Geometry of vortices on Riemann surfaces IV

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1. Kähler–Yang–Mills equations

- M compact complex manifold with $\dim_{\mathbb{C}} M = n$
- $E \to M$ holomorphic vector bundle over M

Kähler–Yang–Mills equations

for a Kähler metric g on M and a Hermitian metric h on E:

$$i\Lambda_g F_h = \lambda \,\mathrm{Id}_E$$
$$S_g - \alpha\Lambda_g^2 \,\mathrm{Tr}\, F_h^2 = c$$

- $F_h \in \Omega^2(M, \mathrm{End}(E,h))$ curvature of Chern connection of h on E
- $\Lambda_g F_h \in \Omega^0(M, \mathrm{End}(E,h))$ contraction of F_h with the Kähler form ω_g determined by g
- S_g scalar curvature of g
- $\alpha > 0$ coupling constant
- $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$ determined by the topology

- Taking traces in the first equation and integrating over M:

$$\lambda = \frac{2\pi}{\mathrm{vol}_g(M)} \frac{\deg_g(E)}{\mathrm{rank}\, E}$$

where $\deg_g(E) = \frac{1}{(n-1)!} \int_M c_1(E)\omega_g^{n-1}$

- Integrating the second equation over M:

$$c\, \mathrm{vol}_g(M) = \int_M (S_g - \alpha\Lambda_g^2 \operatorname{Tr} F_h^2)$$

$$= \deg_g(TM) - \alpha \int_M ch_2(E) \wedge \omega_g^{n-2}$$

- **These equations were introduced in**: [AGG2013] L. Álvarez-Cónsul, M. García-Fernández and O. García-Prada, Coupled equations for Kähler metrics and Yang–Mills connections, *Geometry and Topology* **17** (2013) 2731–2812.

  Based on: Mario García-Fernández's PhD Thesis (2009).

2. Moment map interpretation of KYM equations

Let E be C^∞ complex vector bundle over M and fix:

- h Hermitian metric on E
- ω symplectic form on M

Define ∞-dimensional manifolds:

$$\mathscr{J} := \{\text{complex structures } J : TM \to TM \text{ on } M\}$$
$$\mathscr{A} := \{\text{unitary connections on } (E, h)\}$$

Define $\mathscr{P} :=$ pairs $(J, A) \in \mathscr{J} \times \mathscr{A}$ such that:

- A induces a holomorphic structure on E over (M, J)
- (M, J, ω) is Kähler

$\mathscr{J}$ and $\mathscr{A}$ have canonical symplectic structures $\omega_{\mathscr{J}}$ and $\omega_{\mathscr{A}}$

Symplectic form on $\mathscr{P}$: $\omega_\alpha := (\omega_{\mathscr{J}} + \alpha\omega_{\mathscr{A}})|_{\mathscr{P}}$ for fixed $\alpha \neq 0$

Group action?

Atiyah–Bott–Donaldson:

- $\mathscr{G}$ group of automorphisms of (E, h) covering the identity on M
- $\mathscr{G}$ acts symplectically on $(\mathscr{A}, \omega_{\mathscr{A}})$ with moment map $\mu_{\mathscr{A}} : \mathscr{A} \to (\mathrm{Lie}\,\mathscr{G})^*$ such that

$$\mu_{\mathscr{A}}(A) = 0 \Longleftrightarrow i\Lambda F_h = \lambda\,\mathrm{Id}_E$$

Fujiki–Donaldson–Quillen:

- $\mathscr{H} := \{\text{Hamiltonian symplectomorphisms } (M, \omega) \to (M, \omega)\}$
- $\mathscr{H}$ acts symplectically on $(\mathscr{J}, \omega_{\mathscr{J}})$ with moment map $\mu_{\mathscr{J}} : \mathscr{J} \to (\mathrm{Lie}\,\mathscr{H})^*$ such that

$$\mu_{\mathscr{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

Hamiltonian extended gauge group $\widetilde{\mathscr{G}}$:
Automorphisms of (E,h) covering Hamiltonian symplectomorphims of (M,ω)

$$\begin{array}{ccc} (E,h) & \xrightarrow{g} & (E,h) \\ \downarrow & & \downarrow \\ (M,\omega) & \xrightarrow{\check{g}} & (M,\omega) \end{array}$$

