# Quadrature rules on manifolds: useful results

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# First result: Euclidean approximation of kernels

**Theorem (Gariboldi, G. 2022)**
*There exists a smooth positive function $q$ on $\mathcal{M} \times \mathcal{M}$ such that if $h$ is even, smooth and compactly supported on $\mathbb{R}$, then for every $n$*

$$\sum_m h(\lambda_m/R)\varphi_m(x)\overline{\varphi_m(y)}$$
$$= q(x,y)R^d \mathcal{F}_d h(R|x-y|) + O(R^{d-2}(1+R|x-y|)^{-n})$$

# Ideas in the proof: The operator $\cos(t\sqrt{\Delta})$

- Let $\cos(t\sqrt{\Delta})(x,y) \in \mathcal{D}'(\mathcal{M}\times\mathcal{M})$ be the kernel of the operator $f \mapsto \cos(t\sqrt{\Delta})f = u(t,x)$, where $u(t,x)$ is the solution of problem

$$\begin{cases} (\frac{\partial^2}{\partial t^2} + \Delta)u(t,x) = 0, & (t \in \mathbb{R},\ x \in \mathcal{M}) \\ u(0,x) = f(x) \quad \frac{\partial u}{\partial t}(0,x) = 0. \end{cases}$$

  Notice that

$$\cos(t\sqrt{\Delta})(x,y) = \sum_{m=0}^{+\infty} \cos(t\lambda_m)\varphi_m(x)\overline{\varphi_m(y)}$$

- Similarly, let $\cos(t\sqrt{\Delta_{\mathbb{R}^d}})(|x-y|) \in \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ be the kernel of the operator $f \mapsto \cos(t\sqrt{\Delta_{\mathbb{R}^d}})f = u(t,x)$,

$$\begin{cases} (\frac{\partial^2}{\partial t^2} + \Delta_{\mathbb{R}^d})u(t,x) = 0, & (t \in \mathbb{R},\ x \in \mathbb{R}^d) \\ u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = 0. \end{cases}$$

# Hadamard's parametrix for $\cos(t\sqrt{\Delta})$

The kernel $\cos(t\sqrt{\Delta})(x,y)$ can be approximated (for small $t$) by means of the corresponding kernel on $\mathbb{R}^d$, (and other radial distributions on $\mathbb{R}^d$...)

$$\cos(t\sqrt{\Delta})(x,y) = q_0(x,y)\cos(t\sqrt{\Delta_{\mathbb{R}^d}})(|x-y|) + \sum_{\nu=1}^{M} q_\nu(x,y)B_\nu(t,|x-y|) + R_M(t,x,y)$$

where

- $M > d+3$
- $q_\nu \in \mathcal{D}(\mathcal{M}\times\mathcal{M})$ and $q_0(x,y) > 0$.
- $R_M \in \mathcal{C}^{M-d-3}([-\varepsilon,\varepsilon]\times\mathcal{M}\times\mathcal{M})$, and

$$|\partial^\beta_{t,x,y} R_M(t,x,y)| \leq C|t|^{2M+2-d-|\beta|}$$

(see Sogge, Hangzhou Lectures on Eigenfunctions of the Laplacian, 2014)

# "Proof"

Suppose $h$ even (and smooth and compactly supported). Formally:

$$
\begin{aligned}
\sum_{m=0}^{+\infty} h(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)} &= \sum_{m=0}^{+\infty}\left(\int_{-\infty}^{+\infty}\mathcal{F}_1 h(t)\cos(t\lambda_m)dt\right)\varphi_m(x)\overline{\varphi_m(y)} \\
&= \int_{-\infty}^{+\infty}\mathcal{F}_1 h(t)\left(\sum_{m=0}^{+\infty}\cos(t\lambda_m)\varphi_m(x)\overline{\varphi_m(y)}\right)dt \\
&= \int_{-\infty}^{+\infty}\mathcal{F}_1 h(t)\cos(t\sqrt{\Delta})(x,y)dt \\
&\approx q_0(x,y)\int_{-\infty}^{+\infty}\mathcal{F}_1 h(t)\cos(t\sqrt{\Delta_{\mathbb{R}^d}})(|x-y|)dt \\
&= q_0(x,y)\int_{-\infty}^{+\infty}h(t)\mathcal{F}_1(\cos(\cdot\sqrt{\Delta_{\mathbb{R}^d}})(|x-y|))(t)dt \\
&= C_d q_0(x,y)\int_{-\infty}^{+\infty}h(t)\frac{J_{d/2-1}(|t||x-y|)}{(|t||x-y|)^{d/2-1}}|t|^{d-1}dt \\
&= q_0(x,y)\mathcal{F}_d h(|x-y|)
\end{aligned}
$$

Problem: Need $\mathcal{F}_1 h$ supported in $[-\varepsilon,\varepsilon]$...

