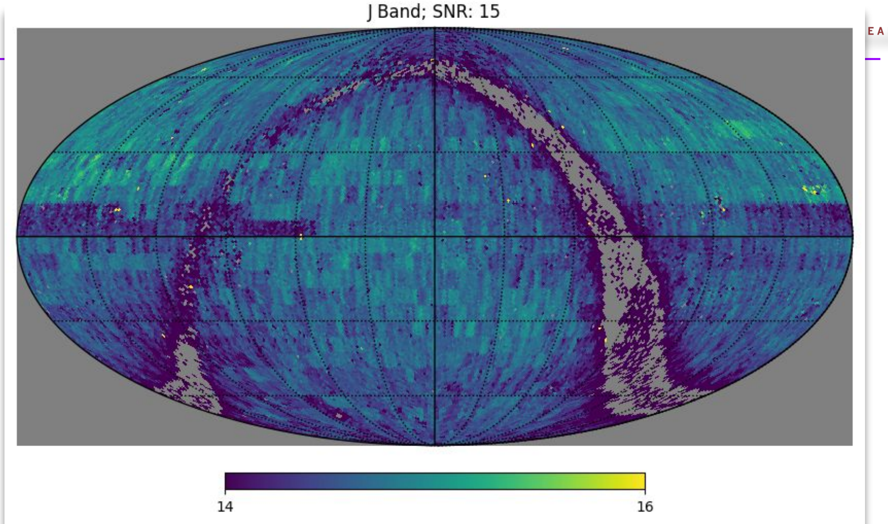
$$\sigma_m = -\frac{2.5}{\ln(10)} \frac{dF}{F}$$



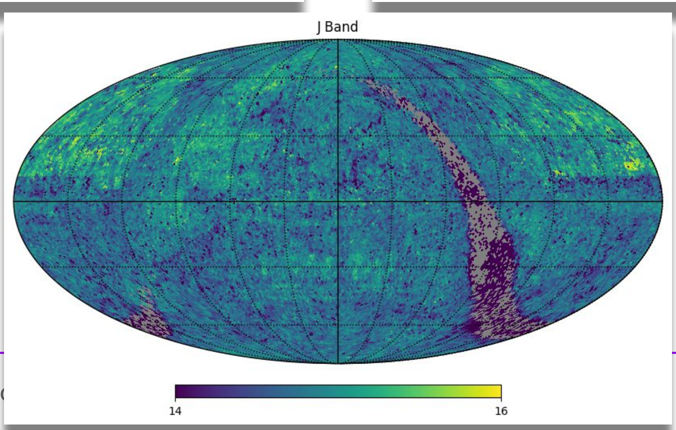
- However currently adopted: use the median of galaxy magnitude in any given pixel to be the threshold.
  - Phenomenological
  - Is not necessarily well motivated

Das, SM, et al. (in prep)





SNR: 10
Galaxies removed: 26.56%



SNR: 15
Galaxies removed: 60.52%

Median values
Galaxies removed:50%

Das, SM, et al. (in prep)

In catalog term

$$
\begin{aligned}
p(z, M, m|G, \Omega_i, \Lambda, s, I) \\
&= \frac{p(z, M, m|G, \Omega_i, \Lambda, I)p(s|z, M, m, G, \Omega_i, \Lambda, I)}{p(s|G, \Omega_i, \Lambda, I)}, \\
&= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} p(M|z, m, G, \Omega_i, \Lambda, I)p(z, m|G, \Omega_i, \Lambda, I)p(s|z, M, \Lambda, I), \\
&= \frac{1}{p(s|G, \Omega_i, \Lambda, I)} \delta(M - M(z, m, \Lambda))p(z, m|G, \Omega_i, I)p(s|z, M, \Lambda, I).
\end{aligned}
$$

Conditional probability

$P(A, B) = P(A|B)P(B)$

Connecting apparent and absolute magnitude via cosmological parameters

Gray et al. 2023

LOS redshift prior (precomputed)

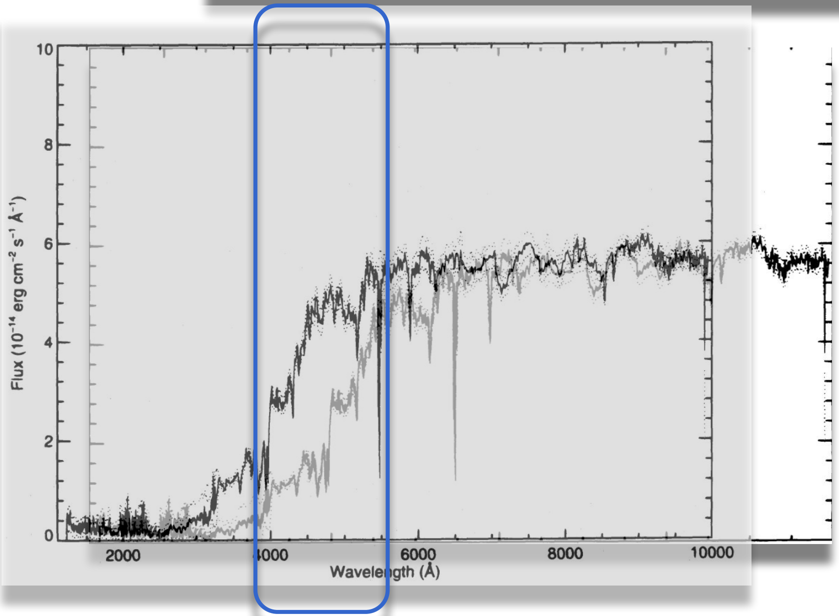
$$p(z|\Omega_i, \Lambda, s, I)$$
$$= \frac{p(s|z, \Lambda_{\rm rate}, I)}{p(s|\Omega_i, \Lambda, I)} \Bigg[p(G|\Omega_i, \Lambda, I) \frac{1}{N_{\rm gal}(\Omega_i)} \sum_k^{N_{\rm gal}(\Omega_i)} p(z|\hat{z}_k) p(s|M(z, \hat{m}_k, \Lambda_{\rm cosmo}), I)$$
$$+ p(z|\Lambda_{\rm cosmo}, I) \Bigg(\Theta[z_{\rm cut} - z] \int_{M(z, m_{\rm th}(\Omega_i), \Lambda_{\rm cosmo})}^{M_{\rm max}(H_0)} p(M|z, \Lambda_{\rm cosmo}, I) p(s|M, I)\, dM$$
$$+ \Theta[z - z_{\rm cut}] \int_{M_{\rm min}(H_0)}^{M_{\rm max}(H_0)} p(M|z, \Lambda_{\rm cosmo}, I) p(s|M, I)\, dM \Bigg) \Bigg].$$

This LOS prior can be pre-computed to save runtime. This is the approach used in GWcosmo.

Gray et al. 2023

Magnitudes of galaxies: k-corrections

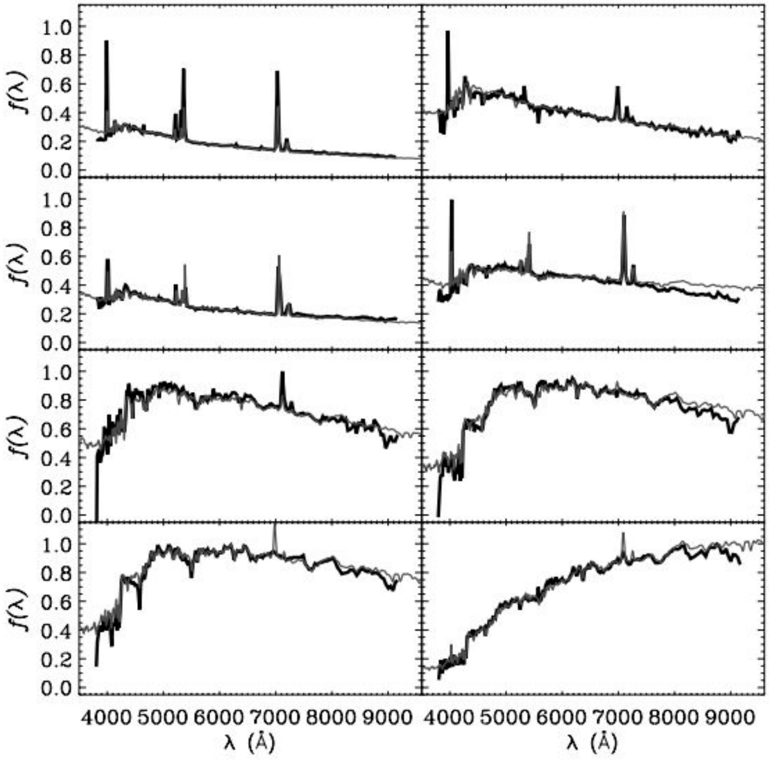
$$M = m - 5\log D_{\mathrm{L}} - 25 - K_{\mathrm{corr}}$$



- Exact same rest frame galaxy spectrum will give a different value for the magnitude in the observed band pass, when galaxies are at different redshifts.
- K-corrections allow you to put galaxies on the same footing by eliminating this effect.

