Gravitating vortices with positive curvature

Mario Garcia-Fernandez

Universidad Autónoma de Madrid
Instituto de Ciencias Matemáticas

Vortex Moduli, Bangalore

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Abelian vortices

The (abelian) vortex equation for a Hermitian metric h on a line bundle L over a Riemann surface Σ with section $\phi \in H^0(\Sigma, L)$

$$*F_h - \frac{i}{2}(|\phi|_h^2 - \tau) = 0$$

is a generalization of the equations on $\mathbb{R}^2$ which were introduced in 1950 by Ginzburg and Landau in the theory of superconductivity.

The equations, often called Bogomol'nyi equations, depend on a choice of background metric g on Σ and a *symmetry breaking parameter*

$$0 < \tau \in \mathbb{R}.$$

In this physical setup, the Chern connection $A_h = h^{-1}\partial h$ represents the electromagnetic field and ϕ is understood as an order parameter for Cooper pairs ($|\phi|_h^2$ being a measure of local density).

Abelian vortices have been extensively studied in the mathematics literature after the seminal work of Jaffe and Taubes on the Euclidean plane, and Witten on the hyperbollic plane.

Assuming that Σ is compact, the existence problem for abelian vortices was solved independently by Noguchi, Bradlow, and Garcia-Prada:

Theorem (Noguchi '87, Bradlow '90, Garcia-Prada '93)

L holomorphic line bundle over a compact Riemann surface Σ with Kähler metric g, with section $0 \neq \phi \in H^0(\Sigma, L)$. Fix $0 < \tau \in \mathbb{R}$. Then,

$$i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0$$

admits a solution h if and only if (where $N := \int_\Sigma c_1(L)$)

$$\frac{4\pi N}{Vol_g} < \tau.$$

In that case, the solution is unique.

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Proof: Assume that h is a solution. Integrating the equation

$$iF_h + \frac{1}{2}(|\phi|_h^2 - \tau)\omega = 0 \qquad (1)$$

we obtain ($\phi \neq 0$)

$$4\pi N + (\|\phi\|_{L^2}^2 - \tau) Vol_g = 0 \qquad \Longrightarrow \qquad 4\pi N / Vol_g < \tau.$$

Assume now $4\pi N / Vol_g < \tau$. Consider the SU(2)-equivariant bundle on $X = \Sigma \times \mathbb{P}^1$

$$0 \to p^* L \to E \to q^* O_{\mathbb{P}^1}(2) \to 0$$

determined by

$$\phi \in H^0(\Sigma, L) \cong H^1(X, p^* L \otimes q^* O_{\mathbb{P}^1}(-2)).$$

Then, for the Kähler form $\omega_\tau = p^*\omega + \frac{4}{\tau} q^* \omega_{FS}$, Garcia-Prada proves that

$$E \text{ is slope } [\omega_\tau] - \text{stable} \Leftrightarrow 4\pi N < \tau Vol_g$$

$$E \text{ admits } \omega_\tau - \text{Hermite-Einstein metric } \Leftrightarrow L \text{ admits a solution of } (1)$$

The proof follows from the Donaldson-Uhlenbeck-Yau Theorem. □

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Bradlow–Garcia-Prada–Noguchi's Theorem identifies the moduli space of abelian vortices on Σ with the symmetric product

$$\mathcal{M} = S^N\Sigma.$$

This moduli space carries an interesting Kähler metric, obtained by infinite-dimensional Kähler reduction, extensively studied in the mathematical physics literature (Manton, Romao, Baptista, ...).

'We assume vortices have no back-reaction on the metric. They are not gravitating. Some vortices - cosmic strings - have a gravitational effect.'

N.S. Manton, Programme on Moduli Spaces, Cambridge '11.

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Gravitating vortices

Motivated by the *Kähler-Yang-Mills equations* in higher dimensions, jointly with Álvarez-Cónsul and García-Prada we introduced a notion of vortex on a Riemann surface with back-reaction of the metric.

- *Gravitating vortices, cosmic strings, and the Kähler–Yang–Mills equations*, **Comm. Math. Phys. 351** (2017) (Álvarez-Cónsul, GF, García-Prada).

Definition (Álvarez-Cónsul, GF, García-Prada '17)

L holomorphic line bundle over a Riemann surface Σ, with section $0 \neq \phi \in H^0(\Sigma, L)$. Fix $0 < \tau \in \mathbb{R}$ and $0 \leqslant \alpha \in \mathbb{R}$.

