Global solutions with asymptotic self-similar behaviour for the cubic wave equation

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Asymptotic **behaviour**. Basic facts

Cauchy initial value problem for the focusing cubic wave equation

$$\partial_t^2 u - \Delta u = u^3, \qquad t \in I, x \in \mathbb{R}^3,$$
$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 = (u_0, \dot{u}_0) \quad \textit{(initial data)}.$$

Notation: $\boldsymbol{u}(t) = (u(t,\cdot), \partial_t u(t,\cdot))$, $\quad \mathcal{H}^s = \dot{H}^s \times \dot{H}^{s-1}$.

- $I \subseteq \mathbb{R}$ maximal time of existence. ***Blow-up*** occurs $\iff I \subsetneq \mathbb{R}$.
  Example: $u = \sqrt{2}(T \pm t)^{-1}$ for $T \geqslant 0$.
- $E = \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\mathcal{H}^1(\mathbb{R}^3)} - \frac{1}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R}^3)}$ (*conserved energy*).
- L.W.P. for $\boldsymbol{u}_0 \in \mathcal{H}^s$ with $s \geqslant \frac{1}{2}$ ($s = \frac{1}{2}$ is *scaling-critical*).
- Ill-posed for $\boldsymbol{u}_0 \in \mathcal{H}^s$ with $s < \frac{1}{2}$ (Lindblad–Sogge 1995).
- If $\|\boldsymbol{u}_0\|_{\mathcal{H}^{1/2}}$ is small, then $I = \mathbb{R}$ and u is asymptotic to a linear solution at $t \to \pm\infty$ (***scattering***).
  Those $\boldsymbol{u}_0$ that scatter are an open set in $\mathcal{H}^{1/2}$ (*scattering is stable*).
- (Dodson-Lawrie 2015) If $\|\boldsymbol{u}(t)\|_{\mathcal{H}^{1/2}} < C$ for all $t \in I$, then $I = \mathbb{R}$ and u scatters.

Digression: Scattering is stable

Linear inhomog. solution: $\partial_t^2 v - \Delta v = F(t,x)$, $\boldsymbol{v}(0) = \boldsymbol{v}_0$, i.e.

$$\boldsymbol{v}(t,\cdot) = \boldsymbol{S}_t \boldsymbol{v}_0 + \int_0^t \boldsymbol{S}_{t-t'} \begin{bmatrix} 0 \\ F(t',\cdot) \end{bmatrix} dt', \text{ where}$$

$$S_t \boldsymbol{v}_0 = \cos(t\sqrt{-\Delta}) v_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \dot{v}_0.$$

The Strichartz estimates (Strichartz 1977; see Keel–Tao 1997 for this version)

$$\|v\|_{L^4(I\times\mathbb{R}^3)} + \|\boldsymbol{v}\|_{L_t^\infty \mathcal{H}^{1/2}(I\times\mathbb{R}^3)} \lesssim \|\boldsymbol{v}_0\|_{\mathcal{H}^{1/2}} + \|F\|_{L^{4/3}(I\times\mathbb{R}^3)}.$$

Scattering criterion

$\partial_t^2 u - \Delta u = u^3$ scatters at $t = +\infty \iff \|u\|_{L^4([0,\infty)\times\mathbb{R}^3)} < \infty$.

By well-posedness, $\boldsymbol{u}_0 \mapsto \|u\|_{L^4}$ is continuous. So scattering is stable.

Proof of the criterion: $(\Rightarrow)$. u scatters means that u is essentially a linear solution for $t > T \gg 1$. For linear solutions, $\|u\|_{L^4([T,\infty)\times\mathbb{R}^3)} < \infty$.

Digression: Scattering is stable

Recall: $\|v\|_{L^4(I\times\mathbb{R}^3)} + \|\boldsymbol{v}\|_{L^\infty_t\mathcal{H}^{1/2}(I\times\mathbb{R}^3)} \lesssim \|\boldsymbol{v}_0\|_{\mathcal{H}^{1/2}} + \|F\|_{L^{4/3}(I\times\mathbb{R}^3)}$.