Blanton et al. 2007

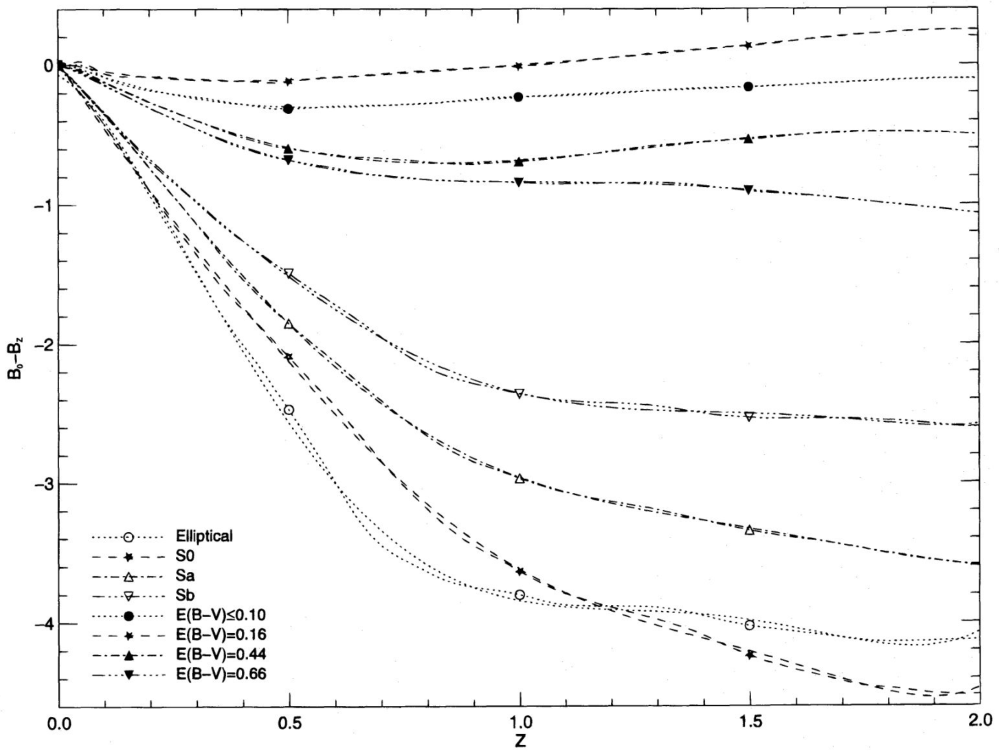
Calculating k-corrections



- Library of PCA templates in order to capture the diversity of spectra in SDSS
- Given any broadband photometry information in a series of bands, fit the photometry as a sum of these spectra weighted by non-negative coefficients
- Reconstruct the spectrum of the galaxy which allows the calculation of the k-correction

Blanton et al. 2007

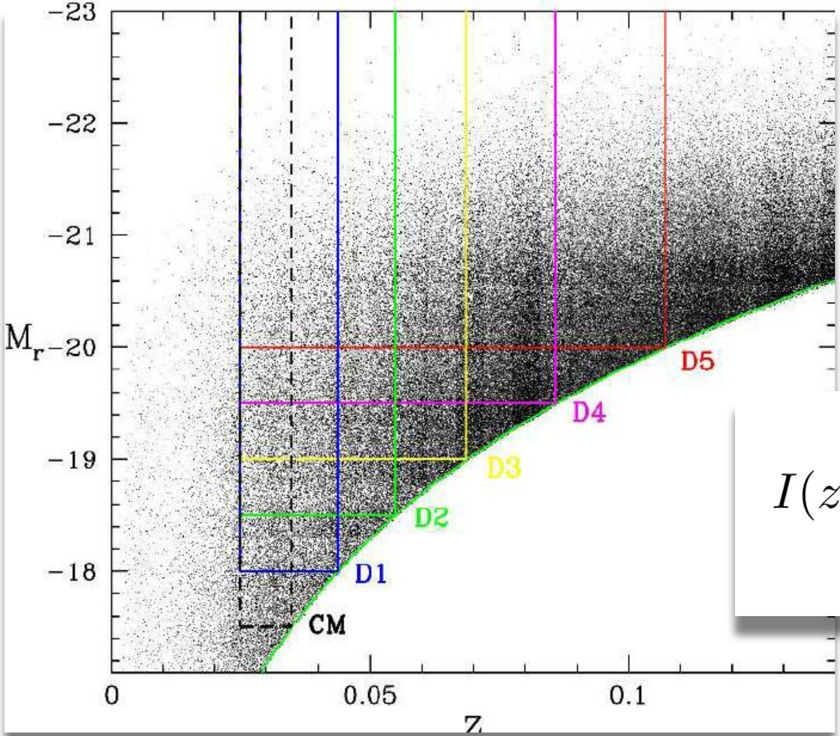
K-corrections



- K-corrections depend upon the type of the galaxy and the galaxy redshift
- Often times redshifts are estimated using photometric redshift, so one has to worry if the K-corrections are reliable or not
- There are publicly available codes which compute these corrections as a function of redshift and color of galaxies

Blanton et al. 2007

Missing galaxy estimates: Luminosity function

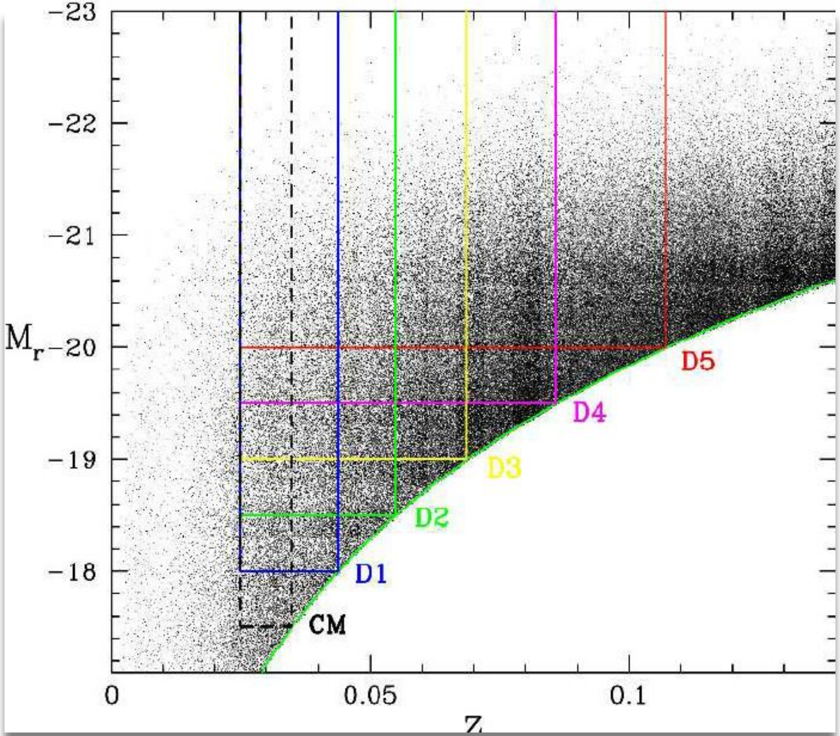


- The number of missing galaxies at any given redshift can be found by integrating the expectation for these number of galaxies below the flux limit of the survey.

$$I(z) = \left[\int_{L_{\min}}^{L_{\rm thr}} \Phi(L,z)dL\right]\left[\int_{L_{\min}}^{L_{\max}} \Phi(L,z)dL\right]^{-1}$$

