Geometry of vortices on Riemann surfaces III

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1. Vortices and Higgs bundles

- X compact Riemann surface
- **Higgs bundle** (**Hitchin** 1987): pair (E, φ)
  - $E \to X$ holomorphic vector bundle
  - $\varphi : E \to E \otimes K$, K **canonical line bundle** of X
- $\mathcal{M}(n, d)$ **Moduli space of polystable Higgs bundles** of rank n and degree d
- **Hitchin equations**:

$$F_h + [\varphi, \varphi^*] = \mu \, \mathrm{Id}_E, \qquad \mu \in \Omega^2(X)$$

- **Non-abelian Hodge correspondence** (Hitchin 1987, Donaldson 1987, Simpson 1988, Corlette 1988): $\mathcal{M}(n, d)$ homeomorphic to moduli of representations of (the universal central extension of) the fundamental group of X in $\mathrm{GL}(n, \mathbb{C})$
- To study topology of $\mathcal{M}(n, d)$ localize at **fixed points** of $\mathbb{C}^*$-**action**:

$$\lambda \cdot (E, \varphi) := (E, \lambda\varphi) \qquad \lambda \in \mathbb{C}^*$$

- Fixed points:
  - $\varphi = 0$, E polystable bundle
  - $\varphi \neq 0 \iff E = \oplus E_i \quad \varphi|_{E_i} : E_i \to E_{i+1} \otimes K$

  Gauge symmetry breaking: $\mathrm{U}(n) \rightsquigarrow \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_m)$
- Moduli spaces of chains!
- Topology of moduli of chains $\longleftrightarrow$ Topology of $\mathcal{M}(r,d)$

  When n and d are coprime $\mathcal{M}(n,d)$ is smooth and can compute:
- $n = 2$ **Hitchin** 1987 (Poincaré polynomial)
- $n = 3$ **Gothen** 1994 (Poincaré polynomial)
- $n = 4$ **GP–Heinloth–Schmitt** 2011 (motive)
- arbitrary n: **GP–Heinloth** 2013 (recursive formula for the motive)
- arbitrary n: **Bradlow-GP–Gothen** 2008 (homotopy groups)
- Variation of vortex parameters play a central role in these works!

2. Higgs pairs

- X compact Riemann surface of genus $g \geq 2$ with canonical line bundle K
- G reductive complex Lie group with Lie algebra $\mathfrak{g}$
- $\rho : G \to \mathrm{GL}(V)$ a representation of G in a complex vector space V
- A (G, V)-**Higgs pair on** X is a pair (E, φ) consisting of a holomorphic principal G-bundle $E \to X$ and
  $\varphi \in H^0(X, E(V) \otimes K)$,
  where $E(V) = E \times_G V$ is the vector bundle associated to the representation ρ.
- There are suitable notions of σ-**(semi,poly)stability** for any $\sigma \in i\mathfrak{z}_{\mathbb{R}}$, where $\mathfrak{z} = \mathfrak{z}_{\mathbb{R}} \oplus i\mathfrak{z}_{\mathbb{R}}$ is the centre of $\mathfrak{g}$.
  $\mathcal{M}_\sigma(G, V)$: **moduli space of** σ**-polystable** (G, V)**-Higgs pairs**.
  We write $\mathcal{M}(G, V)$ when $\sigma = 0$.

- When ρ is the **adjoint representation** $G \to \mathrm{GL}(\mathfrak{g})$ $(G, \mathfrak{g})$-Higgs pairs are the G-**Higgs bundles** introduced by **Hitchin** (1987).

$$\mathcal{M}(G) \ : \ \textbf{moduli space of polystable } G\textbf{-Higgs bundles}$$

- We may twist by any line bundle L in our definition of Higgs pairs of type (G, V), including the trivial line bundle (no twisting! Like in the vortex situation).
  We consider twisting by K in preparation for a relation of Higgs pairs of certain type to G-Higgs bundles that we will discuss.

Theory introduced by **Mikio Sato** in the early 1970s

G Complex reductive Lie group

- A **prehomogeneous vector space** (phvs) for G is a complex finite dimensional vector space V together with a holomorphic representation $\rho : G \to \mathrm{GL}(V)$ such that there exists an **open** G-orbit Ω in V. Such an open orbit is necessarily unique and dense.
- If V is a phvs, let Ω denote the open orbit in V and $S = V \setminus \Omega$ be the **singular set**.
  For $x \in V$, denote the G-stabilizer of x by G^x.
  A phvs vector space V is called **regular** if G^x is reductive for $x \in \Omega$, otherwise it is called **nonregular**.

- **Example 1**:
  The vector space $\mathbb{C}^n$ is a phvs for the standard representation of $\mathrm{GL}(n, \mathbb{C})$. For this example, $\Omega = \mathbb{C}^n \setminus \{0\}$, and it is regular only when $n = 1$.
- **Example 2**:
  The vector space $M_{p,q}$ of $p \times q$-matrices is a phvs for the action of $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$ given by

  $$(A, B) \cdot M = AMB^{-1}.$$

  Here, $\Omega = \{M \in M_{p,q} \mid \mathrm{rank}(M) = \min(p, q)\}$. This example is regular only when $p = q$.
- Example 1 is related to Bradlow pairs, while example 2 is related to triples.
- A phvs V is regular if and only if $S = V \setminus \Omega$ is **hypersurface**.

