Geometry of vortices on Riemann surfaces I

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1. The abelian Higgs model on $\mathbb{R}^2$

- A: $\mathrm{U}(1)$-connection on the trivial complex line bundle $\mathbb{R}^2 \times \mathbb{C}$ — the **potential**
- $\varphi : \mathbb{R}^2 \to \mathbb{C}$ smooth function — the **Higgs field**
- **Yang–Mills–Higgs** functional:

$$\mathrm{YMH}(A, \varphi) = \int_{\mathbb{R}^2} |F_A|^2 + |d_A \varphi|^2 + \frac{\lambda}{4}(1 - |\varphi|^2)^2,$$

- F_A is the curvature of A and d_A is the covariant derivative
- λ is a **real parameter**

- Finite action $\implies$ (A, φ) must satisfy:

$$|\varphi| \to 1, \quad |d_A\varphi| \to 0 \;\text{ and }\; |F_A| \to 0, \quad \text{as } |x| \to \infty$$

- $\varphi/|\varphi|$ defines a map from a large circle in $\mathbb{R}^2$ to the unit circle, whose degree d is the **vortex charge** or **vortex number**.

- This functional first appeared in the **Ginzburg–Landau** model (1950) of **superconductivity**: macroscopic theory
- It is one of the first instances of **gauge symmetry breaking**. Analogous to the gauge symmetry breaking described by **Higgs** (1964) and others in the context of the **electroweak interaction**
- **Bardeen–Cooper–Schrieffer** (1957) gave a microscopic theory: $|\varphi|^2$ is the density of **Cooper pairs**
- $\lambda < 1$: superconductors of **type I**
- $\lambda > 1$: superconductors of **type II**
- **Phase transition** occurs for $\lambda = 1$

- Let $\lambda = 1$
- Considering $\mathbb{R}^2 \cong \mathbb{C}$, we may decompose with respect to the complex structure, to get $d_A = \partial_A + \bar{\partial}_A$
- Assume $d > 0$. The case $d < 0$ is obtained considering the conjugate complex structure of $\mathbb{C}$
- Integrating by parts (**Bogomolny**) (1976)

$$\mathrm{YMH}(A, \varphi) = 2\pi d + \int_{\mathbb{R}^2} |F_A - \frac{1}{2} * (1 - |\varphi|^2)|^2 + |2\bar{\partial}_A \varphi|^2.$$

- Action is bounded below by $2\pi d$. The minimum is attained if and only if (A, φ) satisfy the **vortex equations**:

$$\left. \begin{array}{l} \bar{\partial}_A \varphi = 0 \\ F_A = \frac{1}{2} * (1 - |\varphi|^2) \end{array} \right\}$$

- Here $*$ is the Hodge star-operator and hence $*(1 - |\varphi|^2)$ is a 2-form

- In **real coordinates** (x_1, x_2):

$$A = A_1 dx_1 + A_2 dx_2 \quad \text{and} \quad \varphi = \varphi_1 + i\varphi_2$$

$$F_A = F_{12} dx_1 \wedge dx_2, \quad \text{where} \quad F_{12} = \partial_1 A_2 - \partial_2 A_1$$

- In terms of the **complex coordinate** $z = x_1 + ix_2$:

$$A = \alpha dz + \bar{\alpha} d\bar{z}, \quad \text{where} \quad \alpha = \frac{1}{2}(A_1 - iA_2)$$

$$\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$$

- The vortex equations are

$$\left.\begin{array}{l} (\bar{\partial} - i\bar{\alpha})\varphi = 0 \\ F_{12} - \frac{1}{2}(\varphi_1^2 + \varphi_2^2 - 1) = 0 \end{array}\right\}$$

- The vortex number can be also obtained as:

$$d = \frac{1}{2\pi}\int_{\mathbb{R}^2} F_A$$

- The equations are invariant under gauge transformations, and the **moduli space of solutions** (**vortices**) is defined as the quotient space of all solutions modulo gauge equivalence.
- The basic result is **Taubes's existence theorem** (1980): Given d points $z_i \in \mathbb{R}^2$ (possibly with multiplicities) there exists a solution to the vortex equations, unique up to gauge equivalence, with $\varphi(z_i) = 0$ and $\mathrm{YMH}(A, \varphi) = 2\pi d$.
- This means that the moduli space of vortices is the space of unordered d-tuples $S^d\mathbb{C}$, the d-th symmetric product of $\mathbb{C}$. But this space can be thought of as the space of zeros of a monic polynomial

  $$p(z) = z^d + a_d z^{d-1} + \ldots + a_1.$$

  Hence the moduli space is just the vector space $\mathbb{C}^d$ of coefficients of all such polynomials.
- Details in the beautiful book by **Jaffe and Taubes**: Vortices and Monopoles