Choi et al. 2006

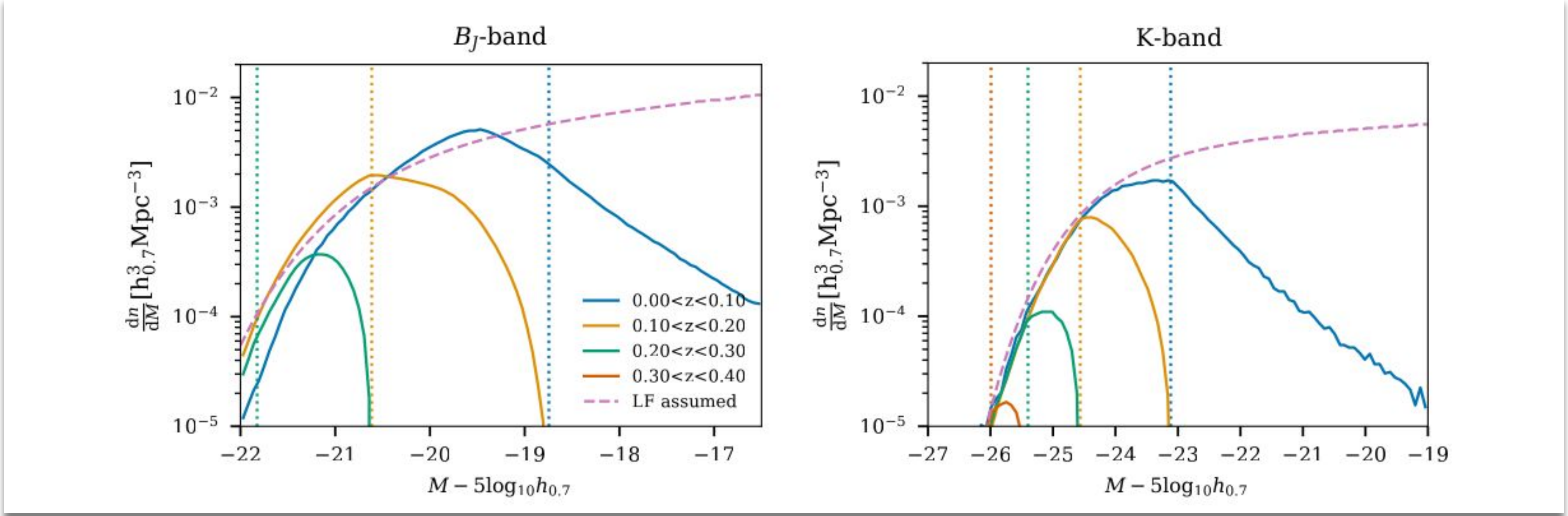
Inferring luminosity function



- Fainter galaxies can be seen only to small redshifts, while brighter galaxies can be seen further out.
- Account for the difference by counting the number of galaxies in a given luminosity bin, using the maximum volume out to which the galaxy would have been detected.
- If the luminosity function does not evolve substantially this method can establish a good baseline for the missing galaxies.

Choi et al. 2006

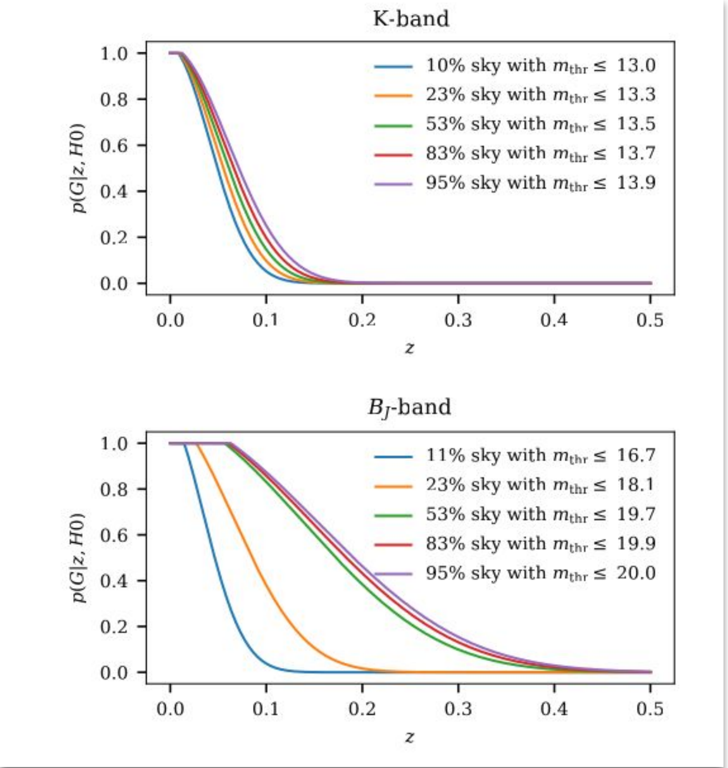
Luminosity function of galaxies



Schechter function fit: Power law at the faint end, and exponential drop at the bright end

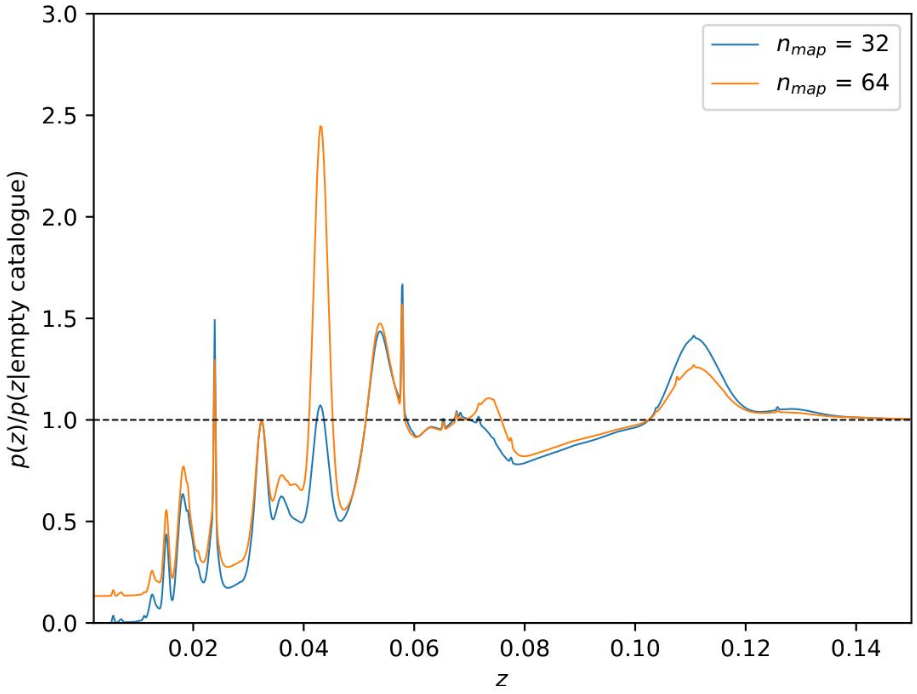
LVK collaboration 2023

Incompleteness of galaxy catalog



- K-band luminosity function related to the stellar mass, but heavily incomplete at higher redshifts.
- B band has more galaxies spanning out to larger redshifts but uncertainties in the LF determinations
- The GW event probability could depend upon either, or some completely different combination.

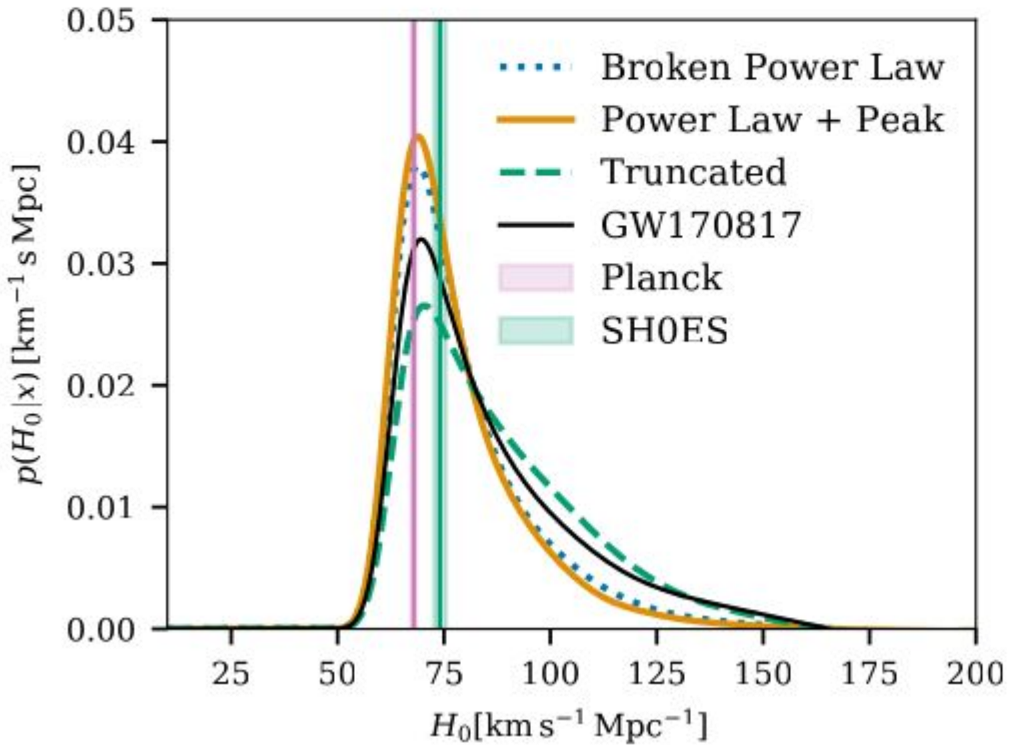
Line of sight prior



- There is some dependence on what resolution the line of sight prior is computed using
- Results should therefore be verified for convergence by changing the resolution.