# Second result: Area regular partitions

The following result is well known

**Theorem**
*For any sufficiently large integer $N$ there is a partition of the sphere $S^d$ into $N$ regions of equal measure and small diameter $\leq cN^{-1/d}$.*

Stated and used by several authors:

- Stolarsky (1973)
- Beck and Chen (1987)
- Bourgain and Lindenstrauss (1988)

Proofs

- Constructive proof for $d = 2$ with a small constant $c$, by Rakhmanov, Saff and Zhou (1994)
- Extension to general $d$ by Leopardi (2007)
- Different proof by Feige and Schechtman (2002).

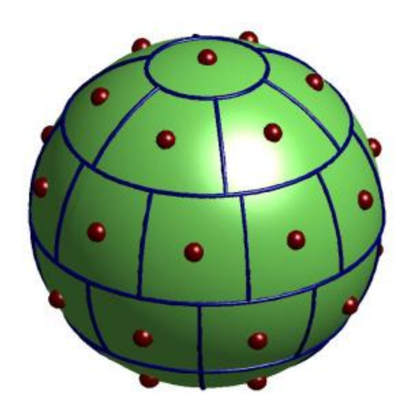

**Figure 1:** $N = 33$. Rakhmanov, Saff and Zhou's construction (picture from Leopardi's thesis)

**Definition (Ahlfors regular metric measure space of dimension $d > 0$)**

A complete metric space $X$ with a Borel measure $\mu$ such that for all open balls $B(x, r)$ with $x \in X$, $0 < r \leq \mathrm{diam}(X)$

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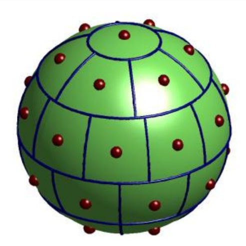
**Example**

- Compact connected Riemannian manifolds (e.g. the sphere).

**Theorem (G., Leopardi, 2017)**
*Let $(X, \rho, \mu)$ be an Ahlfors regular metric measure space of dimension $d$:*

- *connected,*
- *with finite measure.*

*Then for any integer $N$ there is a partition of $X$ into $N$ regions of measure $\mu(X)/N$, each contained in a ball of radius $\asymp N^{-1/d}$ and containing a ball of radius $\asymp N^{-1/d}$.*



**Theorem (G. David 1988, M. Christ 1990)**

*An Ahlfors regular metric measure space of dimension $d$ admits a family of dyadic cubes: a collection of subsets of $X$, $\{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ s. t.*

- $X = \cup_{\alpha \in I_k} Q^k_\alpha$ *for all $k$ (each generation covers $X$).*
- $Q^k_\alpha \cap Q^k_\beta = \emptyset$ *for all $k$ and $\alpha \neq \beta$ (disjoint).*
- *If $\ell > k$ then either $Q^\ell_\beta \subset Q^k_\alpha$ or $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$ (dyadic).*
- *Each $Q^k_\alpha$ contains a ball $b^k_\alpha$ (inner ball) of radius $a_0 2^{-k}$.*
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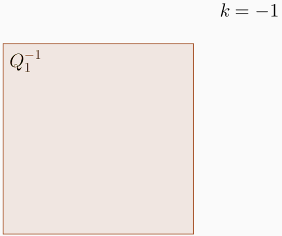

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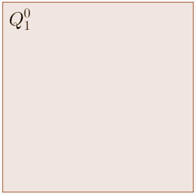
$k = 0$



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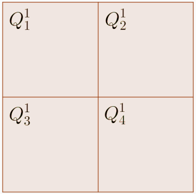

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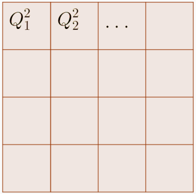
$k = 2$



