Geometry of vortices on Riemann surfaces II

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1. Vortices and the Hermitian–Yang–Mills equations

- (M, ω_M) compact Kähler manifold with $\dim_{\mathbb{C}} M = n$ (normalisation $\mathrm{vol}(M) = 2\pi$)
- $E \to M$ holomorphic vector bundle over M

Hermitian–Yang–Mills equations

for a Hermitian metric h on E:

$$i\Lambda F_h = \mu \, \mathrm{Id}_E$$

- $F_h \in \Omega^2(M, \mathrm{End}(E,h))$ curvature of **Chern connection** of h on E
- $\Lambda F_h \in \Gamma(M, \mathrm{End}(E,h))$ contraction of F_h with ω
- Taking traces in the equation and $\int_M (-) \, \mathrm{dvol}_M$:

$$\mu = \mu(E) = \frac{\deg_{\omega_M}(E)}{\mathrm{rank}\, E}$$

where $\deg_{\omega_M}(E) = \int_M c_1(E) \omega_M^{n-1}$

Recall **Hitchin–Kobyashi correspondence**:

Definition (Mumford–Takemoto)

- E is **stable** if $\mu(E') < \mu(E)$ for every coherent subsheaf $0 \neq E' \subsetneq E$
- E is **polystable** if $E \cong \oplus E_i$ with E_i stable of the same slope

Donaldson–Uhlenbeck–Yau (1986–87)

$\exists$ of solutions to the HYM equations on $E \iff E$ polystable

Irreducible solution $\iff E$ stable

Relation of the Hermitian–Yang–Mills equations to vortices

- First, we will look at the vortex equations from an equivalent point of view: Fix a **holomorphic** line bundle L and a **holomorphic** section $\varphi \in H^0(X, L)$, and look for a Hermitian metric h satisfying

$$i\Lambda F_h + |\varphi|_h^2 - \tau = 0$$

  where $F_h \in \Omega^2(X)$ is the curvature of the Chern connection of h on L

- Fix a pair (L, φ) as above over compact Riemann surface X Associated to (L, φ) there is rank 2 holomorphic vector bundle E over $X \times \mathbb{P}^1$:

$$0 \to p^* L \to E \to q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

$p : X \times \mathbb{P}^1 \to X$ and $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ natural projections

- Extensions as above are parametrized by

$$H^1(X \times \mathbb{P}^1, p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(X, L) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$$
$$\cong H^0(X, L)$$

  by Künneth formula, and Serre duality
  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong \mathbb{C}$
- SU(2)-**action** on $X \times \mathbb{P}^1$: Trivially on X and $\mathbb{P}^1 = \mathrm{SU}(2)/\mathrm{U}(1)$
- SU(2)-action can be lifted to E: trivially on p^*L and standard on $q^*\mathcal{O}_{\mathbb{P}^1}(2) \Longrightarrow E$ is a SU(2)-**equivariant** holomorphic vector bundle
- SU(2)-**invariant Kähler metric** on $X \times \mathbb{P}^1$

$$\omega_\tau = p^*\omega_X \oplus \frac{4}{\tau} q^*\omega_{\mathbb{P}^1}^{FS}$$

  $\tau > 0$; $\omega_{\mathbb{P}^1}^{FS}$ **Fubini–Study** metric (volume normalised to 2π)

Theorem (existence of vortices on $L \longrightarrow X$)

$$\begin{array}{ccc}
\text{solution to } \tau\text{-vortex on } X & \Longleftrightarrow & \tau > \deg L \\
\Big\Updownarrow \text{GP} & & \Big\Updownarrow \text{GP} \\
SU(2)\text{-inv. } \omega_\tau\text{-HYM metric} & \xLeftrightarrow{\text{Donaldson–Uhlenbeck–Yau}} & E \ \omega_\tau\text{-polystable}
\end{array}$$

- vortices on X $\longleftrightarrow$ SU(2)-**dimensional reduction** of **instantons** on $X \times \mathbb{P}^1$
- Generalises results of **Witten (1977)** for vortices on the **hyperbolic real plane** and **Taubes** for vortices on $\mathbb{R}^2$ (1980)

2. Non-abelian vortices and Bradlow pairs

- Replace L by a **higher rank** vector bundle $V \to X$ $(V, \varphi), \varphi \in H^0(X, V)$
- **Bradlow** (1989): Notion of τ-**stability** for (V, φ) and correspondence existence theorem with solutions to **non-abelian** τ**-vortex equations**:

$$i\Lambda F_h + \varphi \otimes \varphi^{*_h} - \tau \,\mathrm{Id}_V = 0$$

- $(V, \varphi) \quad \longleftrightarrow \quad E \to X \times \mathbb{P}^1$ with

$$0 \to p^* V \to E \to q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

- **GP** 1991: Bradlow τ-stability of $(V, \varphi) \Longleftrightarrow$ Stability of E wrt $\omega_\tau = p^*\omega_X \oplus f(\tau) q^* \omega_{\mathbb{P}^1}^{FS}$ for a certain function $f(\tau) > 0$.
- Moduli of rank n vortices on $X \subset$ Moduli of rank $n+1$ stable bundles on $X \times \mathbb{P}^1$