Gray et al. 2023

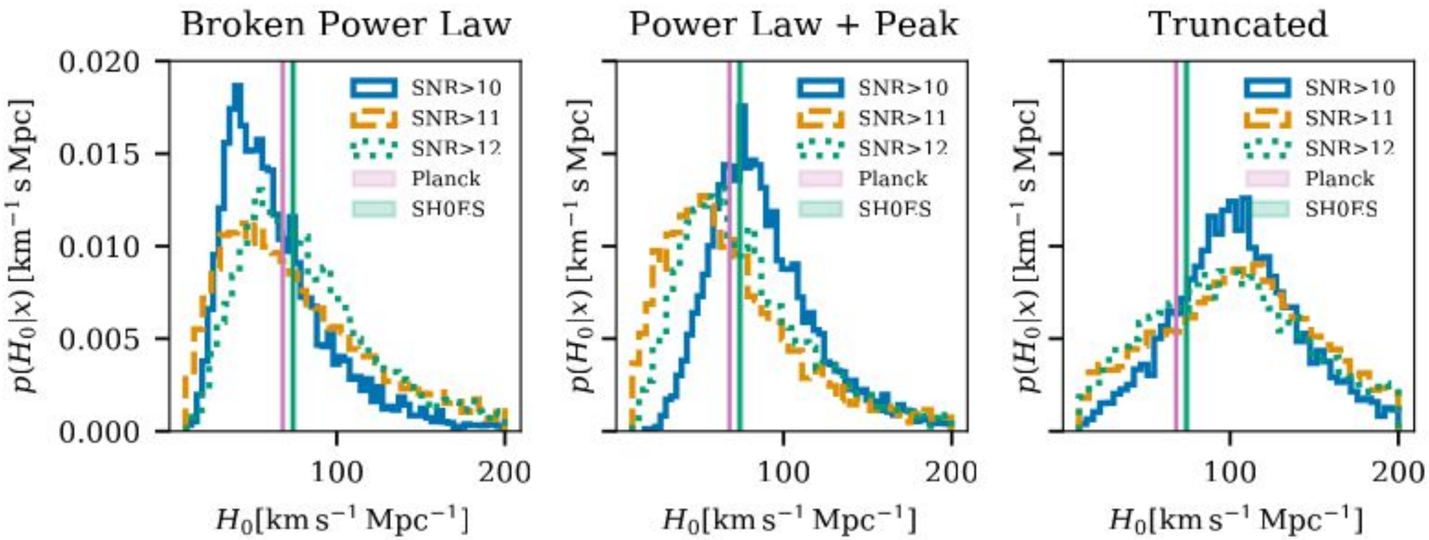
Sensitivity to mass model



- Mass model parameterization uncertainties can bias the posterior distribution of the expansion rate.
- Models can be ruled out based on Bayesian evidence ratios in favor of others.

LVK collaboration 2023

Event selection thresholds

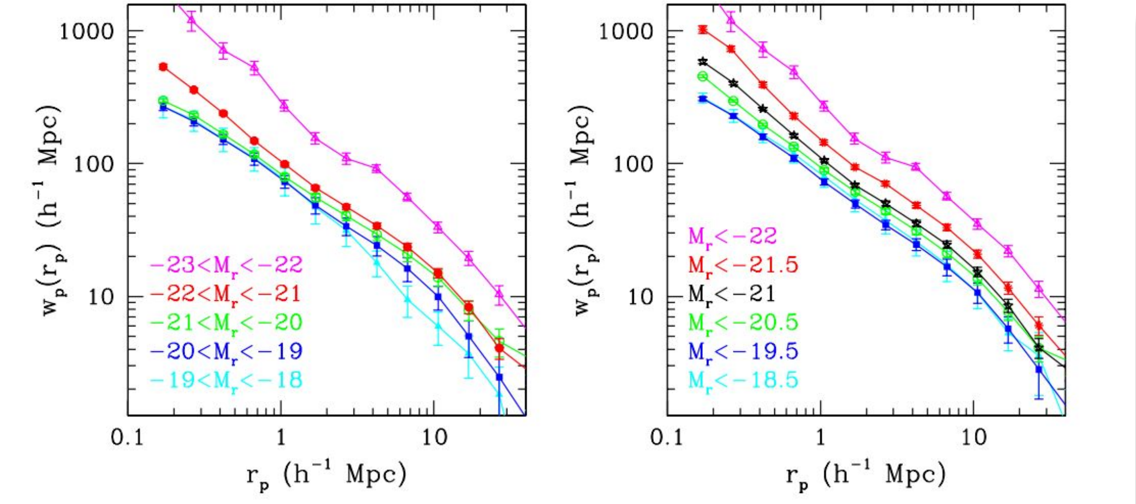


Threshold selection can also impact the posteriors, important to perform blind analysis

Issues with current methods

- Spectral sirens:
  - Is the mass distribution of BBH events expected to be invariant with redshift?
  - Could it carry over when the Hubble constant can be measured to percentage precision?
- Galaxy catalogs:
  - Currently impact is dependent upon galaxy catalog incompleteness
  - Obtaining spectroscopic redshifts for galaxies within localization regions is difficult
  - Do GW events really follow the luminosity of galaxies, or star formation rate, or something entirely different?
- Schutz et al. 1986 seemed to talk about clustering of galaxies, which is completely ignored in the statistical host identification. Can we do better with more information?

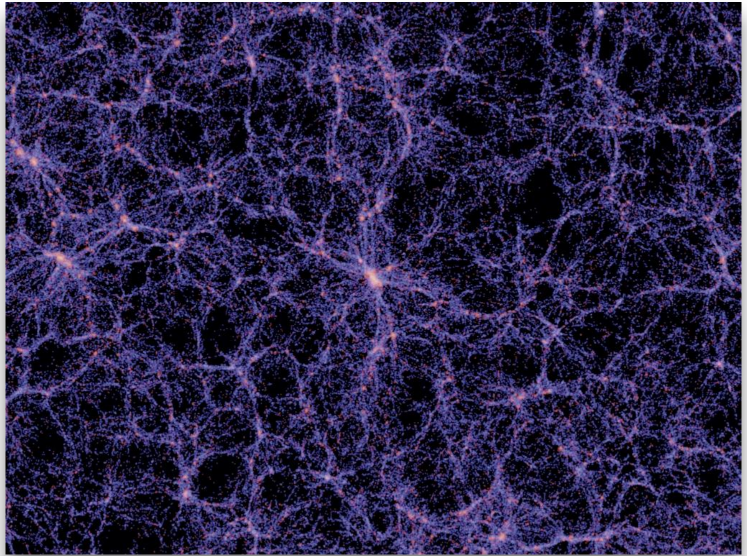
Clustering of galaxies

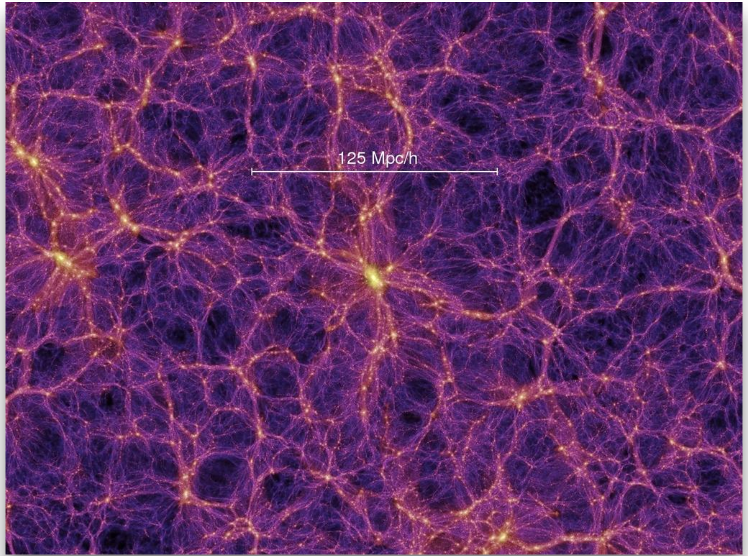


- Galaxy distribution in the Universe is not entirely homogeneous.
- Brighter galaxies tend to cluster strongly compared to fainter galaxies.
- This behaviour is driven by brighter galaxies living in massive dark matter halos which are highly clustered with respect to the matter distribution.