# A consequence

**Corollary (Sierpinski)**
*Let $(X, \rho, \mu)$ be an Ahlfors regular metric measure space of dimension $d$ and let $S$ be a measurable subset of $X$ with finite measure. Then, for any $0 \le t \le \mu(S)$ there exists a subset $T \subset S$ with measure $\mu(T) = t$.*

**Proof.**
This is true for all non-atomic measures. Here it follows easily from the dyadic cube decomposition of $X$. $\square$

**Lemma**
*For all $k$ and $\alpha \in I_k$*

$$Q_\alpha^k \cup \bigcup_{B_\beta^k \cap B_\alpha^k \neq \emptyset} Q_\beta^k \subset 3B_\alpha^k.$$

*Here $3B_\alpha^k$ is the ball with the same center as $B_\alpha^k$ and triple radius.*

**Proof.**

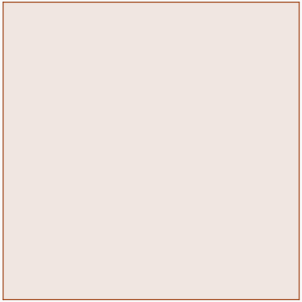
$$Q_\alpha^k \cup \bigcup_{B_\beta^k \cap B_\alpha^k \neq \emptyset} Q_\beta^k \subset B_\alpha^k \cup \bigcup_{B_\beta^k \cap B_\alpha^k \neq \emptyset} B_\beta^k \subset 3B_\alpha^k.$$

□

We will say that cubes $Q_\alpha^k$ and $Q_\beta^k$ are *neighbours* if the corresponding outer balls intersect.

# Proof of the Theorem, 1

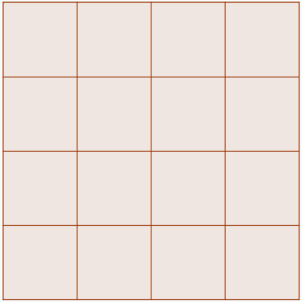
- Assume for simplicity that $\mu(X) = 1$.



# Proof of the Theorem, 1

- Assume for simplicity that $\mu(X) = 1$.
- For any big enough $N$, let $n$ be the greatest generation of cubes of $X$ such that

$$\mu(Q^n_\alpha) \geq \frac{2}{N} \quad \forall \alpha \in I_n$$



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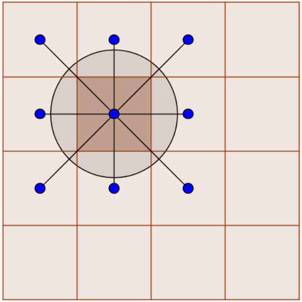
- Let $\Gamma$ be the graph with vertices in the centers of the outer balls of the cubes, and edges corresponding to pairs of neighbouring cubes.



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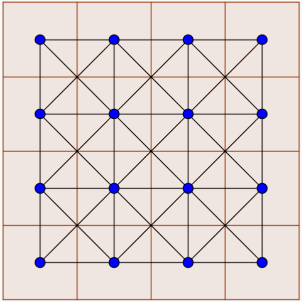
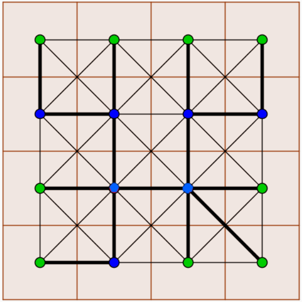
- Let $\Gamma$ be the graph with vertices in the centers of the outer balls of the cubes, and edges corresponding to pairs of neighbouring cubes.
- $X$ connected $\Rightarrow$ $\Gamma$ connected



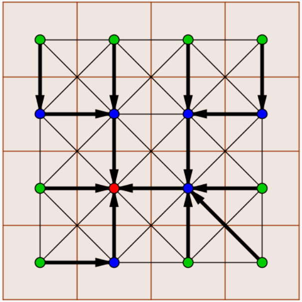
# Proof of the Theorem, 2

- Take a spanning tree $S$ of $\Gamma$. It has leaves.



