Galois representations: Lecture 5

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Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$. Let b be the unique root of the characteristic polynomial of $\rho(g_0)$ such that $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if $g \in G_{\mathbb{Q},S}$ and $h \in I_p$, we have

$$\mathrm{tr}(\rho(ghg_0)) - b\,\mathrm{tr}(\rho(gh)) - \det(\rho(h))\,\mathrm{tr}(\rho(gg_0)) + b\det(\rho(h))\,\mathrm{tr}(\rho(g)) = 0.$$

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We call this condition as *ordinary condition* and denote it by (ord).

It is easy to verify that ordinary condition is indeed a deformation condition.

Note that for a CNL $W(\mathbb{F})$-algebra, $D_{\bar{\rho},\mathrm{ord}}(R)$ is the set of deformations of $\bar{\rho}$ to R satisfying the ordinary condition.

Lemma

ρ is ordinary if and only if $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_1 and η_2 are characters lifting $\bar{\eta}_1$ and $\bar{\eta}_2$ and η_2 is an unramified character.

Proof: The reverse direction was checked last time.

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We will now prove the other direction. We first observe that, after changing basis if necessary, we can assume that $\rho(g_0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

So if $\rho(h) = \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix}$ for $h \in I_p$, then, under this basis,

$$(\rho(h) - \det(\rho(h)))(\rho(g_0) - b) = \begin{pmatrix} (a_h - \det(\rho(h)))(a-b) & 0 \\ c_h(a-b) & 0 \end{pmatrix}.$$

Putting $g = g_0$ in the ordinary condition, we get

$$\mathrm{tr}(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} (a_h - \det(\rho(h)))(a-b) & 0 \\ c_h(a-b) & 0 \end{pmatrix}) = a(a_h - \det(\rho(h)))(a-b) = 0.$$

This means that $a_h = \det(\rho(h))$ as $a(a-b)$ is a unit.

Now for $g \in G_{\mathbb{Q},S}$, suppose $\rho(g) = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$. Let $g_1 \in G_{\mathbb{Q},S}$ be such that b_{g_1} is a unit.

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Therefore, we have $c_h = 0$ and $a_h = \det(\rho(h))$ for all $h \in I_p$.

Since $\rho(g_0)$ is diagonal, it follows that $\rho|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$. From the choices of a and b, it follows that η_1 and η_2 are lifts of $\bar{\eta}_1$ and $\bar{\eta}_2$.

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$R = \mathbb{T}$ theorem

On the arithmetic side, Galois representations attached to modular forms are interpolated by Hecke algebras.

An $R = \mathbb{T}$ theorem compares a suitable deformation ring R with a Hecke algebra $\mathbb{T}$ acting faithfully on a suitable space of modular forms (or more generally, a space of automorphic forms) and asserts that they are isomorphic.

A key step in the proof of Fermat's last theorem was an $R = \mathbb{T}$ theorem in the setting of modular forms of weight 2.

They have played a crucial role in proving some very important results, like:

Modularity of elliptic curves (due to Wiles, Taylor, Diamond, Breuil and Conrad),

Serre's conjecture (due to Khare and Winterberger),

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Setup

Let $p \geq 5$, let $\bar{\rho} : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_p)$ be an odd and absolutely irreducible representation such that:

- $\bar{\rho} = \mathrm{Ind}_{G_{K,S'}}^{G_{\mathbb{Q},\{p,\infty\}}} \bar{\chi}$, for some unramified character $\bar{\chi}$ of $G_{K,S'}$, where $K = \mathbb{Q}(\sqrt{(-1)^{p-1/2}p})$ and S' is the set of primes of K lying above $\{p,\infty\}$,
- $\mathrm{Im}(\bar{\rho}) \simeq D_{2h}$ (i.e. dihedral group of order $2h$) with $p \nmid h$,
- The dihedral extension L of $\mathbb{Q}$ fixed by $\ker(\bar{\rho})$ has class number prime to p,
- $H^2(G_{\mathbb{Q},\{p,\infty\}}, \mathrm{Ad}(\bar{\rho})) = 0$.

Example: Neat S_3-representations constructed by Mazur.

Note that $\bar{\rho}(I_p) = \bar{\rho}(G_{\mathbb{Q}_p})$ and $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ factors through a totally ramified quotient of $G_{\mathbb{Q}_p}$ of order 2 and under a suitable basis, the non-trivial element in the quotient acts via the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. So $\bar{\rho}$ is an ordinary representation.

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By a classical theorem of Hecke, ρ is the p-adic Galois representation to a cuspidal Hecke eigenform f_0 of level $\Gamma_1(p)$ and weight 1. Moreover, f has U_p-eigenvalue 1 which means f is ordinary.

By Hida's theorem, there exists an ordinary Λ-adic eigenform F of level 1 and some character χ which specializes to f_0.