The *gravitating vortex equations* for (g, h), where g is a Kähler metric on Σ and h is a hermitian metric on L, are given by

$$\begin{aligned} i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\ S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= c. \end{aligned} \tag{2}$$

The constant $c \in \mathbb{R}$ is topological, given explicitly by

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{Vol_g}. \tag{3}$$

Consider the SU(2)-equivariant bundle on $X = \Sigma \times \mathbb{P}^1$ determined by $\phi \in H^0(\Sigma, L)$:

$$0 \to p^*L \to E \to q^*O_{\mathbb{P}^1}(2) \to 0.$$

Proposition (Álvarez-Cónsul, GF, García-Prada '17)

Fix $0 < \tau \in \mathbb{R}$ and $0 \leqslant \alpha \in \mathbb{R}$. The gravitating vortex equations admit a solution with Kähler form ω if and only if (X, E) admits a solution (g_X, H) of the Kähler-Yang-Mills equations

$$\begin{aligned} i\Lambda F_H &= \lambda \,\mathrm{Id}, \\ S_{g_X} - \alpha\Lambda^2 \operatorname{tr} F_H \wedge F_H &= c_X. \end{aligned} \quad (4)$$

with Kähler form $\omega_X = p^*\omega + \frac{4}{\tau} q^*\omega_{FS}$.

Upshot: the gravitating vortex equations inherit a **moment map interpretation** from the Kähler-Yang-Mills equations.

Coupled equations for Kähler metrics and Yang–Mills connections, **Geometry and Topology 17** (2013) (Álvarez-Cónsul, GF, García-Prada).

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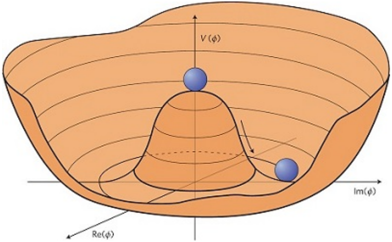
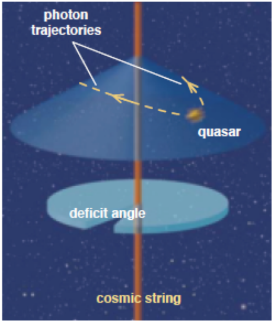
When $c = 0$ ($\Sigma = \mathbb{P}^1$ and $\alpha = 1/\tau N$), we find *self-dual Einstein-Maxwell-Higgs equations*, describing phase transitions in the early universe:

$$\begin{aligned} i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\ S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= 0. \end{aligned} \tag{5}$$

The higgs field ϕ decays to a minima of the potential $V(\phi) = (|\phi|^2 - \tau)^2$ 'breaking the symmetry'. The 'choice of point' in the circle $|\phi|^2 = \tau$ may vary in space-time causing *topological defects* (cosmic strings).



- R. Abbott et al, Constraints on Cosmic Strings Using Data from the Third Advanced LIGO-Virgo Observing Run, Physical Review Letters (2021).



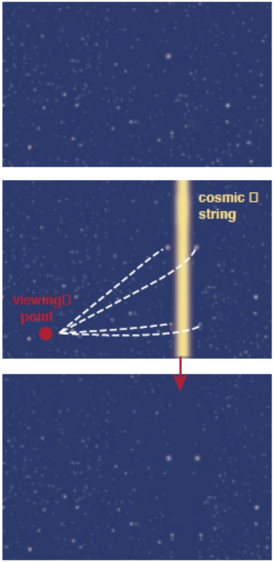


Figure 6. Cosmic strings affect surrounding spacetime by removing a small angular segment, creating a conelike geometry *(above)*. Space remains flat everywhere, but a circular path around the string encompasses slightly less than 360 degrees. The deficit angle is tiny, about 10^{-5} radian. To an observer, the presence of a cosmic string would be betrayed by its effect on the trajectory of passing light rays, which are deflected by an amount equal to the deficit angle. The resultant gravitational lensing reveals itself in the doubling of images of objects behind the string *(bottom panel, right)*.