Zehavi et al. 2011

Galaxy dark matter connection





Clustering

$$\delta_g = b\,\delta_m$$

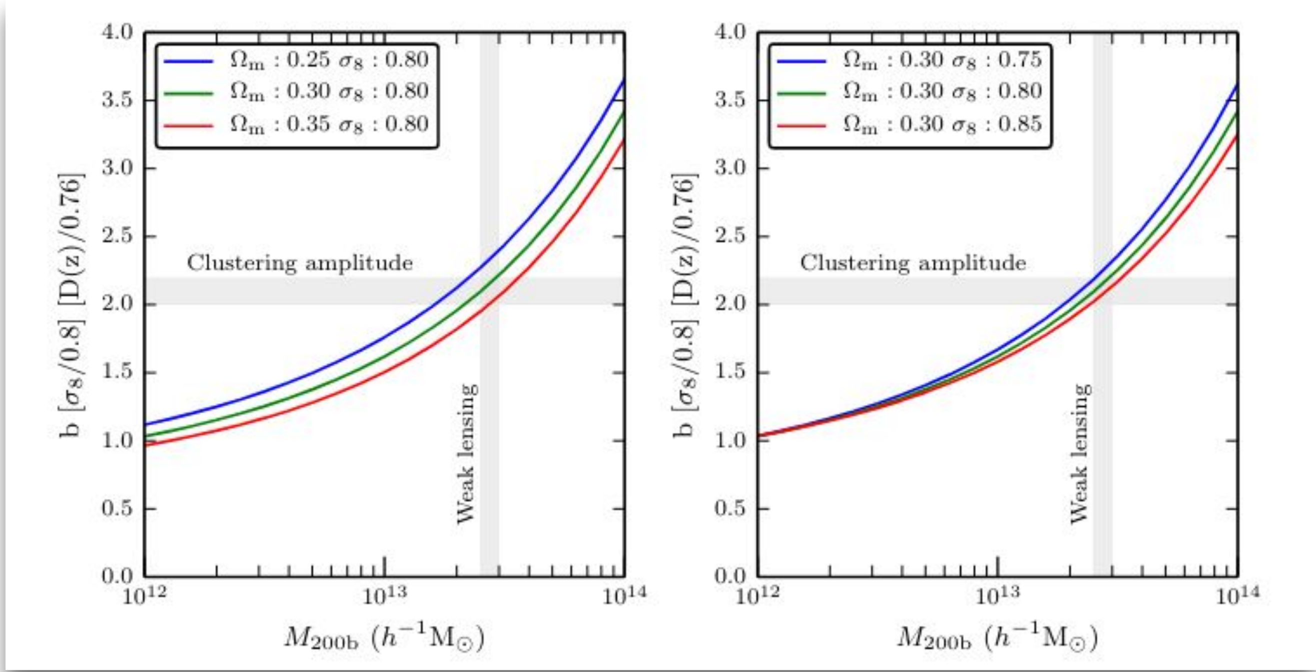
Lensing

$$P_{gg}(k) = b^2\,P_{mm}(k)$$

$$P_{gm}(k) = b\,P_{mm}(k)$$

Millennium simulation (Credit: Volker Springel)

Bias of dark matter halos



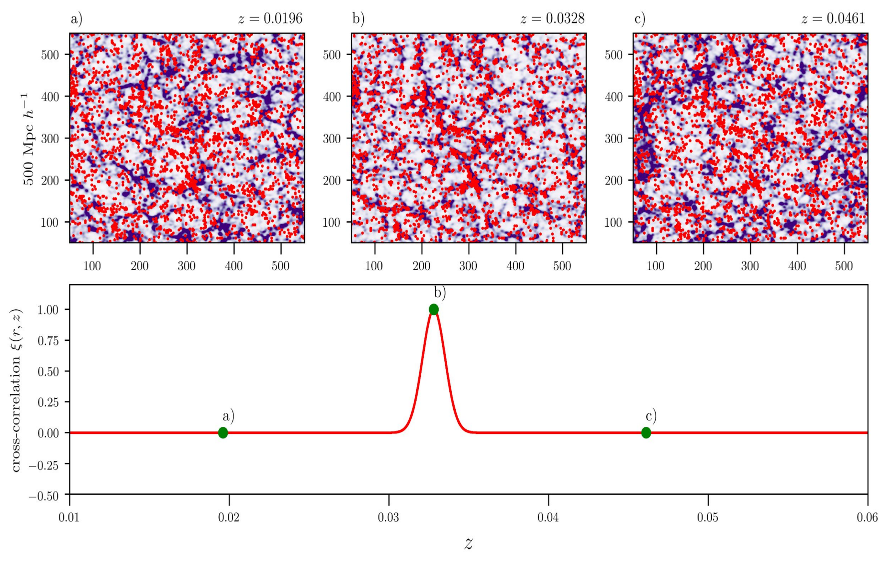
Galaxies borrow the bias of the dark matter halos in which they reside

More et al. 2013

Large scale structure correlations

Red: Fixed BBH events with unknown redshifts

Blue: Galaxy distribution at different redshift slices.



Angular cross-correlation captures the similarity in the large scale structure.

Newman (2008)

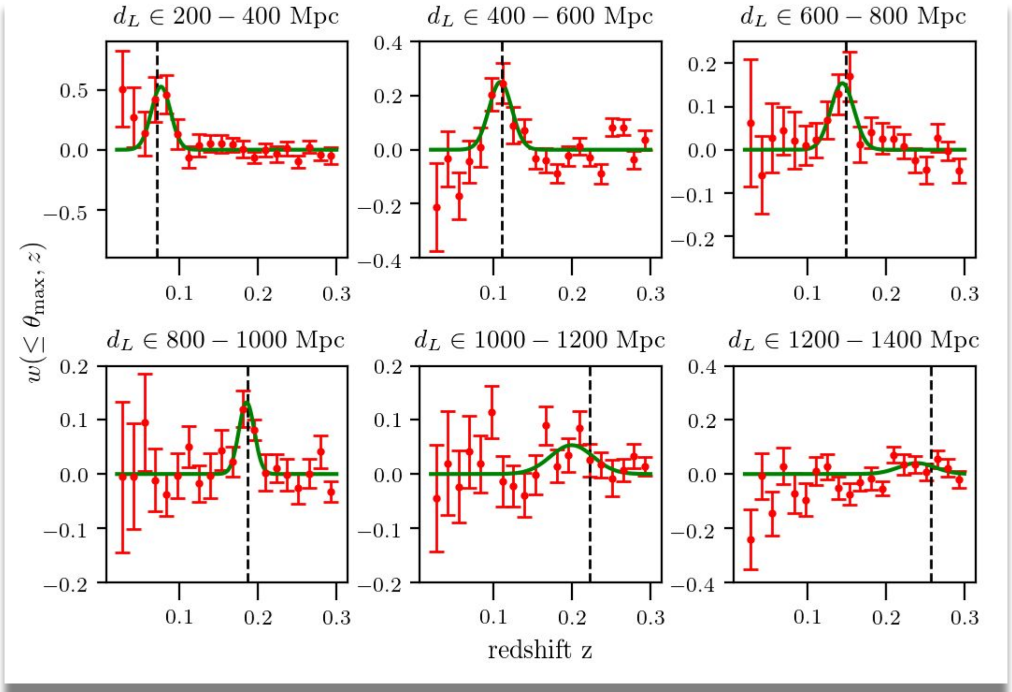
In the context of GWs:

Schutz (1986)
Macleod and Hogan (2007)
Oguri (2016), Nair et al. (2018), Mukherjee et al. (2018), Bera et al. 2020

- Merging BBHs share the same large scale structure as the galaxies.
- Possible to statistically obtain the redshift of the merging BBHs.