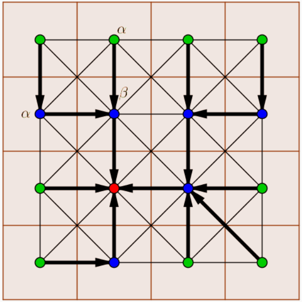
# Proof of the Theorem, 2

- Take a spanning tree $S$ of $\Gamma$. It has leaves.
- Mark one vertex as the root, and direct $S$ from the leaves towards the root. $(\alpha, \beta) \in S$ means that there is an edge from $\alpha$ towards $\beta$.



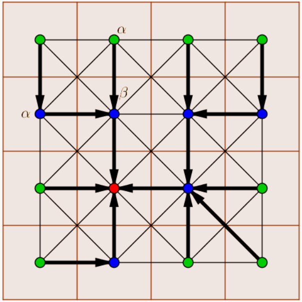
# Proof of the Theorem, 3

- For any $\beta \in I_n$, $\quad \mu\left(Q^n_\beta \cup \bigcup_{(\alpha,\beta)\in S} Q^n_\alpha\right) \leq \frac{C}{N}$, $\quad C$ depending on $X$.



# Proof of the Theorem, 3

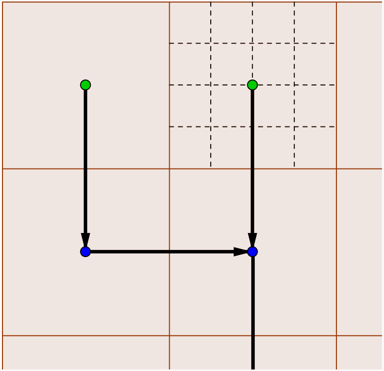
- For any $\beta \in I_n$, $\quad \mu(Q_\beta^n \cup \bigcup_{(\alpha,\beta)\in S} Q_\alpha^n) \leq \frac{C}{N}$, $\quad C$ depending on $X$.
- In particular, $Q_\beta^n \cup \bigcup_{(\alpha,\beta)\in S} Q_\alpha^n$ may contain at most $C$ disjoint sets of measure $N^{-1}$.



# Proof of the Theorem, 4

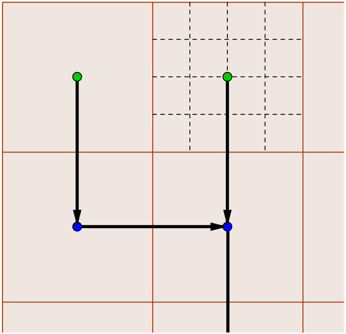
- One can fix an integer $k$, depending only on $X$, such that all cubes of generation $m = n + k$ have measure

$$\mu(Q_\alpha^m) \leq \frac{1}{CN}$$



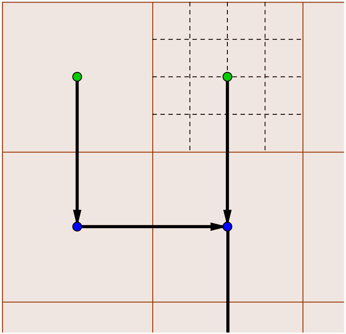
# Proof of the Theorem, 5. Leaves

- Let $\alpha$ be a leaf of generation $n$.



# Proof of the Theorem, 5. Leaves

- Let $\alpha$ be a leaf of generation $n$.
- Since $2/N \leq \mu(Q_\alpha^n) \leq C/N$, $Q_\alpha^n$ can contain $2 \leq N_\alpha \leq C$ subsets of measure $1/N$.



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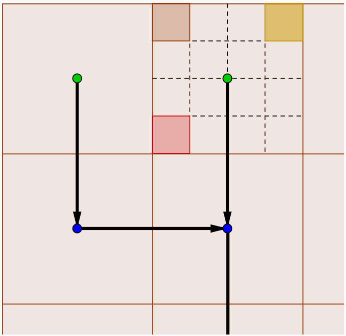
- Let $\alpha$ be a leaf of generation $n$.
- Since $2/N \le \mu(Q^n_\alpha) \le C/N$, $Q^n_\alpha$ can contain $2 \le N_\alpha \le C$ subsets of measure $1/N$.
- Take $N_\alpha$ cubes of generation $m$ inside $Q^n_\alpha$.