Figure: Comtet-Gibbons' solution '88 on $\mathbb{R}^{1,1} \times \mathbb{C}^*$ (pure gravity): $-dt^2 + dx^3 + |z - z_0|^{-\frac{\delta}{\pi}} dz d\bar{z}$, deficit angle δ. From A. Gangui 'Superconducting strings', Am. Scientists, 2000

Existence for $c \leq 0$

Take functions $u, f \in C^\infty(\Sigma)$ and consider $g = (1 - \Delta u) g_0$ and $h = e^{2f} h_0$, for a suitable background geometry (g_0, h_0). Then, the gravitating vortex equations for (g, h) are equivalent to

$$\Delta f + \frac{1}{2}(e^{2f}|\phi|^2 - \tau) e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2 - 2cu} = -N,$$
$$\Delta u + e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2 - 2cu} = 1.$$

Theorem (Álvarez-Cónsul, GF, García-Prada, Pingali '21)

Let Σ be a compact connected Riemann surface with $g(\Sigma) \geqslant 2$. Then, there exists a solution of the gravitating vortex equations with volume 2π provided that

$$0 \leqslant \alpha \leqslant \frac{2g(\Sigma) - 2}{\tau(\tau - 2N)}, \qquad 0 < \tau - 2N$$

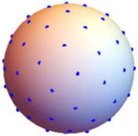
- *Gravitating vortices and the Einstein-Bogol'nyi Equations*, **Math. Annalen (2021)**, (Álvarez-Cónsul, GF, García-Prada, Pingali).

Assume now that

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{Vol_g} = 0.$$

Then $\Sigma \cong \mathbb{P}^1$ and $\alpha = 1/\tau N$, and the *self-dual Einstein-Maxwell-Higgs equations* reduce to a single PDE for a function on the two-sphere

$$\Delta f + \frac{1}{2}(e^{2f}|\phi|^2 - \tau)e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2} = -N.$$



Theorem (Yang '95 '97, Han-Sohn '19)

Let $D = \sum_i n_i p_i$ be the effective divisor on $\mathbb{P}^1$ corresponding to the pair (L, ϕ).

1. Assume that $n_i < \frac{N}{2}$ for all i. Then, for any $V > 4\pi N/\tau$ there exists a solution of the *self-dual Einstein-Maxwell-Higgs equations* such that $Vol_g > V$.
2. Assume that $D = \frac{N}{2}p_0 + \frac{N}{2}p_1$. Then, the *self-dual Einstein-Maxwell-Higgs equations* admit a solution on any Kähler class such that $Vol_g > 4\pi N/\tau$. Furthermore, the solution is S^1-symmetric.

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Yang's method:

1) Yang considers the ε-rescaled PDE (for $((1-\Delta_\epsilon u)g_\epsilon, e^{2f}h_0)$ and $g_\varepsilon = \frac{1}{\varepsilon} g_0$)

$$\Delta f + \frac{1}{2\varepsilon}(e^{2f}|\phi|^2 - \tau)e^{4\alpha\tau f - 2\alpha e^{2f}|\phi|^2} = -N.$$

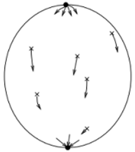
Smoothing $\log|\phi|^2_{h_0}$, takes sequence of supersolutions. C^0-estimate requires

$$|\phi|_{h_0}^{-2/N} \in L^p, \; p > 1.$$

2) Follows by reduction to ODE.

Of course, the conditions in Yang's Theorem correspond to Mumford GIT stability for the linearised $\mathrm{SL}(2,\mathbb{C})$-action on $S^N\mathbb{P}^1$

- $n_i < \frac{N}{2}$, for every $i \iff D \in S^N\mathbb{P}^1$ stable
- $D = \frac{N}{2}p_0 + \frac{N}{2}p_1 \iff D \in S^N\mathbb{P}^1$ strictly polystable



Theorem (Álvarez-Cónsul - GF - García-Prada - Pingali - Yao '21, '23)

Let $0 < \tau, \alpha \in \mathbb{R}$. Let $D = n_0 p_0 + n_1 p_1$ be the effective divisor on $\mathbb{P}^1$, with support given by two points, corresponding to the pair (L, ϕ). If $(\mathbb{P}^1, L, \phi)$ admits a solution of the gravitating vortex equations, then

1. $Vol_g > 4\pi N/\tau$,
2. D is polystable with respect to the $\mathrm{SL}(2,\mathbb{C})$-action on $S^N\mathbb{P}^1$, that is, $n_0 = n_1 = N/2$.