Bera, Rana, SM ,Bose 2020

Angular cross-correlations for redshifts



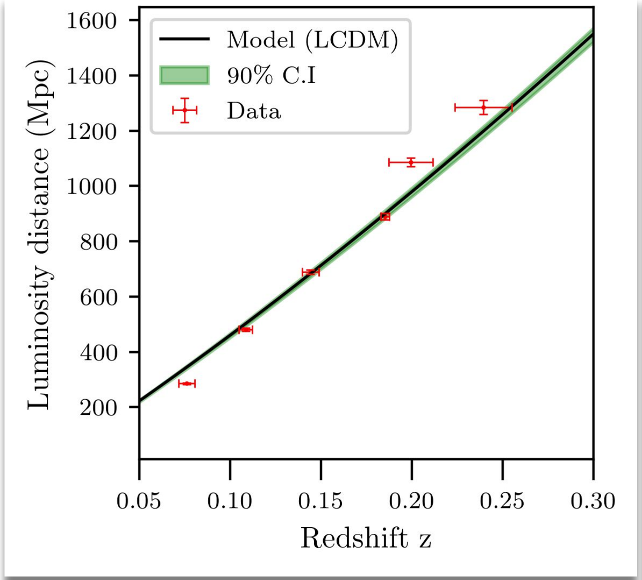
- ~5000 BBH mergers, divided into 6 bins in *inferred* luminosity distance
- Mock galaxies divided in 20 bins according to their redshift
- Each panel shows the angular cross-correlation function between GW sources in a d_L bin cross-correlated with mock galaxies

- **Angular cross-correlation function peaks at the correct redshift of the sources**

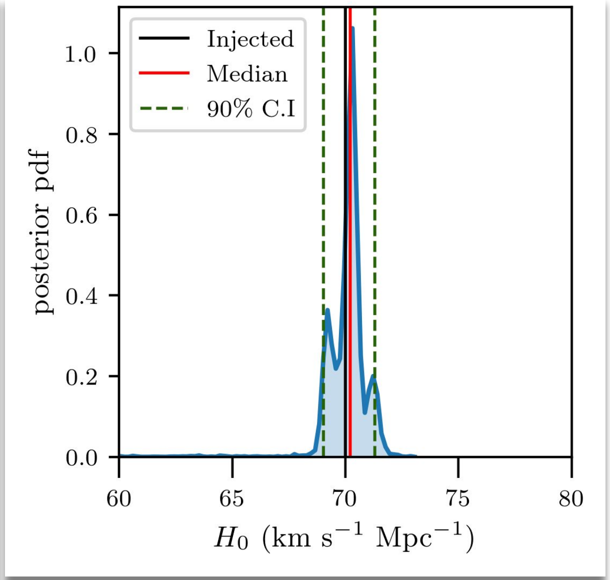
Dashed lines: true average redshift of the sources in each d_L bin

Hubble diagram

Red points: d_L inferred from BBH merger waveforms, redshift from cross-correlations



High precision with ~5000 sources

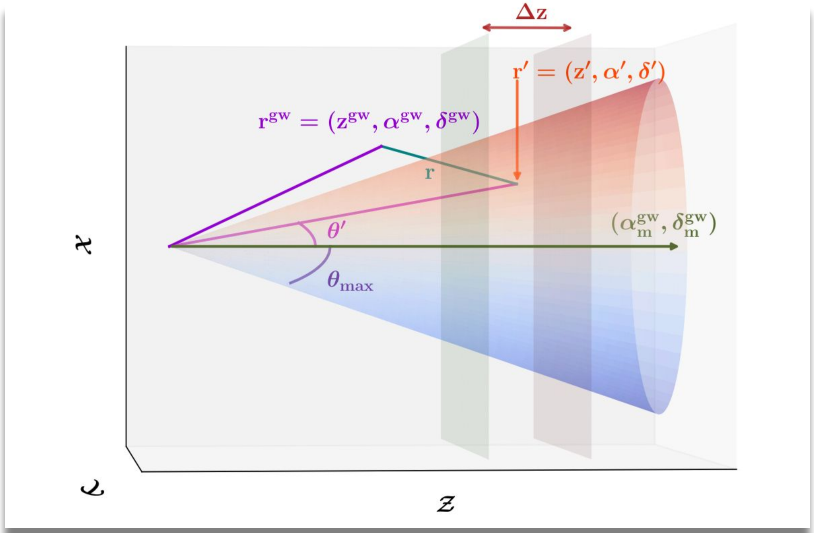


Black solid line: the true value of H_0 in the simulation

Dashed lines: 90 percent credible interval $P(H_0|D)$

- **Even with Incomplete galaxy catalogs, the Hubble constant can be accurately recovered.**

Cross-correlation: individual events

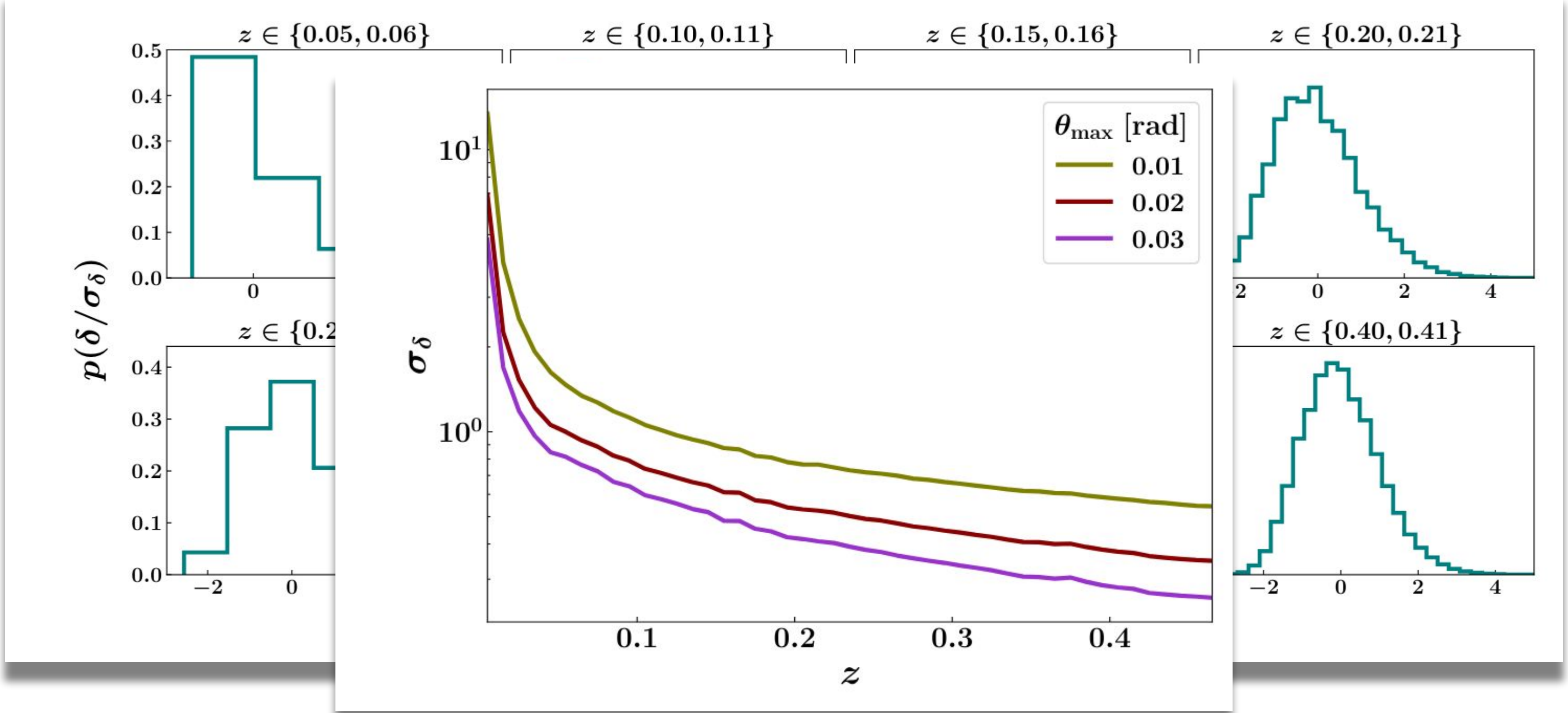


- Measure overdensity of galaxies along the line of sight towards a GW event in different bins of redshift
- The overdensity should account for the occasional chance alignments.
- But it will also capture the
lustered with
laxies if any.

$$\delta_g^{\mathrm{mod}}(z \mid \mathbf{r}^{\mathrm{gw}}) = \frac{\int_{z-\Delta z/2}^{z+\Delta z/2} \int_0^{\theta_{\max}} \int_0^{2\pi} \bar{n}_V(z')\xi_{\mathrm{gw,g}}(|\mathbf{r}(\mathbf{r}^{\mathrm{gw}}, \mathbf{r}')|) \sin\theta' d_c'^{\,2} J\left(\frac{d_c'}{z'}\right) d\alpha' d\theta' dz'}{\int_{z-\Delta z/2}^{z+\Delta z/2} \int_0^{\theta_{\max}} \int_0^{2\pi} \bar{n}_V(z') \sin\theta' d_c'^{\,2} J\left(\frac{d_c'}{z'}\right) d\alpha' d\theta' dz'}$$