3. Non-abelian vortices and triples

(**GP**, 1991; **Bradlow–GP**, 1996; **Bradlow–GP–Gothen**, 2004)

- Consider **triples** $T = (E_1, E_2, \varphi)$ over X consisting of holomorphic vector bundles E_1, E_2 over X of ranks n_1 and n_2, and degrees d_1 and d_2 and a homomorphism $\varphi : E_2 \to E_1$.
- These are in one-to-one correspondence with vector bundles over $X \times \mathbb{P}^1$ of the form

$$0 \to p^* E_1 \to E \to p^* E_2 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

- **Vortex equations** for hermitian metrics h_1 and h_2 on E_1 and E_2 respectively

$$i\Lambda F_{h_1} + \varphi\varphi^* = \tau_1 \operatorname{Id}_{E_1},$$
$$i\Lambda F_{h_2} - \varphi^*\varphi = \tau_2 \operatorname{Id}_{E_2},$$

where τ_1 and τ_2 are **real parameters** satisfying

$$d_1 + d_2 = n_1\tau_1 + n_2\tau_2.$$

Here φ^* is the adjoint of φ with respect to the Hermitian metrics h_1 and h_2.

- A homomorphism from $T' = (E_1', E_2', \varphi')$ to $T = (E_1, E_2, \varphi)$ is a commutative diagram

$$\begin{array}{ccc} E_2 & \stackrel{\varphi}{\rightarrow} & E_1 \\ \uparrow & & \uparrow \\ E_2' & \stackrel{\varphi'}{\rightarrow} & E_1'. \end{array}$$

- A triple $T' = (E_1', E_2', \varphi')$ is a **subtriple** of $T = (E_1, E_2, \varphi)$ if the sheaf homomorphims $E_1' \to E_1$ and $E_2' \to E_2$ are injective.
- For any $\alpha \in \mathbb{R}$ the α-**slope** of T is defined as

$$\mu_\alpha(T) = \mu(E_1 \oplus E_2) + \alpha \frac{\operatorname{rank}(E_2)}{\operatorname{rank}(E_1) + \operatorname{rank}(E_2)},$$

where $\deg(E)$, $\operatorname{rank}(E)$ and $\mu(E) = \deg(E)/\operatorname{rank}(E)$ are the degree, rank and slope of E, respectively.

- We say $T = (E_1, E_2, \varphi)$ is α-**stable** if
$$\mu_\alpha(T') < \mu_\alpha(T).$$
for any proper subtriple $T' = (E_1', E_2', \varphi')$.
- We define α-**semistability** by replacing the above strict inequality with a weak inequality.
- A triple is called α-**polystable** if it is the direct sum of α-stable triples of the same α-slope.

Theorem (GP-Bradlow, 1996)

Let τ_1 and τ_2 so that $d_1 + d_2 = n_1\tau_1 + n_2\tau_2$, let $\alpha = \tau_1 - \tau_2$. Then a triple $T = (E_1, E_2, \varphi)$ of type (n_1, n_2, d_1, d_2) has a solution to the vortex equations if and only if T is α-polystable.

- The proof is again an application of the Donaldson–Uhlenbeck–Yau theorem. By showing that the α-polystability of T is equivalent to the polystability of the vector bundle over $X \times \mathbb{P}^1$ associated to T with respect to a Kähler form depending on α.

- Let
$$\mathcal{M}_\alpha = \mathcal{M}_\alpha(n_1, n_2, d_1, d_2)$$
be the **moduli space** of isomorphism classes of α-polystable triples of type (n_1, n_2, d_1, d_2). $\mathcal{M}_\alpha$ is a complex analytic variety, which is **projective** when α is rational (GIT construction by **A. Schmitt**).
- Let $\mu_i = d_i/n_i$ for $i = 1, 2$. We define
$$\begin{aligned}\alpha_m =& \mu_1 - \mu_2, \\ \alpha_M =& (1 + \frac{n_1 + n_2}{|n_1 - n_2|})(\mu_1 - \mu_2), \quad n_1 \neq n_2.\end{aligned}$$
- A **necessary condition** for $\mathcal{M}_\alpha(n_1, n_2, d_1, d_2)$ to be non-empty is
$$\begin{aligned}&0 \leq \alpha_m \leq \alpha \leq \alpha_M \quad \text{if} \quad n_1 \neq n_2, \\ &0 \leq \alpha_m \leq \alpha \quad \text{if} \quad n_1 = n_2.\end{aligned}$$

Fix (n_1, n_2, d_1, d_2).