Method: direct application of the Futaki invariant

$$\mathcal{F}_{\alpha,\tau} = 2\alpha(\tau Vol_g - 4\pi N)(N - 2n_i).$$

Of course, the conditions in Yang's Theorem correspond to Mumford GIT stability for the linearised $\mathrm{SL}(2,\mathbb{C})$-action on $S^N\mathbb{P}^1$

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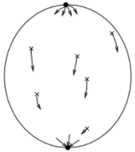

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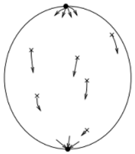

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2. D is polystable with respect to the $SL(2, \mathbb{C})$-action on $S^N\mathbb{P}^1$, that is, $n_0 = n_1 = N/2$.

In the original proof of this result in 2021, it was claimed the polystability of D with arbitrary support. However, this proof has a **serious gap**

- *Gravitating vortices and the Einstein-Bogol'nyi Equations*, **Math. Annalen (2021)**, (Álvarez-Cónsul, GF, García-Prada, Pingali).

The theorem above has been reproved with an alternative explicit proof in

- *Obstructions to the existence of solutions of the self-dual Einstein-Maxwell-Higgs equations on a compact surface*, **Bulletin des Sciences Mathématiques (2023)**, (Álvarez-Cónsul, GF, García-Prada, Pingali, Yao).

Thanks to a collaborative effort with **Chengjian Yao**, a complete proof of the general case seems to be now within reach.

Existence for $c > 0$

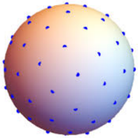
(positive curvature)

Assume now that $\alpha > 0$ and

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha\tau N)}{Vol_g} > 0.$$

Then $\Sigma \cong \mathbb{P}^1$ and $0 < \alpha < 1/\tau N$.

$$\begin{aligned} i\Lambda_g F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\ S_g + \alpha(\Delta_g + \tau)(|\phi|_h^2 - \tau) &= c. \end{aligned}$$



Theorem (GF - Pingali - Yao '21)

Let $D = \sum_i n_i p_i$ be the effective divisor on $\mathbb{P}^1$ corresponding to the pair (L, ϕ). Fix $\tau > 0$ and α such that $0 < \alpha < 1/\tau N$ ($c > 0$).

Assume that $D \in S^N\mathbb{P}^1$ is $\mathrm{SL}(2, \mathbb{C})$-polystable. Then, the gravitating vortex equations admit a solution with coupling constant α for any Kähler class such that $Vol_g > 4\pi N/\tau$.

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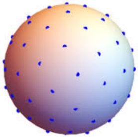

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Assume that $D \in S^N\mathbb{P}^1$ is $\mathrm{SL}(2, \mathbb{C})$-polystable. Then, the gravitating vortex equations admit a solution with coupling constant α for any Kähler class such that $Vol_g > 4\pi N/\tau$.

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Idea: apply continuity method in the parameter α starting at Yang's solution for $\alpha = 1/\tau N$ ($c = 0 \sim$ cosmic strings).

Difficulties:

- Yang's result only holds for $V_g \gg 4\pi N/\tau$ when $D \neq \frac{N}{2}p_0 + \frac{N}{2}p_1$,
- S^1-symmetry potentially obstructs openness when $D = \frac{N}{2}p_0 + \frac{N}{2}p_1$,
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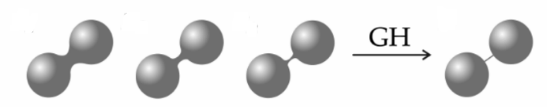

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Proposition (GF - Pingali - Yao '21)

Let $D = \sum_i n_i p_i$ be the effective divisor on $\mathbb{P}^1$ corresponding to (L, ϕ). Assume that D is polystable with respect to the $\mathrm{SL}(2, \mathbb{C})$-action.

Then, the existence of gravitating vortices is an open condition for $\alpha > 0$.

Proof: when D is stable, the proof follows from the moment map framework applying the Implicit Function Theorem.

For $D = \frac{N}{2} p_0 + \frac{N}{2} p_1$ we apply the following theorem (Lebrun-Simanca type argument)

Theorem (Álvarez-Cónsul - GF - García-Prada '13, GF - Pingali - Yao '21)

Given a solution of the gravitating vortex equations with $\alpha > 0$, for any nearby $\alpha' \sim \alpha$ there exists an *extremal pair* with coupling constant α'.