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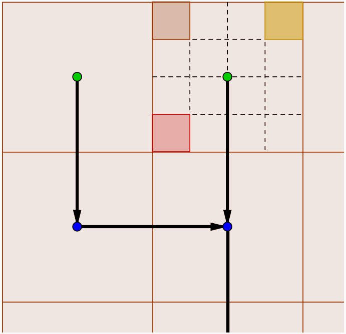
- Take $N_\alpha$ cubes of generation $m$ inside $Q_\alpha^n$.
- Their total measure is bounded by $N_\alpha \times (1/(CN)) \leq 1/N$.



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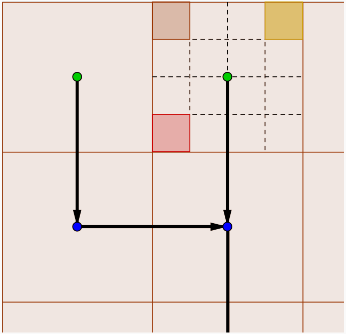
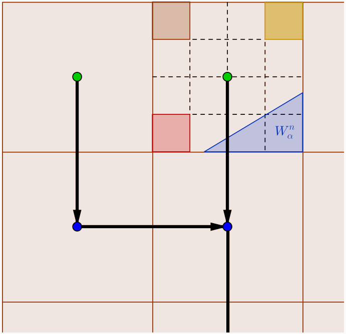
- Take $N_\alpha$ cubes of generation $m$ inside $Q_\alpha^n$.
- Their total measure is bounded by $N_\alpha \times (1/(CN)) \leq 1/N$.
- The measure of the remaining part of $Q_\alpha^n$ is at least $1/N$.



# Proof of the Theorem, 6. Leaves

- Take a subset $W_\alpha^n \subset Q_\alpha^n$, disjoint from the $N_\alpha$ cubes of generation $m$, of measure $\mu(X_\alpha^n) - N_\alpha / N$



# Proof of the Theorem, 6. Leaves

- Take a subset $W_\alpha^n \subset Q_\alpha^n$, disjoint from the $N_\alpha$ cubes of generation $m$, of measure $\mu(X_\alpha^n) - N_\alpha / N$
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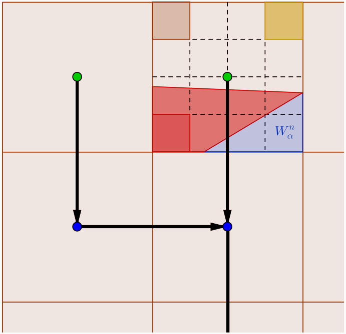

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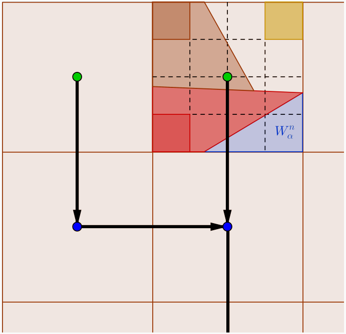

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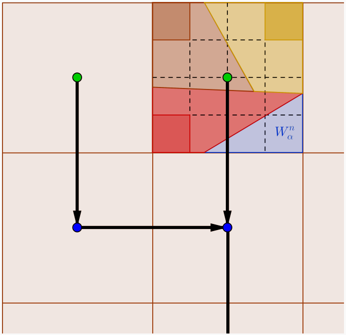
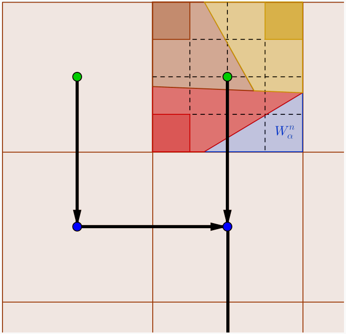


# Proof of the Theorem, 7. Generic node

- Let $\beta$ be a generic node of generation $n$.