Extension

$$1 \to \mathscr{G} \to \widetilde{\mathscr{G}} \to \mathscr{H} \to 1$$

$\mathscr{G}$: group of automorphisms of (E,h) covering the identity on M
$\mathscr{H}$: group of Hamiltonian symplectomorphisms of (M,ω)

- $\widetilde{\mathscr{G}}$ acts on $\mathscr{J}$ via $\widetilde{\mathscr{G}} \to \mathscr{H}$: $g \mapsto \check{g}$
- $\widetilde{\mathscr{G}}$ acts on $\mathscr{A}$ in the usual way

Proposition

The action of $\widetilde{\mathscr{G}}$ on $(\mathscr{P}, \omega_\alpha)$ has moment map

$$\mu_\alpha : \mathscr{P} \to (\mathrm{Lie}\,\widetilde{\mathscr{G}})^*$$

such that

$$\mu_\alpha(J, A) = 0 \iff \text{solution to Kähler–Yang–Mills equations}$$

For $\alpha > 0$, $(\mathscr{P}, \omega_\alpha)$ has a canonical $\widetilde{\mathscr{G}}$-invariant Kähler structure

Moduli $\mathcal{M}_\alpha :=$ {solutions to Kähler–Yang–Mills equations}$/\widetilde{\mathscr{G}}$ is Kähler for $\alpha > 0$

Remarks:

- We recover the Hermitian–Yang–Mills equations, while the equation $S_{\omega,J} =$ constant (Yau–Tian–Donaldson theory) is deformed

- The coupling term in the second equation comes precisely from the non-triviality of the extension defining $\widetilde{\mathscr{G}}$.

- Equations 'decouple' for $\dim_{\mathbb{C}} M = 1$ (as $F_h^2 = 0$ in this case)

- Solutions to the Kähler–Yang–Mills equations are absolute minima of a certain **Calabi–Yang–Mills** functional

Programme: Study existence of solutions

- Very hard problem: In general, this is a system of coupled fourth-order fully non linear partial differential equations!
- Motivation: Analytic approach to the algebraic geometric problem of studying the moduli space classifying pairs (X, E) consisting of a projective variety and a holomorphic vector bundle
- In the paper [AGG2013] we give some existence results for small α, by perturbation from constant scalar curvature Kähler metrics and Hermitian–Yang–Mills connections
- More concrete and interesting solutions — over a polarised threefold not admitting any constant scalar curvature Kähler metric — were obtained by Keller and Tønnesen–Friedman (2012)
- Garcia-Fernandez–Tipler (2013) added new examples to this short list by simultaneous deformation of the complex structures of M and E

- In [AGG2013], we also study **obstructions** for the existence of solutions, generalizing the Futaki invariant, the Mabuchi K-energy and geodesic stability (Chen, Donaldson) that appear in the constant scalar curvature theory
- **Conjecture**: Existence of solutions to Kähler–Yang–Mills equations is equivalent to geodesic stability.
- Test this in a simpler situation where there is a group of symmetries acting on the picture: **dimensional reduction**
- One of the simplest cases to consider is $M = X \times \mathbb{P}^1$, where X is a Riemann surface, with SU(2) as group of symmetries
- **This study was initiated in**:

  [AGG2017] L. Álvarez-Cónsul, M. García-Fernández and O. García-Prada, Gravitating vortices, cosmic strings and the Kähler–Yang–Mills equations, *Comm. Math. Phys.* **351** (2017) 361–385.

  Building upon:

  O. García-Prada, Invariant connections and vortices, *Comm. Math. Phys.*, **156** (1993) 527–546.

3. Dimensional reduction: Gravitating vortex equations

- X compact Riemann surface
  $L \to X$ holomorphic line bundle over X
  $\varphi \in H^0(X, L)$ holomorphic section of L
- To the pair (L, φ) we can sssociate a rank 2 holomorphic vector bundle E over $X \times \mathbb{P}^1$:

$$0 \to p^* L \to E \to q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

  where $p : X \times \mathbb{P}^1 \to X$ and $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ natural projections
- Extensions as above are parametrized by

$$\begin{aligned} H^1(X \times \mathbb{P}^1, p^* L \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-2)) &\cong H^0(X, L) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \\ &\cong H^0(X, L) \end{aligned}$$

  by Künneth formula, and Serre duality
  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong \mathbb{C}$