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Proof of the criterion: $(\Leftarrow)$. By definition,

$$\boldsymbol{u}(t,\cdot) = \boldsymbol{S}_t\left(\boldsymbol{u}_0 + \int_0^t \boldsymbol{S}_{-t'}\begin{bmatrix}0\\ u^3(t',\cdot)\end{bmatrix} dt'\right).$$

The integral is Cauchy, hence convergent in $\mathcal{H}^{1/2}$, because

$$\begin{aligned}\left\|\int_{t_0}^t \boldsymbol{S}_{-t'}\begin{bmatrix}0\\ u^3(t',\cdot)\end{bmatrix} dt'\right\|_{\mathcal{H}^{1/2}} &= \left\|\boldsymbol{S}_t\int_{t_0}^t \boldsymbol{S}_{-t'}\begin{bmatrix}0\\ u^3(t',\cdot)\end{bmatrix} dt'\right\|_{\mathcal{H}^{1/2}} \\ &\lesssim \|u^3\|_{L^{4/3}([t_0,t]\times\mathbb{R}^3)} = \|u\|^3_{L^4([t_0,t]\times\mathbb{R}^3)}.\end{aligned}$$

So $\|\boldsymbol{u}(t,\cdot) - \boldsymbol{S}_t\boldsymbol{u}_+\|_{\mathcal{H}^{1/2}} \to 0$ with $\boldsymbol{u}_+ = \boldsymbol{u}_0 + \int_0^\infty \boldsymbol{S}_{-t'}\begin{bmatrix}0\\ u^3(t',\cdot)\end{bmatrix} dt'$.$\square$

Asymptotic **behaviour**. Basic facts (repeat)

Cauchy initial value problem for the cubic wave equation

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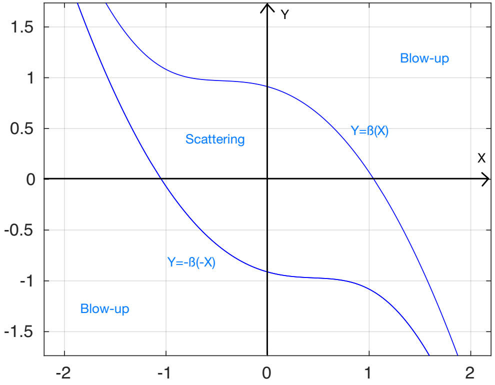
The new result in pictures. Future(+) times

There is a *threshold function* $\beta\colon \mathbb{R} \to \mathbb{R}$ such that, letting

$$\boldsymbol{u}_0^{X,Y}(x) \cong \left(\frac{X}{1+|x|^2}, \frac{Y}{(1+|x|^2)^2} \right), \qquad \text{for } (X,Y) \in \mathbb{R}^2,$$

we have the pictured behaviours for future times $t \geqslant 0$:



At the threshold $Y = \pm\beta(\pm X)$: **non-scattering, $(+)$-global** ($[0,\infty) \subseteq I$).

The complete picture. Future(+) & past(-)

Remark 1. $\sqrt{X^2 + Y^2} = \|\boldsymbol{u}_0^{X,Y}\|_{\mathcal{H}^{1/2}}$.

Remark 2. All our solutions are finite energy.

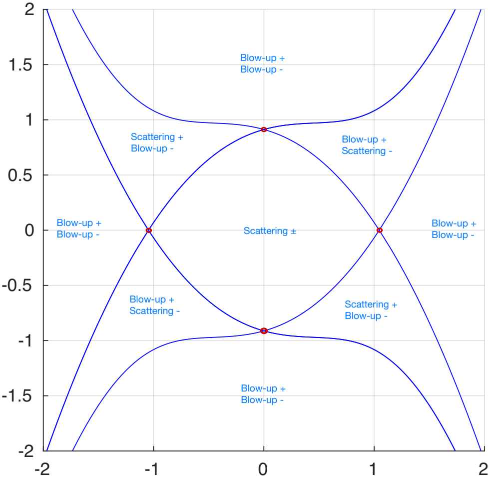


Figure: 9 behaviours. Red dots: **($\pm$)-global**, **non-scattering** solutions.