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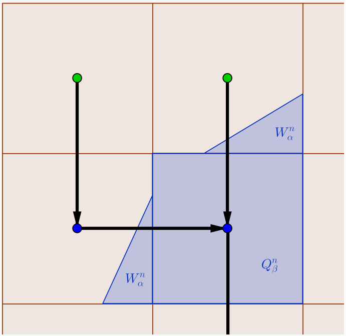

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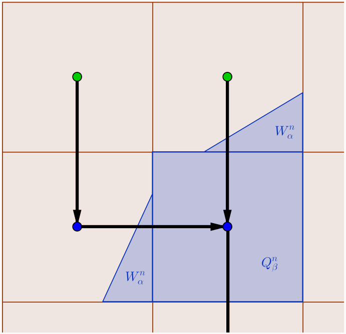

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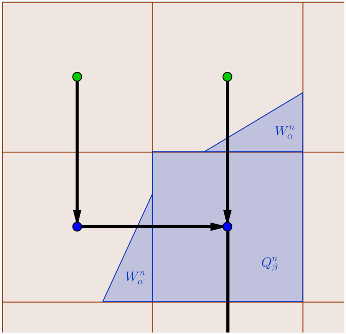

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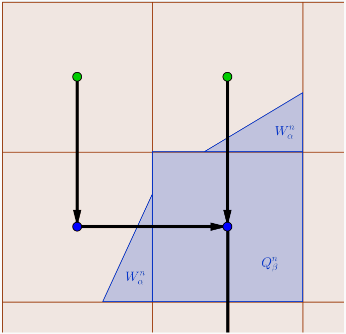
- Repeat the above argument for the set $X_\beta^n = Q_\beta^n \cup \bigcup_{(\alpha,\beta)\in S} W_\alpha^n$.
- Can do it so that the remainder $W_\beta^n \subset Q_\beta^n$.




- This construction due to Feige-Schechtman (2002, for the sphere), modified with the use of the David-Christ dyadic decomposition.

# Third result: Cassels inequality

As reported by H. L. Montgomery. For any choice of $N$ real numbers $x_1, \ldots, x_N$ and for any positive integer $X$,

$$\sum_{m=1}^{X} \left| \sum_{j=1}^{N} e^{2\pi i m x_j} \right|^2 \geq \frac{1}{2} N(X+1) - \frac{1}{2} N^2.$$

For the sake of simplicity, throughout this talk we will always

- consider $X \gtrsim N$,
- disregard precise constants.

Thus we can simplify the form of the inequality to

$$\sum_{m=1}^{X} \left| \sum_{j=1}^{N} e^{2\pi i m x_j} \right|^2 \geq C\, NX.$$

# A quick interpretation of the inequality

$$\sum_{m=1}^{X} \left| \sum_{j=1}^{N} e^{2\pi i m x_j} \right|^2 \geq C\, NX$$

can be rewritten as

$$\left( \frac{1}{X} \sum_{m=1}^{X} \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i m x_j} - \int_0^1 e^{2\pi i m x}\, dx \right|^2 \right)^{1/2} \geq C\, \frac{1}{\sqrt{N}}.$$

Thus, no matter how well I choose the points $x_1, \ldots, x_N$, the corresponding Riemann sums give an error of order at least $N^{-1/2}$ when tested on the first $X$ exponentials, in $\ell^2$ average.

In other words, it can be seen as a result on irregularity of distributions.

# Montgomery's generalizations

- Positive weights $\omega_j$:

$$\sum_{m=1}^{X} \left| \sum_{j=1}^{N} \omega_j e^{2\pi i m x_j} \right|^2 \geq C \left( \sum_{j=1}^{N} \omega_j^2 \right) X.$$

- Higher dimensions: for any $x_1, \ldots, x_N \in \mathbb{T}^2$, and for any $X_1$, $X_2$ with $X_1 X_2 \gg N$

$$\sum_{|m_1| \leq X_1,\, |m_2| \leq X_2,\, m \neq 0} \left| \sum_{j=1}^{N} e^{2\pi i m \cdot x_j} \right|^2 \geq C\, N X_1 X_2.$$

# Generalizing to manifolds. Previous results.

D. Bilyk, F. Dai and S. Steinerberger (2019) proved the following version of Cassels-Montgomery inequality for manifolds.

For any choice of points $x_1, \dots, x_N$ in $\mathcal{M}$ and for any choice of positive weights $\omega_1, \dots, \omega_N$,

$$\sum_{\lambda_m \le L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \ge C \left( \sum_{j=1}^{N} \omega_j^2 \right) L^d (\log L)^{-d/2}.$$

It is easy to show that there is a choice of points $x_1, \dots, x_N$ in $\mathcal{M}$ such that

$$\sum_{\lambda_m \le L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \approx \left( \sum_{j=1}^{N} \omega_j^2 \right) L^d$$

Thus the result of Bilyk, Dai and Steinerberger is sharp up to a logarithmic factor.