- We say that $\alpha \in [\alpha_m, \infty)$ is a **critical value** if there exist integers n_1', n_2', d_1' and d_2' such that

$$\frac{d_1' + d_2'}{n_1' + n_2'} + \alpha \frac{n_2'}{n_1' + n_2'} = \frac{d_1 + d_2}{n_1 + n_2} + \alpha \frac{n_2}{n_1 + n_2},$$

  that is,

$$\alpha = \frac{(n_1 + n_2)(d_1' + d_2') - (n_1' + n_2')(d_1 + d_2)}{n_1' n_2 - n_1 n_2'},$$

  with $0 \leq n_i' \leq n_i$, $(n_1', n_2', d_1', d_2') \neq (n_1, n_2, d_1, d_2)$, $(n_1', n_2') \neq (0, 0)$ and $n_1' n_2 \neq n_1 n_2'$. We say that α is **generic** if it is not critical.
- The critical values of α form a **discrete subset** of $\alpha \in [\alpha_m, \infty)$,
- If $n_1 \neq n_2$ the number of critical values is **finite** and lies in the interval $[\alpha_m, \alpha_M]$.

- The stability criteria for two values of α lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.
- If α is generic and $\mathrm{GCD}(n_2, n_1 + n_2, d_1 + d_2) = 1$, then α-semistability is equivalent to α-stability, and the moduli space $\mathcal{M}_\alpha$ is a **smooth projective variety**.
- Using the vortex interpretation of the moduli space of triples one can easily identify the moduli space of triples for $\alpha = \alpha_m$. A triple $T = (E_1, E_2, \varphi)$ is α_m-polystable if and only if $\phi = 0$ and E_1 and E_2 are polystable. We thus have

$$\mathcal{M}_{\alpha_m}(n_1, n_2, d_1, d_2) \cong M(n_1, d_1) \times M(n_2, d_2),$$

where $M(n_i, d_i)$ is the **moduli space of semistable bundles** of rank n_i and degree d_i.

Theorem (Bradlow–GP–Gothen, 2004)

Assume that $n_1 > n_2$ and $d_1/n_1 > d_2/n_2$. Let α_c the largest critical value before α_M. Let $\mathcal{M}_L$ the moduli space of α-polystable triples for $\alpha_c < \alpha < \alpha_M$.
Then the moduli space $\mathcal{M}_L^s = \mathcal{M}_L^s(n_1, n_2, d_1, d_2)$ is smooth of dimension

$$(g-1)(n_1^2 + n_2^2 - n_1 n_2) - n_1 d_2 + n_2 d_1 + 1,$$

and is **birationally equivalent** to a $\mathbb{P}^N$-fibration over $M^s(n_1 - n_2, d_1 - d_2) \times M^s(n_2, d_2)$, where $M^s(n, d)$ is the moduli space of stable bundles of rank n and degree d, and

$$N = n_2 d_1 - n_1 d_2 + n_1(n_1 - n_2)(g-1) - 1.$$

In particular, $\mathcal{M}_L^s(n_1, n_2, d_1, d_2)$ is **non-empty and irreducible**. If $\mathrm{GCD}(n_1 - n_2, d_1 - d_2) = 1$ and $\mathrm{GCD}(n_2, d_2) = 1$, the birational equivalence is an isomorphism. Moreover, in all cases, $\mathcal{M}_L$ is **irreducible** and hence birationally equivalent to $\mathcal{M}_L^s$.

3. Chains (**Álvarez-Cónsul–GP** 2001, **AC–GP–Schmitt** 2006, **GP–Heinloth–Schmitt** 2014, **GP–Heinloth** 2013)

- SU(2)-equivariant holomorphic vector bundles on $X \times \mathbb{P}^1$ are in bijective correspondence with **chains** over X

$$E_m \xrightarrow{\varphi_{m-1}} E_{m-1} \xrightarrow{\varphi_{m-2}} \cdots \xrightarrow{\varphi_1} E_1$$

- Vortex equations and stability involve $m-1$ **real parameters**. Moduli spaces of chains are compact, and are smooth for generic values of the stability parameters.
- **Quiver bundles** (**Álvarez-Cónsul–GP** 2003)
Let G be a complex reductive group and $P \subset G$ be a parabolic subgroup. Consider the flag variety G/P.
G-**equivariant bundles** over $X \times G/P$ are in bijective correspondence with certain **quiver bundles with relations** on X for a quiver and relations determined by the parabolic subgroup P.