The global, non-scattering solutions at the threshold

In this talk, we will focus on the threshold solutions only.

Let $u = u^{X,Y}$ be a threshold solution for $t \to +\infty$.

Inside light cone - ODE:

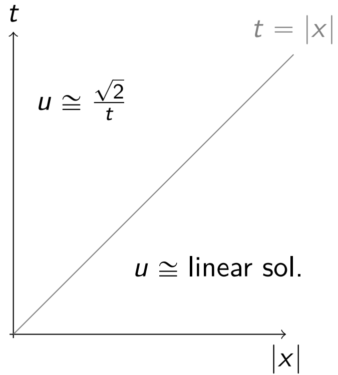
- $u = \frac{\sqrt{2}}{t} + O(t^{-3})$.
- $\left\| u(t,x) - \frac{\sqrt{2}}{t}\mathbf{1}_{|x|\leqslant t} \right\|_{L^p_x(\mathbb{R}^3)} \lesssim t^{\frac{2}{p}-1}$, $p > \frac{3}{2}$.

Outside light cone - Scattering:

- $\| \boldsymbol{u}(t) - \boldsymbol{v}_L(t) \|_{\mathcal{H}^1(|x|>t)} \to 0$, where v_L solves $\partial_t^2 v_L = \Delta v_L$.

Grow-up at ∞ (recall Dodson–Lawrie):

- $\| \boldsymbol{u}(t) \|^2_{\mathcal{H}^{1/2}} = 64\pi^3 \log t + O(\sqrt{\log t})$.



Comparison with known results (only for $\partial_t^2 u - \Delta u = u^3$ on $\mathbb{R}^{1+3}$)

Bizoń–Zenginoğlu's 2009 conjecture (numerical)

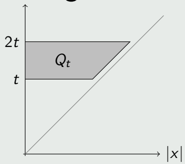
The general threshold between scattering and blowup should be given by codimension-1 global solutions u such that, (up to symmetries)

$$u = \frac{\sqrt{2}}{t} + O(t^{-4}), \qquad t \to \infty.$$

Donninger–Zenginoğlu 2014

There is a codimension-4 manifold of global non-scattering u such that

$$\left\| u - \frac{\sqrt{2}}{t} \right\|_{L^4(Q_t)} = O(t^{-\frac{1}{2}+\epsilon}).$$



Remark 1. DZ solutions: not finite energy, other Initial Value Problem.
Remark 2. Our solutions: first theoretical example of global, non-scattering solutions for Cauchy IVP (to the best of our knowledge).

Main ingredient of proof: conformal invariance

Recall. *Conformal =(Lorentzian) angle-preserving*.
Consider a conformal coordinate change $\mathcal{P}$ with factor Ω, i.e.:

$$(\tilde{t}, \tilde{x}) = \mathcal{P}(t, x), \quad \Omega = |\det D\mathcal{P}|^{\frac{1}{4}}, \quad \mathcal{P} \colon \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}.$$

Fundamental property:

Conformal change of the D'Alembert operator

$$(\partial_t^2 - \Delta)u = \Omega^3(\partial_{\tilde{t}}^2 - \tilde{\Delta})(\Omega^{-1}u).$$

So letting $\tilde{u}(\tilde{t}, \tilde{x}) := (\Omega^{-1}u)(t, x)$, we have $u^3 = \Omega^3 \tilde{u}^3$ and

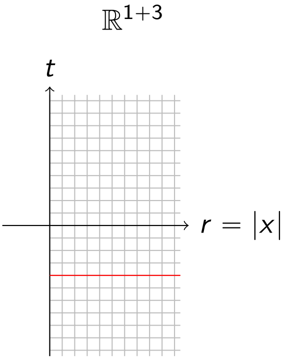
$$(\partial_t^2 - \Delta)u = u^3 \iff \Omega^3(\partial_{\tilde{t}}^2 - \tilde{\Delta})\tilde{u} = \Omega^3 \tilde{u}^3.$$

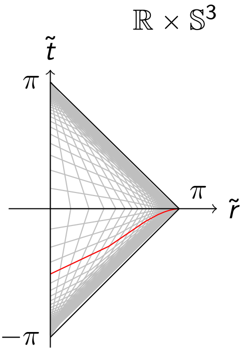
This can be done with manifold-valued $\mathcal{P} \colon \mathbb{R}^{1+3} \to \mathbb{R} \times M^3$, too.
We will have $M^3 = \mathbb{S}^3$. Thus the D'Alembertian is $\partial_{\tilde{t}}^2 - \Delta_{\mathbb{S}^3} + 1$.