# The proof of Bilyk, Dai and Steinerberger

Take the heat kernel

$$p_t(x,y) = \sum_{m=0}^{+\infty} e^{-\lambda_m^2 t} \varphi_m(x)\overline{\varphi_m(y)} \approx \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(d(x,y))^2}{4t}} > 0 \quad (t \text{ small})$$

- $e^{-\lambda_m^2 t} \leq 1$.
- Taking $t = cL^{-2}\log L$, get $p_t(x,x) \geq C\, L^d (\log L)^{-d/2}$.

Thus

$$\sum_{\lambda_m \leq L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \geq \sum_{m=0}^{+\infty} e^{-\lambda_m^2 t} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 - \sum_{\lambda_m > L} \cdots$$

$$= \sum_{j=1}^{N} \omega_j \sum_{k=1}^{N} \omega_k \sum_{m=0}^{+\infty} e^{-\lambda_m^2 t} \varphi_m(x_j)\overline{\varphi(x_k)} - \ldots = \sum_{j=1}^{N}\sum_{k=1}^{N} \omega_j \omega_k p_t(x_j, x_k) - \ldots$$

$$\geq \sum_{j=1}^{N} \omega_j^2\, p_t(x_j, x_j) - \ldots \geq C \left( \sum_{j=1}^{N} \omega_j^2 \right) L^d (\log L)^{-d/2} - \ldots .$$

# Our contribution

L. Brandolini, B. Gariboldi, G. (2020): sharp version of the Cassels-Montgomery inequality for manifolds.

**Theorem**
*For any choice of points* $x_1, \ldots, x_N$ *in* $\mathcal{M}$ *and for any choice of positive weights* $\omega_1, \ldots, \omega_N$*,*

$$\sum_{\lambda_m \leq L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \geq C \left( \sum_{j=1}^{N} \omega_j^2 \right) L^d .$$

# Proof

- Assume WLOG that $\omega_1 \geq \omega_2 \geq \dots$.
- Call $X = \sharp\{m : \lambda_m \leq L\} \approx L^d$
- Take an area regular partition of $\mathcal{M} = \cup_{i=1}^{X} U_i$

$$\mu(U_i) = 1/X, \quad B(z_i, c_1 X^{-1/d}) \subseteq U_i \subseteq B(z_i, c_2 X^{-1/d})$$

- Assume, for the sake of simplicity, that each region $U_i$ contains at most 1 point $x_j$. Call $V_j$ the region that contains $x_j$.
- Let $h$ be a smooth even compactly supported function on $[-1, 1]$, with $h(t) \leq 1$ and $\mathcal{F}_d h(s) \geq 0$ ($d$–dimensional Fourier transform of $h$, thought of as a radial function).
- Recall the kernel theorem: there exists $q_0 > 0$ s.t.

$$\sum_m h\left(\frac{\lambda_m}{L}\right) \varphi_m(x)\overline{\varphi_m(y)}$$
$$= q_0(x, y) L^d \mathcal{F}_d h(L|x - y|) + O\left(\frac{L^{d-2}}{(1 + L|x - y|)^M}\right)$$

# Proof

$$\sum_{\lambda_m \le L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \ge \sum_{m=0}^{+\infty} h\left(\frac{\lambda_m}{L}\right) \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} \omega_j \omega_k \sum_{m=0}^{+\infty} h\left(\frac{\lambda_m}{L}\right) \varphi_m(x_j) \overline{\varphi_m(x_k)}$$

$$\ge \sum_{j=1}^{N} \sum_{k=1}^{N} \omega_j \omega_k q_0(x_j, x_k) L^d \mathcal{F}_d h(L|x_j - x_k|)$$

$$- C \sum_{j=1}^{N} \sum_{k=1}^{N} \omega_j \omega_k \frac{L^{d-2}}{(1 + L|x_j - x_k|)^M}$$

Main term:

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \omega_j \omega_k q_0(x_j, x_k) L^d \mathcal{F}_d h(L|x_j - x_k|) \ge C \sum_{j=0}^{N} \omega_j^2 L^d \mathcal{F}_d h(0)$$