Let $\mathbb{T}^\circ$ be the Λ-Hecke algebra acting on $S^\circ(1,\chi)$ generated by the Hecke operators T_ℓ for primes $\ell \neq p$ and U_p over Λ.

So f_0 defines a primes ideal P_0 of $\mathbb{T}^\circ$ such that $\mathbb{T}^\circ/P_0 \simeq \mathbb{Z}_p$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}^\circ$ generated by p and P_0. Let $\mathbb{T}^\circ_\mathfrak{m}$ be the completion of $\mathbb{T}^\circ$ at $\mathfrak{m}$.

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There exists a continuous, odd, irreducible Galois representation $\rho : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{T}^{\circ}_{\mathfrak{m}})$ such that

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Let $R^{\mathrm{ord}}_{\bar{\rho}}$ be the ordinary deformation ring of $\bar{\rho}$ with maximal ideal $\mathfrak{m}_0$. As ρ is an ordinary deformation of $\bar{\rho}$, it induces a morphism $\phi : R^{\mathrm{ord}}_{\bar{\rho}} \to \mathbb{T}^{\circ}_{\mathfrak{m}}$ by the universal property.

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The morphism $\phi : R^{\mathrm{ord}}_{\bar{\rho}} \to \mathbb{T}^{\circ}_{\mathfrak{m}}$ induced by the representation ρ is an isomorphism and both are isomorphic to $\mathbb{Z}_p[[T]]$.

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We claim that $\dim_{\mathbb{F}_p}(\mathfrak{m}_0/(p, \mathfrak{m}_0^2)) \leq 1$.

To prove the claim, note that $\mathrm{Ad}(\bar{\rho}) = 1 \oplus \chi_0 \oplus V$, where V is an irreducible 2-dimensional representation of D_{2h}. Note that the subrepresentation $1 \oplus \chi_0$ of $\mathrm{Ad}(\bar{\rho})$ corresponds to the subspace of diagonal matrices of $M_2(\mathbb{F}_p)$.

Therefore, if $\rho' : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a non-trivial deformation of $\bar{\rho}$, then ρ' corresponds to an element of $H^1(G_{\mathbb{Q},\{p,\infty\}}, 1 \oplus \chi_0)$ if and only if $\rho' = \mathrm{Ind}_{G_{K,S'}}^{G_{\mathbb{Q},\{p,\infty\}}} \chi'$ for a character $\chi' : G_{K,S'} \to \mathbb{F}[\epsilon]^\times$ lifting $\bar{\chi}$.

As class number of K is co-prime to p, χ' is ramified at the prime of K lying above p. Therefore, ρ' is not ordinary.

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We claim that $\dim_{\mathbb{F}_p}(\mathfrak{m}_0/(p, \mathfrak{m}_0^2)) \leq 1$.

To prove the claim, note that $\mathrm{Ad}(\bar{\rho}) = 1 \oplus \chi_0 \oplus V$, where V is an irreducible 2-dimensional representation of D_{2h}. Note that the subrepresentation $1 \oplus \chi_0$ of $\mathrm{Ad}(\bar{\rho})$ corresponds to the subspace of diagonal matrices of $M_2(\mathbb{F}_p)$.

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If $\ell \neq p$ is a prime, then $T_\ell = \mathrm{tr}(\rho(\mathrm{Frob}_\ell)) = \phi(\mathrm{tr}(\rho^{\text{ord}}(Frob_\ell)))$.
If $\rho^{\text{ord}}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with χ_2 unramified at p, then $\phi(\chi_2(\mathrm{Frob}_p)) = U_p$.

This image of ϕ contains all the Hecke operators which topologically generate $\mathbb{T}^\circ_{\mathfrak{m}}$ over $\mathbb{Z}_p$. Hence, $\phi : R^{\text{ord}}_{\bar\rho} \to \mathbb{T}^\circ_{\mathfrak{m}}$ is surjective as it is continuous.

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Small $R = \mathbb{T}$ theorem

Let p be an odd prime, $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, absolutely irreducible representation such that:

- $\bar{\rho}|_{G_L}$ is absolutely irreducible, where $L = \mathbb{Q}(\sqrt{(-1)^{p-1/2}p})$,
- $\bar{\rho}$ arises from modular forms,
- $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is ordinary i.e. $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^{\times}$ is an unramified character,
- If ℓ is a prime such that $\bar{\rho}$ is ramified at ℓ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, then $\bar{\rho}$ is minimally ramified at ℓ.

Let $S = \{\ell \mid \bar{\rho} \text{ is ramified at } \ell\} \cup \{p, \infty\}$. So we will consider $\bar{\rho}$ as a representation of $G_{\mathbb{Q},S}$ in what follows.