Remark

Cubic wave equation = conformal wave equation.

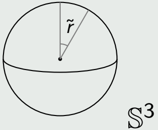
The Penrose map $\mathcal{P}\colon \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$



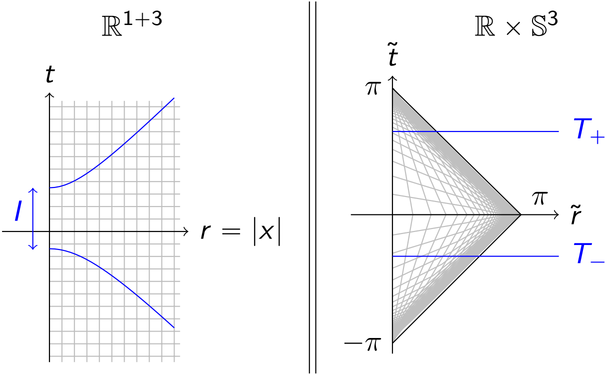


The map $(\tilde{t}, \tilde{r}) = \mathcal{P}(t, r)$ and the definition of $\tilde{r}$

$$\tilde{t} = \arctan(t + r) + \arctan(t - r),$$
$$\tilde{r} = \arctan(t + r) - \arctan(t - r).$$



Constructing our solutions



Recall: $\tilde{u} = \Omega^{-1} u$. Ansatz: $\tilde{u} = \tilde{u}(\tilde{t})$. We get an ODE:

$$\partial_t^2 u - \Delta u = u^3 \iff \partial_{\tilde{t}}^2 \tilde{u} - \Delta_{\mathbb{S}^3} \tilde{u} + \tilde{u} = \tilde{u}^3 \iff \tilde{u}'' + \tilde{u} = \tilde{u}^3$$

Let (T_-, T_+) ODE time of existence of $\tilde{u}$. Then u exists for $t \in I$ (picture).

Behaviour of u as $t \to +\infty$

$T_+(\tilde{\boldsymbol{u}}_0) < \pi$: blows-up. $T_+(\tilde{\boldsymbol{u}}_0) > \pi$: scatter. $T_+(\tilde{\boldsymbol{u}}_0) = \pi$: thresh..

What about other nonlinearities? Quintic is special

Only for $p = 5$ you have **stationary solutions** to $\partial_t^2 u - \Delta u = |u|^{p-1}u$.
Proof:

Baby Pohozaev identity

If $(*) - \Delta Q = |Q|^{p-1}Q$ then $\int_{\mathbb{R}^3} |\nabla Q|^2 = \frac{6}{p+1}\int_{\mathbb{R}^3} |Q|^{p+1}$.

On the other hand, multiplying (*) by Q and integrating by parts we see that $\int_{\mathbb{R}^3} |\nabla Q|^2 = \int_{\mathbb{R}^3} |Q|^{p+1}$. So $p + 1 = 6$. $\square$

$$\text{For } p = 5,\ Q(x) = (1 + \tfrac{|x|^2}{3})^{-\frac{1}{2}}.$$

Radial soliton resolution (Duyckaerts–Kenig–Merle 2013)

If $\partial_t^2 u - \Delta u = u^5$ is global then $\|\boldsymbol{u}\|_{\mathcal{H}^1} \leqslant C < \infty$ and

$$\boldsymbol{u}(t,x) = \sum_{j=1}^{J} \left(\frac{Q(\frac{x}{\lambda_j(t)})}{\lambda_j(t)^{\frac{1}{2}}}, 0 \right) + \boldsymbol{v}_{\text{lin}}(t,x) + o(1), \quad t \to \infty.$$