SU(2)-**action**:

- SU(2) acts on $X \times \mathbb{P}^1$:
  Trivially on X and $\mathbb{P}^1 = \mathrm{SU}(2)/\mathrm{U}(1)$
- SU(2)-action can be lifted to E:
  Trivially on p^*L and standard on $q^*\mathcal{O}_{\mathbb{P}^1}(2)$
  Trivial action of SU(2) on $H^0(X, L)$
  $\Longrightarrow$ E is a SU(2)-equivariant holomorphic vector bundle
- SU(2)-invariant Kähler metrics on $X \times \mathbb{P}^1$ have the shape

$$\omega_\tau = p^*\omega_X \oplus \frac{4}{\tau} q^*\omega_{\mathbb{P}^1}^{FS}$$

$\tau > 0$; $\omega_{\mathbb{P}^1}^{FS}$ Fubini–Study metric (volume normalised to 2π)

- **Proposition**: Let E be the SU(2)-equivariant rank 2 holomorphic vector bundle over $X \times \mathbb{P}^1$ defined by (L, φ). A SU(2)-invariant solution to the Kähler–Yang–Mills equations on $E \to X \times \mathbb{P}^1$ is equivalent to a solution of the following equations:

Gravitating vortex equations

for a metric g on X and a Hermitian metric h on L

$$i\Lambda_g F_h + |\varphi|_h^2 - \tau = 0$$
$$S_g + \alpha(\Delta_g + \tau)(|\varphi|_h^2 - \tau) = c$$

Gravitating vortex = solution of the gravitating vortex equations

- $\tau > 0$, $\alpha > 0$ real parameters
- c is determined by the topology

- The first equation

$$i\Lambda_g F_h + |\varphi|_h^2 - \tau = 0$$

is the **abelian vortex equation**

- Integrating it, we obtain

$$2\pi \deg L + \|\varphi\|_{L^2}^2 = \tau \operatorname{vol}_g(X)$$

Assuming $\varphi \neq 0$, this implies that in order to have solutions we must have

$$\deg L \leq \frac{\tau \operatorname{vol}_g(X)}{2\pi}$$

Theorem (Noguchi, 1987; Bradlow, 1990; GP, 1991)

Existence of solutions to the vortex equation
$\iff \quad \deg L \leq \frac{\tau \operatorname{vol}_g(X)}{2\pi}$

- Coming back to the gravitating vortex equations, by integrating the first equation we have

$$\int_X (\varphi|_h^2 - \tau) = \frac{2\pi \deg(L)}{\mathrm{vol}_g(X)}$$

- On the other hand

$$\int_X \Delta_g(|\varphi|_h^2 - \tau) = 0 \quad \text{and} \quad \int_X S_g = \frac{2\pi\chi(X)}{\mathrm{vol}_g(X)}$$

- With this, and integrating the second equation we have

$$c = \frac{2\pi}{\mathrm{vol}_g(X)}(\chi(X) - \alpha\tau \deg(L))$$

In particular we have

$$c \geq 0 \Longrightarrow X = \mathbb{P}^1$$

Physics: cosmic strings and topological defects

- When $c = 0$ our equations are known in the physics literature as **Einstein–Bogomol'nyi equations** also known as **self-dual Einstein–Maxwell–Higgs equations** and their solutions are called **Nielsen–Olesen cosmic strings**
- Cosmic strings are a model (by spontaneous symmetry breaking) for **topological defects** in the early universe.
- $\alpha = 2\pi G$, $G > 0$ is universal gravitation constant
- The abelian vortex equation appears in the **theory of superconductivity**

Physics literature: Linet (1988), Comtet–Gibbons (1988), Spruck–Yisong Yang (1995), Yisong Yang (1995) ...