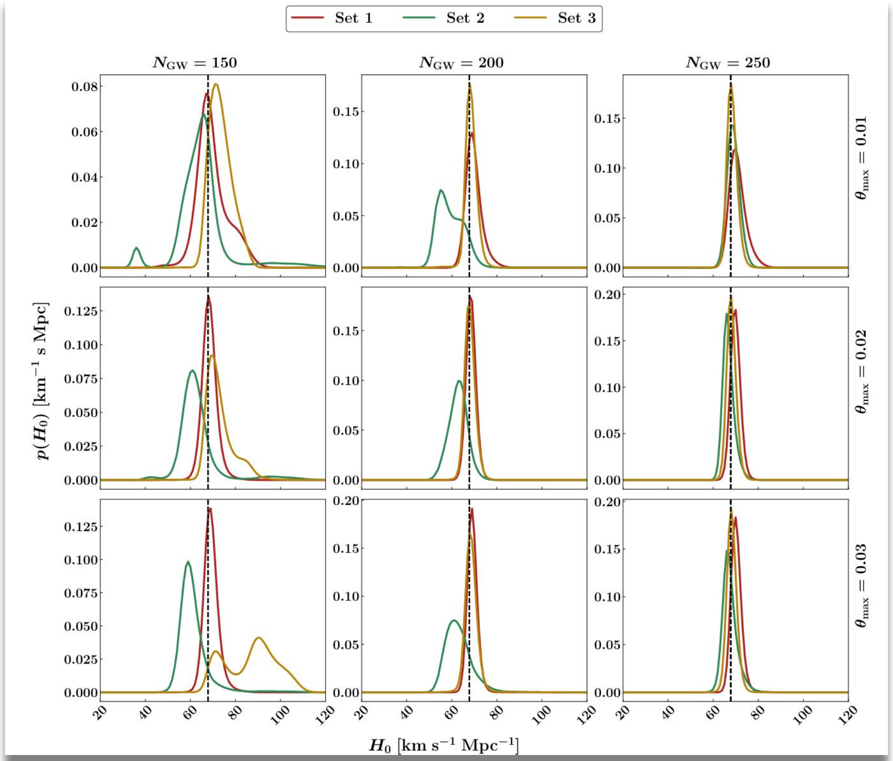
Ghosh, SM et al. 2025

Random lines of sight overdensity



Ghosh, SM et al. 2025

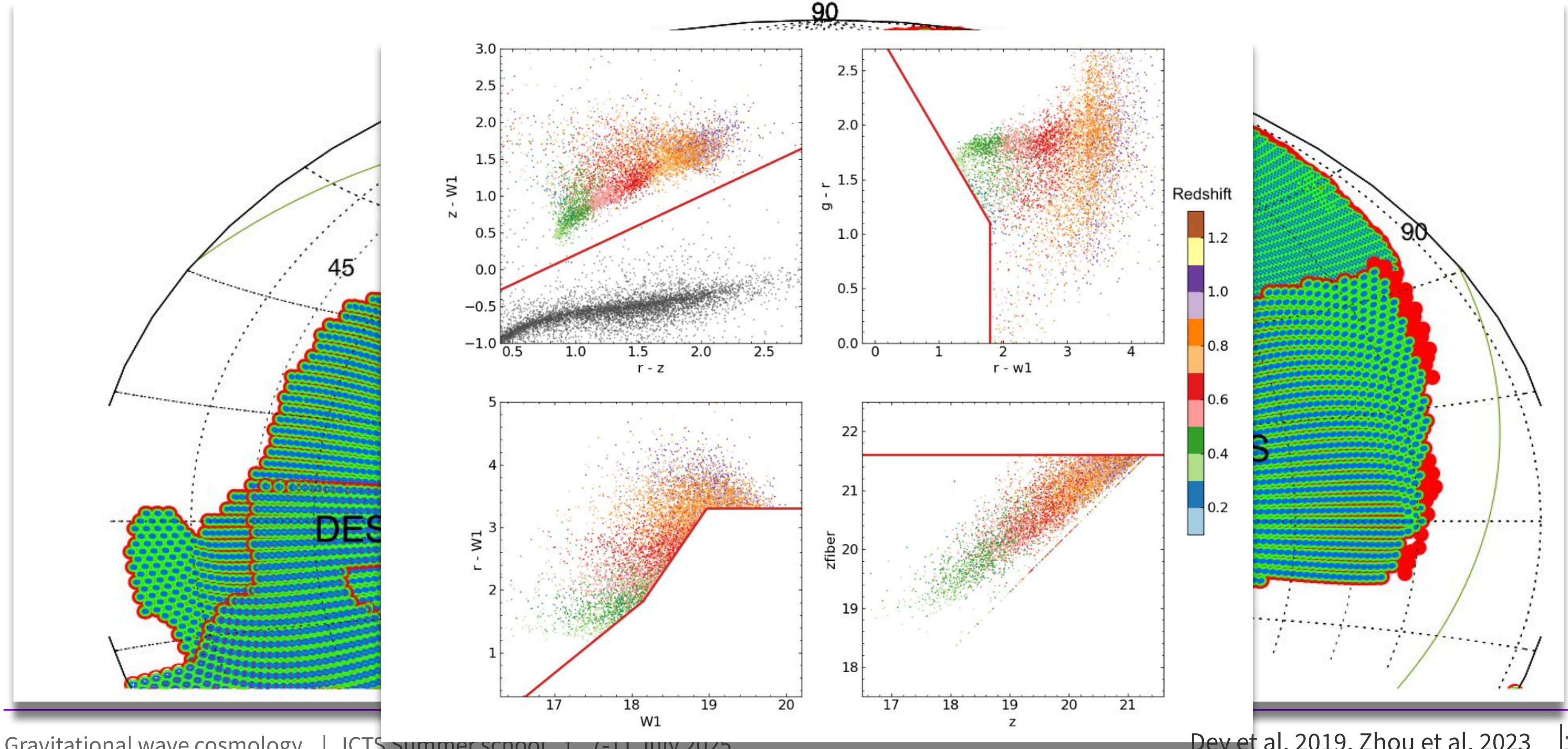
Hubble constant constraints



- Proof of concept that individual cross-correlations can also yield measurements of expansion history.
- Currently run only on simulations.
- Yet to be demonstrated on real GW events and real galaxy catalogs
- Simulated galaxy catalogs with flux limit currently under investigation.