- Let V be a prehomogeneous vector space for G with representation ρ. A non-constant rational function $F : V \to \mathbb{C}$ is called a **relative invariant** for the action of G if there exists a **character** $\chi : G \to \mathbb{C}^*$ such that

$$F(\rho(g)x) = \chi(g)F(x) \quad \text{for every} \quad g \in G \quad \text{and} \quad x \in V.$$

- Up to a constant, a relative invariant is uniquely determined by its corresponding character. In particular, any relative invariant is a **homogeneous function**.
- Let $\chi : G \to \mathbb{C}^*$ be a character. Then there is a relative invariant for χ if and only if χ is trivial on the stabilizers of points in Ω, i.e., $\chi|_{G^x} = 1$ for all $x \in \Omega$.

- **Example**: The regular phvs $M_{p,p}$ from Example 2 has a relative invariant $F : M_{p,p} \to \mathbb{C}$ given by $F(M) = \det(M)$. The associated character $\chi : G \to \mathbb{C}^*$ is given by

$$\chi(A, B) = \det(A) \det(B)^{-1}$$

since

$$F((A, B) \cdot M) = \det(AMB^{-1}) = \chi(A, B) F(M).$$

4. $\mathbb{Z}$-gradings and prehomogeneous vector spaces

- G semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and Killing form B.
- A $\mathbb{Z}$-**grading** of $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \qquad \text{such that} \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

  There is an element $\zeta \in \mathfrak{g}_0$ (**grading element**) such that $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [\zeta, x] = ix\}$
- Let $G_0 < G$ be the centralizer of ζ; G_0 acts on each $\mathfrak{g}_i$ by the adjoint action.

  Important result due to **Vinberg** (1975): For $i \neq 0$, $\mathfrak{g}_i$ is a **prehomogeneous vector space** for the action of G_0.

5. $\mathbb{Z}$-gradings and the Toledo character

In the remaining I describe recent results due to **Biquard–Collier–GP–Toledo**.

- Without loss of generality, we can consider the prehomogeneous vector space $(G_0, \mathfrak{g}_1)$. Let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit.
- Since $\mathfrak{g}_0$ is the centralizer of ζ, $B(\zeta, -) : \mathfrak{g}_0 \to \mathbb{C}$ defines a character. The **Toledo character** $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by
$$\chi_T(x) = B(\zeta, x)B(\gamma, \gamma) \ ,$$
where γ is the longest root such that $\mathfrak{g}_\gamma \subset \mathfrak{g}_1$.
- Let $e \in \mathfrak{g}_1$ and (h, e, f) be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. We define the **Toledo rank** of e by
$$\mathrm{rank}_T(e) = \frac{1}{2}\chi_T(h),$$
and the **Toledo rank** of $(G_0, \mathfrak{g}_1)$ by
$$\mathrm{rank}_T(G_0, \mathfrak{g}_1) = \mathrm{rank}_T(e) \ \text{ for } e \in \Omega.$$

6. $\mathbb{Z}$-gradings and Hodge bundles

- For a $\mathbb{Z}$-grading we consider $(G_0, \mathfrak{g}_i)$-**Higgs pairs** over X. Let (E, φ) be a $(G_0, \mathfrak{g}_i)$-Higgs pair. Extending the structure group defines a G-Higgs bundle (E_G, φ), where $E_G = E \times_{G_0} G$, and we use $E(\mathfrak{g}_i) \subset E_G(\mathfrak{g})$.
- A G-Higgs bundle (E, φ) is called a **Hodge bundle** of type $(G_0, \mathfrak{g}_i)$ if it reduces to a $(G_0, \mathfrak{g}_i)$-Higgs pair.
- A result of **Simpson** (1992) states that the $\mathbb{C}^*$-**fixed points** in the moduli space of G-Higgs bundles (under the action of rescaling the Higgs field) are Hodge bundles for some $\mathbb{Z}$-grading.
- Via de non-abelian Hodge correspondence, Hodge bundles correspond to holonomies of **complex variations of Hodge structure**.

7. Toledo invariant

- Let (E, φ) be a $(G_0, \mathfrak{g}_1)$-Higgs pair and $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ be the **Toledo character** associated to $(G_0, \mathfrak{g}_1)$.
  For a **rational number** q sufficiently large $q\chi_T$ lifts to a character $\widetilde{\chi}_T : G_0 \to \mathbb{C}^*$.
- The **Toledo invariant** $\tau(E, \varphi)$ is defined by

$$\tau(E, \varphi) = \frac{1}{q} \deg_{\widetilde{\chi_T}}(E).$$

Theorem (Biquard–Collier–GP–Toledo, 2021)
Let (E, φ) be $(G_0, \mathfrak{g}_1)$-Higgs pair over X. Assume for simplicity that there is no twisting by K. Let $\zeta \in \mathfrak{g}_0$ be the grading element and $\sigma = \alpha\zeta$ for $\alpha \in \mathbb{R}$.

- If (E, φ) is α-semistable then the Toledo invariant $\tau(E, \varphi)$ satisfies the following inequality

$$\alpha(B(\gamma,\gamma)B(\zeta,\zeta) - \operatorname{rank}_T(\varphi)) \leq \tau(E,\varphi) \leq \alpha B(\gamma,\gamma)B(\zeta,\zeta).$$

- In particular, let $\alpha_m = \frac{\tau(E,\varphi)}{B(\gamma,\gamma)B(\zeta,\zeta)}$ and

$$\alpha_M = \frac{\tau(E,\varphi)}{B(\gamma,\gamma)B(\zeta,\zeta) - \operatorname{rank}_T(\hat{G}_0, \hat{\mathfrak{g}}_1)} \quad \text{if } (G_0, \mathfrak{g}_1) \text{ is non-regular}$$

or ∞ in the regular case (where $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ is maximal regular sub phvs of $(\hat{G}_0, \hat{\mathfrak{g}}_1)$. Then

$$\alpha_m \leq \alpha \leq \alpha_M.$$