The proof follows from the vanishing of the Futaki invariant, which implies that an *extremal pair* with coupling constant α' is a gravitating vortex. □

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- Need to rule out the formation of singularities on a sequence of solutions when taking $\alpha_k \to \alpha_0 > 0$.

This is achieved in several steps. The first is the following:

Lemma (GF - Pingali - Yao '21)

A tuple $(\Sigma, L, \phi, g, h, \alpha, \tau)$ solving the gravitating vortex equations with $g(\Sigma) = 0$ is equivalent to (g, η, Φ) such that

1. g is a smooth Riemannian metric on S^2,
2. η is a smooth closed real 2-form on S^2 such that $\int_{S^2} \eta = 2\pi N$,
3. $\Phi \in C^\infty(S^2)$ is a non-negative function $\Phi \geqslant 0$ (*state function*), vanishing precisely at the $p_j \in S^2$, and such that $\log \Phi \in L^1_{loc}(S^2)$,

solving the *RGV system*

$$\eta + \frac{1}{2}(\Phi - \tau) vol_g = 0, \qquad \Delta_g \log \Phi = (\tau - \Phi) - 4\pi \sum_j n_j \delta_{p_j} \qquad (6)$$

$$S_g + \alpha(\Delta_g + \tau)(\Phi - \tau) = c.$$

$\Sigma = (S^2, J)$ is the Riemann surface determined by g, $|\phi|_h^2 = \Phi$, and $\eta = iF_h$.

- Need to rule out the formation of singularities on a sequence of solutions when taking $\alpha_k \to \alpha_0 > 0$.

The second step is the following key *a priori estimates*:

Theorem (GF - Pingali - Yao '21)

Let (g, η, Φ) be a smooth solution of RGV. Then,

1. $0 \leqslant \Phi \leqslant \tau$
2. $\frac{1}{2\pi}\int_{S^2} \Phi vol_g = \tau - 2N$
3. (*scalar curvature estimate*)
$$c \leqslant S_g \leqslant \frac{(3 + 2\alpha\tau)\tau}{2}$$
4. (*state function estimate*)
$$-\frac{\tau^2}{4} \leqslant -\Delta_g \Phi = \frac{|\nabla\Phi|^2}{\Phi} - \Phi(\tau - \Phi) \leqslant \frac{1}{\alpha}\left(\frac{3\tau}{2} - c\right)$$

The third step is given by *Cheeger-Gromov's Theory*:

Proposition (GF - Pingali - Yao '21)

Let (g_n, η_n, Φ_n) be a sequence of solutions of RGV with $\alpha_n \to \alpha_0$, where $0 < \alpha_0 < 1/\tau N$.

Then, there exists a sequence $\sigma_n \in \mathrm{SL}(2, \mathbb{C})$ (for a suitable complex structure) such that $\sigma_n^*(g_n, \eta_n, \Phi_n)$ converges in $C^{1,\beta}$ sense to a smooth solution of RGV with constant α_0 and divisor $D_\infty \in \mathrm{SL}(2, \mathbb{C}) \cdot D$.

Proof: By a priori estimates, the volume is bounded along the sequence

$$2\pi \leqslant Vol_{g_n} \leqslant 2\pi e^{2\alpha_n \tau},$$

and the diameter of g_n is bounded from above by Bonnet's estimate

$$diam(S^2, g_n) \leqslant \frac{\pi}{\sqrt{c_n e^{-2\alpha_n \tau}}} =: D_n.$$

By Relative Volume Comparison, the volume ratio is bounded from below:

$$\frac{Vol_{g_n} B(p, r)}{\pi r^2} \geqslant \frac{Vol_{g_n} B(p, D_n)}{\pi D_n^2} \geqslant \frac{2}{D_n^2}, \quad \forall p \in S^2, r \in (0, D_n],$$

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By Cheeger-Gromov-Taylor we have an estimate on the injectivity radius:

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independent of n.

Finally, by a priori estimates we have uniform bounds for S_n and $|\nabla_n S_n|_n$ along the sequence. The statement follows by Cheeger-Gromov compactness combined with a slice theorem for complex structures.

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Remark: the uniform bounds for $|\nabla_n S_n|_n$ require to introduce a sequence of rescaled metric $k_n = e^{2\alpha_n \Phi_n}$, but let me ignore that ...

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Thank you!