# Proof: the remainder term

We say that $V_j$ is near $V_k$ (and write $j \sim k$) if $|z_j - z_k| \leq CL^{-1}$. Otherwise we will write $j \nsim k$.

$$\sum_{j=1}^{N}\sum_{k=1}^{N}\omega_j\omega_k\frac{L^{d-2}}{(1+L|x_j-x_k|)^M}$$
$$\leq 2L^{d-2}\sum_{j=1}^{N}\sum_{k=j,k\sim j}^{N}\omega_j\omega_k + 2L^{d-2}\sum_{j=1}^{N-1}\sum_{k=j+1,k\nsim j}^{N}\omega_j\omega_k(L|x_j-x_k|)^{-M}$$

There is only a uniformly bounded (by, say, $\kappa$) number of balls $V_k$ near $V_j$. Also $\omega_j \geq \omega_k$ when $k \geq j$. Thus

$$L^{d-2}\sum_{j=1}^{N}\sum_{k=j,k\sim j}^{N}\omega_j\omega_k \leq \kappa L^{d-2}\sum_{j=1}^{N}\omega_j^2.$$

## Proof: the remainder term

For any big $r$ and for any fixed $j$, how many regions $V_k$ are there such that $|z_k - z_j| \leq rL^{-1}$ ?

$$\approx \frac{(rL^{-1})^d}{(X^{-1/d})^d} \approx r^d$$

Therefore,

$$L^{d-2} \sum_{j=1}^{N-1} \sum_{k=j+1, k \nsim j}^{N} \omega_j \omega_k (L|x_j - x_k|)^{-M}$$

$$\leq L^{d-2} \sum_{j=1}^{N-1} \omega_j \sum_{s=1}^{+\infty} \sum_{k>j,\, 2^{s-1}<L|z_j - z_k| \leq 2^s} \omega_k (L|x_j - x_k|)^{-M}$$

$$\leq cL^{d-2} \sum_{j=1}^{N-1} \omega_j \sum_{s=1}^{+\infty} 2^{-sM} \sum_{k>j,\, L|z_j - z_k| \leq 2^s} \omega_k$$

$$\leq cL^{d-2} \sum_{j=1}^{N-1} \omega_j \sum_{s=1}^{+\infty} 2^{-sM} 2^{sd} \omega_j \leq cL^{d-2} \sum_{j=1}^{N} \omega_j^2$$

# An application

Let $L > 0$ and assume there exist points $\{x_j\}_{j=1}^{N}$ and weights $\{\omega_j\}_{j=1}^{N}$ such that for all $\lambda_m^2 \leq L^2$,

$$\int_{\mathcal{M}} \varphi_m(x)\, dx = \sum_{j=1}^{N} \omega_j \varphi_m(x_j).$$

Then there exists a constant $C > 0$ independent of $L$ and $N$ such that

$$1 \geq CL^d \sum_{j=1}^{N} \omega_j^2.$$

In particular

$$CL^d \leqslant N.$$

# Proof

Let

$$P(x) = \sum_{\lambda_m \le L} \sum_{i=1}^{N} \omega_i \overline{\varphi_m(x_i)} \varphi_m(x),$$

then

$$\int_{\mathcal{M}} P(x)\, dx = \int_{\mathcal{M}} \sum_{\lambda_m \le L} \sum_{i=1}^{N} \omega_i \overline{\varphi_m(x_i)} \varphi_m(x)\, dx$$

$$= \sum_{\lambda_m \le L} \sum_{i=1}^{N} \omega_i \overline{\varphi_m(x_i)} \int_{\mathcal{M}} \varphi_m(x)\, dx = \sum_{i=1}^{N} \omega_i = 1.$$

On the other hand, by Cassels inequality,

$$1 = \sum_{j=1}^{N} \omega_j P(x_j) = \sum_{j=1}^{N} \omega_j \sum_{\lambda_m \le L} \sum_{i=1}^{N} \omega_i \overline{\varphi_m(x_i)} \varphi_m(x_j)$$

$$= \sum_{\lambda_m \le L} \left| \sum_{j=1}^{N} \omega_j \varphi_m(x_j) \right|^2 \ge C L^d \sum_{j=1}^{N} \omega_j^2.$$