4. Existence of solutions when $c = 0$

Theorem (Yisong Yang, 1995, 1997)

Let $D = \sum n_i p_i$ be an effective divisor on $\mathbb{P}^1$ corresponding to a pair (L, φ) s.t. $c = 0$ and $\deg L = N = \sum n_i < \frac{\tau \operatorname{vol}_g(X)}{2\pi}$. Then the Einstein–Bogomol'nyi equations on $(\mathbb{P}^1, L, \varphi)$ have solutions if

$$n_i < \frac{N}{2} \text{ for all } i. \qquad (*)$$

A solution also exists if $D = \frac{N}{2}p_1 + \frac{N}{2}p_2$, with $p_1 \neq p_2$ and N even.

- Fix metrics g_0 on X and h_0 on L and solve for $g = e^{2u}g_0$ and $h = e^{2f}h_0 \implies$ the gravitating vortex equations are equivalent to equations for $f, u \in C^\infty(X)$:

$$\Delta_{g_0} f + e^{2u}(e^{2f}|\varphi|^2_{h_0} - \tau) = -\frac{2\pi \deg L}{\operatorname{vol}_{g_0}(X)}$$

$$\Delta_{g_0}(u + \alpha e^{2f} - 2\alpha\tau f) + c(1 - e^{2u}) = 0$$

- $c = 0 \Longrightarrow u = \text{const.} - \alpha e^{2f} + 2\alpha\tau f \Longrightarrow$ plug u in the first equation.
- Yang applies the continuity method to solve the resulting Kazdan–Warner type equation, finding it suffices to assume

$$n_i < \frac{N}{2} \text{ for all } i, \qquad (*)$$

or $D = \frac{N}{2}p_1 + \frac{N}{2}p_2$, with $p_1 \neq p_2$ and N even.
- Yang (1995) mentions that $(*)$ "*is a technical restriction on the local string number. It is not clear at this moment whether it may be dropped*".
- In fact $*)$ has an algebraic-geometric meaning that comes from the **geometry** of the problem.
- **This is done in the paper**:
[AGGP2021] L. Álvarez-Cónsul, M. García-Fernández, O. García-Prada, and V.P. Pingali, Gravitating vortices and the Einstein–Bogomol'nyi equations, *Math. Ann.* **379** (2021) 1651–1684.

- Yang's "technical restriction" has an **algebro-geometric meaning**, for the natural action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{Sym}^N \mathbb{P}^1 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(N))$ — the **moduli space of vortices** on $\mathbb{P}^1$ with vortex number N:

  $n_i < \frac{N}{2}$ for all $i \Longleftrightarrow D \in \mathrm{Sym}^N \mathbb{P}^1$ is GIT stable
  $D = \frac{N}{2}p_1 + \frac{N}{2}p_2 \Longleftrightarrow D \in \mathrm{Sym}^N \mathbb{P}^1$ is strictly GIT polystable
  In [AGGP2021] we claim:

Theorem

If there exists a solucion to the gravitating vortex equations on $(\mathbb{P}^1, L, \varphi)$ then the divisor D defined by (L, φ) is GIT polystable. In particular, the converse of Yang's theorem also holds.

- However, few months ago we discovered a **gap of our proof** and this is now in the process of being fixed in collaboration with Chengjian Yao. This will appear soon.

5. Gravitating vortices for $g \geq 2$

Theorem ([AGGP2021])

Let X be a compact Riemann surface with $g \geq 2$. Let L a holomorphic line bundle over X of degree N equipped with a holomorphic section $\varphi \neq 0$. Let $\tau > 0$ a real constant such that $0 < N < \tau$. Define

$$0 < \alpha_* = \frac{2g-2}{\tau(\tau - N)}.$$

The the set of $\alpha \geq 0$ for which the gravitating vortex equations have a smooth solution with volume 2π is open and contains $[0, \alpha_*]$. Furthermore, the solution is unique in $[0, \alpha_*]$.

- The proof involves the continuity method. Openness uses the symplectic point of view presented above, while closedness requires an *a priori* estimates as usual. The bound on α is needed for this.

- Analogue of the uniformization theorem for pairs (X, D) consisting of a compact Riemann surface X of $g \geq 0$ and an effective divisor D on X

6. Gravitating vortices for $g = 0$ and $c \neq 0$

In the more recent paper

M. Garcia-Fernandez, V.P. Pingali and C. Yao, Gravitating vortices with positive curvature, *Advances in Mathematics* **388** (2021).

the authors give a complete solution to the case $c > 0$.

- More on gravitating vortices in talks next week!

THANK YOU!