2. Vortices on a compact Riemann surface

- X compact Riemann surface
- ω_X Kähler form on X (conveniently normalised)
- $L \to X$ smooth complex line bundle over X
- h Hermitian metric on L
- τ is a **real parameter**
- The **vortex equations** are equations for A a **unitary connection** on (L, h) and φ a **smooth section** of L:

$$\left.\begin{array}{l} \bar{\partial}_A \varphi = 0 \\ i\Lambda F_A + |\varphi|_h^2 = \tau \end{array}\right\}$$

  $F_A \in \Omega^2(X, i\mathbb{R})$ curvature of A
  $\Lambda F_A \in C^\infty(X, i\mathbb{R})$ contraction of F_A with ω_X
  $|\cdot|_h \in C^\infty(X, \mathbb{R})$ positive norm on L associated to h

- Like for $\mathbb{R}^2$, solutions to the vortex equations are minima of a **Yang–Mills–Higgs** functional (Kähler identities)

- The action of the **gauge group** $\mathscr{G} = C^\infty(X, \mathrm{U}(1))$ of (L, h) acts on the space of pairs (A, φ): for $g \in \mathscr{G}$

$$g \cdot d_A = g d_A g^{-1} \quad \text{and} \quad g \cdot \varphi = g\varphi$$

preserving the solutions to the vortex equations

- **Moduli space of vortices** $= \{\text{solutions}\}/\mathscr{G}$
- Integrating the second equation (i.e. applying $\int_X (-)\omega_X$ to the equation), and relating **Chern–Weil** theory the curvature of F_A to the first Chern class of L, we obtain

$$2\pi \deg L + \|\varphi\|^2_{L^2} = \tau \operatorname{vol}(X)$$

- Normalising $\operatorname{vol}(X) = 2\pi$, this implies

$$\deg L \leq \tau$$

- If $\deg L = \tau$, then $\varphi = 0$ and a solution exists by Hodge theory

Theorem

Let $D = \sum_i z_i \in S^d X$. Then there exists a unique solution to the vortex equations modulo gauge equivalence, with $\varphi \neq 0$ and $\varphi(z_i) = 0$, if and only if $\deg L < \tau$

- Several independent proofs:
- **Noguchi** (1987, $\tau = 1$): direct proof using tools of non-linear analysis
- **Bradlow** (1989): reduces to Kazdan–Warner equation in Riemannian geometry
- **GP** (1991): Two proofs: symplectic geometry and dimensional reduction of Hermitian–Yang–Mills equations

3. Vortices and symplectic geometry

- $\mathscr{A}$ space of unitary connections on (L, h)
- $\Omega^0(L)$ space of smooth sections of L
- $\mathscr{A}$ and $\Omega^0(L)$ are infinite dimensional Kähler manifolds
- The gauge group $\mathscr{G}$ acts symplectically on $\mathscr{A} \times \Omega^0(L)$ with **moment map** $\mu : \mathscr{A} \times \Omega^0(L) \to \mathrm{Lie}\,\mathscr{G} = C^\infty(X, \mathbb{R})$ given by

$$\mu(A, \varphi) = i\Lambda F_A + |\varphi|_h^2$$

- One has the Kähler subvariety

$$\mathscr{N} = \{(A, \varphi) \in \mathscr{A} \times \Omega^0(L) \quad : \quad \bar{\partial}_A \varphi = 0\}$$

  and the restriction $\mu_{\mathscr{N}}$ of the moment map
- Moduli space of vortices coincides with the **Kähler quotient**

$$\mu_{\mathscr{N}}^{-1}(\tau)/\mathscr{G}$$

  Finite dimensional smooth Kähler manifold (non-empty if $\deg L < \tau$)

- $\mathscr{A}$ can be identified with the space $\mathscr{C}$ of holomorphic structures on L: **Chern correspondence**

$$\begin{array}{ccc} \mathscr{A} & \to & \mathscr{C} \\ d_A & \mapsto & \bar{\partial}_A \end{array}$$

- The **complex gauge group** $\mathscr{G}^c = C^\infty(X, \mathbb{C}^*)$ acts on $\mathscr{C}$ by $g \cdot \bar{\partial}_A = g\bar{\partial}_A g^{-1}$ for $g \in \mathscr{G}^c$ preserving $\mathscr{N}$
- The existence theorem for the vortex equation simply says that

$$\mu_{\mathscr{N}}^{-1}(\tau)/\mathscr{G} \cong \mathscr{N}/\mathscr{G}^c \cong S^d(X)$$

  if and only if $d = \deg(L) < \tau$.
- infinite dimensional version of the **Theorem of Kempf-Ness**
- The study of the Kähler metric on the moduli space of vortices $S^d(X)$ is a very rich subject of research pursued over the years by **Nick Manton and his school**. Including the case of $S^d(\mathbb{P}^1) \cong \mathbb{P}^n$
  Excellent account in the book **Topological solitons** by **Manton and Sutcliffe**