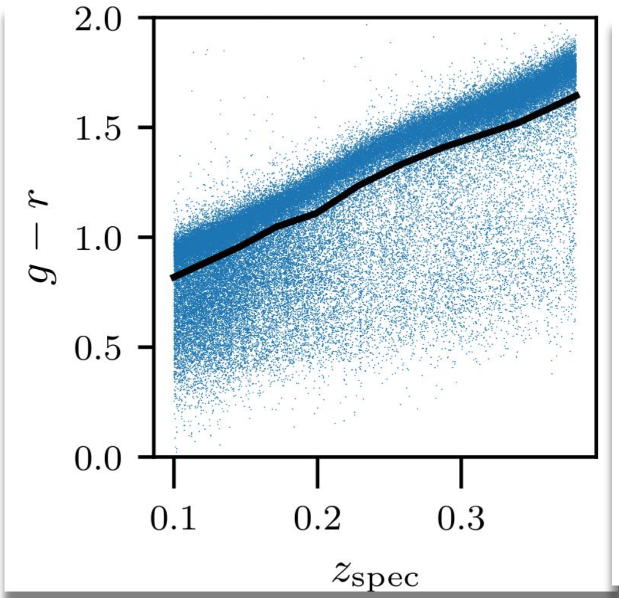
Ghosh, SM et al. 2025

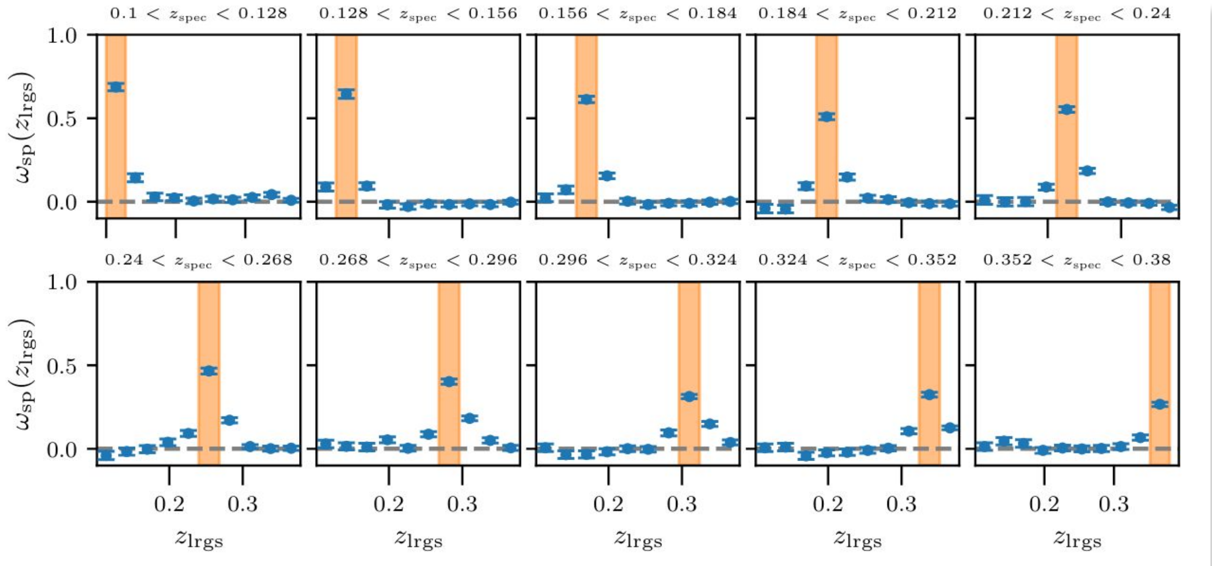
DESI Imaging legacy survey



Dey et al. 2019, Zhou et al. 2023

Galaxy catalogs for cross-correlation

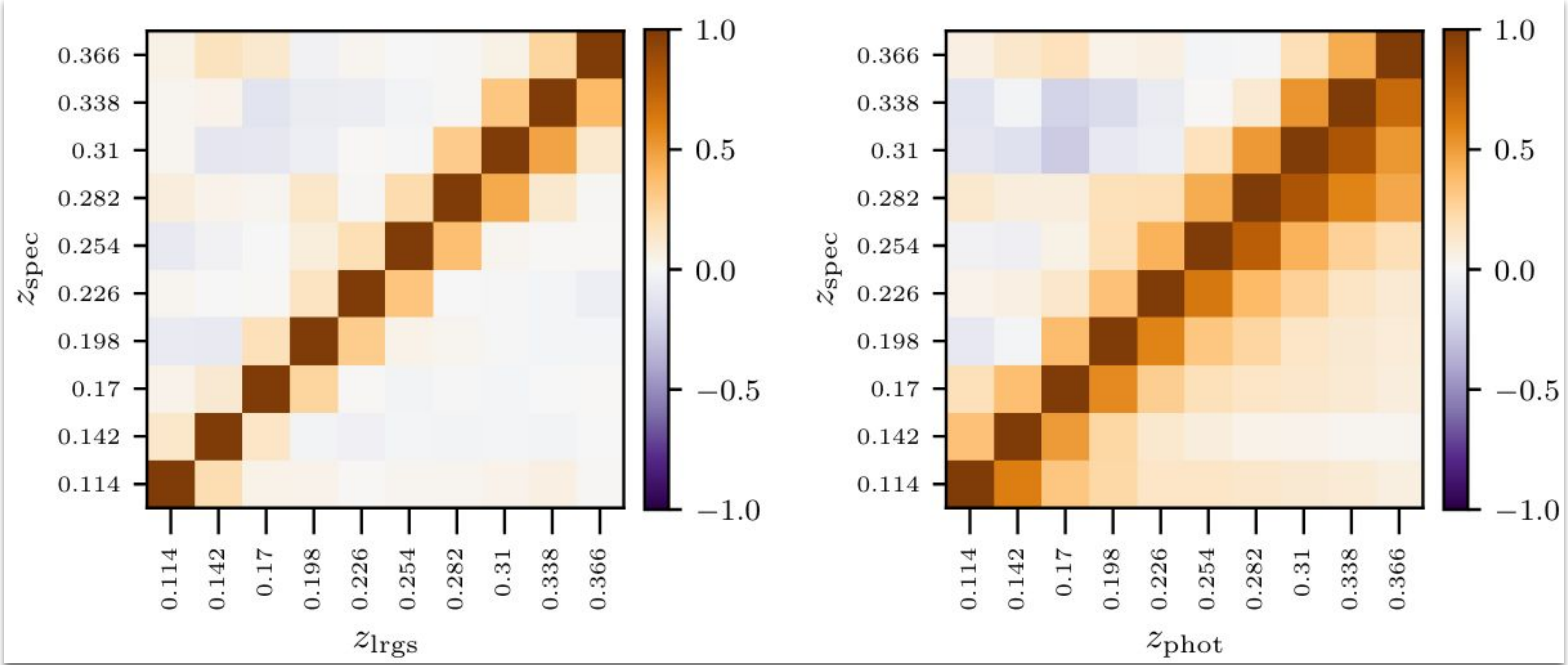




Select galaxies based on their color to select luminous red galaxies

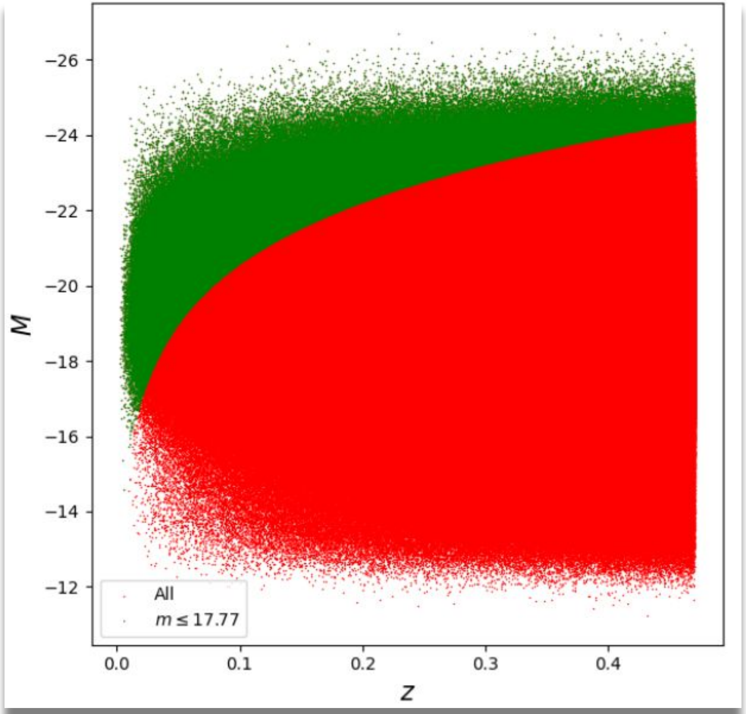
Rana, SM (in prep)

Improvement in redshift estimates



Rana, SM (in prep)

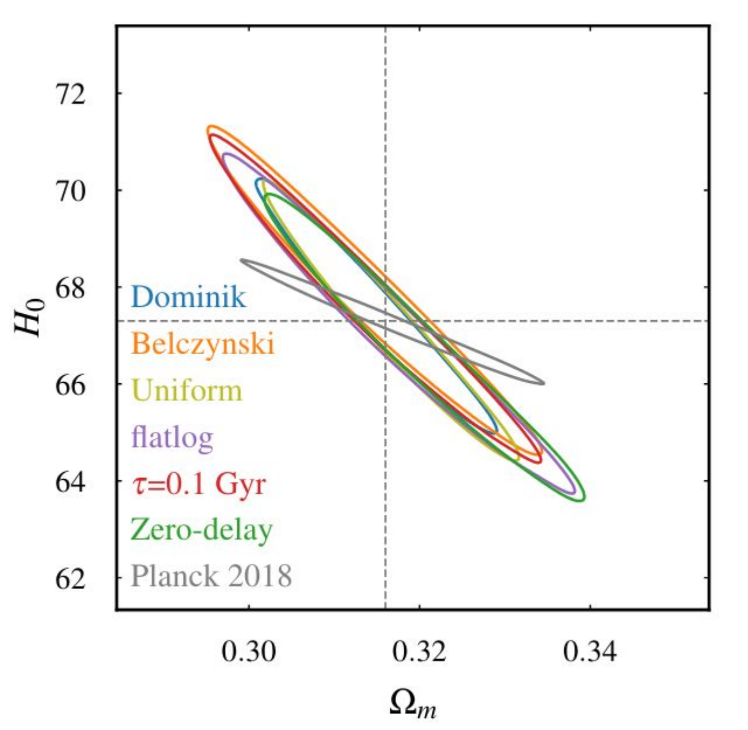
Filling in missing galaxies



- The missing galaxies at the faint end could be filled in by using the large scale structure traced by the brighter galaxies.
- Measure the overdensity of galaxies on a grid, and assign missing galaxies based on this overdensity and the ratio of the biases of the two populations.
- Can reproduce the clustering measurement and hence a low resistance way of incorporating the clustering information.

Gravitational lensing time delays

$$p\left(\vec{\Omega} \mid N, \{\Delta t_i\}, \ldots \quad \ldots t_i\} \mid \vec{\Omega}, T_{\rm obs}, \mathcal{M}\right) \quad \ldots T_{\rm obs}, \mathcal{M})$$



- Time of arrival delay b… …tional waves from the same source can also … …ion.
- Gravitational waves a… …ed precisely
- Identifying lensed eve…
- Requires a reliable m…
- Simple models can de…

Jana et al. 2023

Big questions

- How old is the Universe?
- What is it made up of?
- How did the Universe come into existence?
- What is the fate of the Universe?

Now can be addressed scientifically

Flammarion's *L'atmosphère : météorologie populaire* (Paris: Hachette, 1888)