Let $\det(\bar{\rho}) = \epsilon\omega_p^{k-1}$, where k is an integer and ϵ is a character which is unramified at p.

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- If ℓ is a prime such that $\bar{\rho}$ is ramified at ℓ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, then $\bar{\rho}$ is minimally ramified at ℓ.

Let $S = \{\ell \mid \bar{\rho} \text{ is ramified at } \ell\} \cup \{p, \infty\}$. So we will consider $\bar{\rho}$ as a representation of $G_{\mathbb{Q},S}$ in what follows.

Let $\det(\bar{\rho}) = \epsilon\omega_p^{k-1}$, where k is an integer and ϵ is a character which is unramified at p.

Small $R = \mathbb{T}$ theorem

Let p be an odd prime, $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, absolutely irreducible representation such that:

- $\bar{\rho}|_{G_L}$ is absolutely irreducible, where $L = \mathbb{Q}(\sqrt{(-1)^{p-1/2}p})$,
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Let N be the tame Artin conductor of $\bar{\rho}$. Let $\mathcal{O}$ be the ring of integers of the finite extension of $\mathbb{Q}_p$ containing Hecke eigenvalues of all eigenforms of level Np and weight 2.

Let $R_{\bar{\rho}}$ be the universal deformation ring of $\bar{\rho}$ in the category of CNL $\mathcal{O}$-algebras which parameterizes deformations of $\bar{\rho}$ which:

- are ordinary,
- are minimally ramified at all primes ℓ at which $\bar{\rho}$ is minimally ramified,
- have determinant $\hat{\varepsilon}\hat{\omega}_p^{k-2}\chi_p$.

Let $S_2(Np, \mathcal{O})$ be the space of modular forms of level Np and weight 2 with Fourier coefficients in $\mathcal{O}$.

Let $S_2^{\circ}(Np, \mathcal{O})$ be the $\mathcal{O}$-submodule of $S_2(Np, \mathcal{O})$ generated by the ordinary eigenforms of level Np and weight 2.

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The representation $\bar{\rho}$ defines a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$.
Let $\mathbb{T}_{\mathfrak{m}}$ be the completion of $\mathbb{T}$ at $\mathfrak{m}$. So $\mathbb{T}_{\mathfrak{m}}$ is finite and flat over $\mathcal{O}$.

Gluing the p-adic Galois representations attached to ordinary eigenforms of weight 2 and level pN lifting $\bar{\rho}$, we get a deformation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$ of $\bar{\rho}$.
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Therefore, it induces a morphism $\phi : R_{\bar{\rho}} \to \mathbb{T}_{\mathfrak{m}}$.

Theorem (Taylor–Wiles, Diamond)

Suppose we are in the setup as above. The morphism $\phi : R_{\bar{\rho}} \to \mathbb{T}_{\mathfrak{m}}$ induced by ρ is an isomorphism of local complete intersection rings.

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Let p be a prime, $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, $p \nmid N\phi(N)$ be an integer, $S = \{\ell \mid \ell \mid N\} \cup \{p, \infty\}$ and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible representation arising from a modular eigenform of level N.

Let $\mathbb{T}(N)$ be the sub-algebra of $\mathrm{End}_{W(\mathbb{F})}(\sum_{i=0}^{\infty} M_i(N, W(\mathbb{F})))$ generated by the Hecke operators $\{T_\ell, \langle \ell \rangle \mid \ell \nmid Np\}$ over $W(\mathbb{F})$.

The representation $\bar{\rho}$ defines a maximal ideal $m_{\bar{\rho}}$ of $\mathbb{T}(N)$.

Let $\mathbb{T}(N)_{\bar{\rho}} := \varprojlim_n \mathbb{T}(N)/m_{\bar{\rho}}^n$.

There exists a deformation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{T}(N)_{\bar{\rho}})$ of $\bar{\rho}$. Hence it induces a morphism $\Phi : R_{\bar{\rho}}^{\mathrm{univ}} \to \mathbb{T}(N)_{\bar{\rho}}$.

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Theorem (Gouvêa–Mazur)

If $\bar{\rho}$ is unobstructed, then the morphism $\Phi : R_{\bar{\rho}}^{\text{univ}} \to \mathbb{T}(N)_{\bar{\rho}}$ is an isomorphism and both are isomorphic to $W(\mathbb{F})[[X, Y, Z]]$.

Theorem (Böckle)

Under some mild hypotheses (Taylor–Wiles hypotheses), the morphism $\Phi : R_{\bar{\rho}}^{\text{univ}} \to \mathbb{T}(N)_{\bar{\rho}}$ is an isomorphism.

Corollary

Suppose the hypotheses of one of the previous theorems hold. If $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(W(\mathbb{F}))$ is a lift of $\bar{\rho}$, then ρ arises from a p-adic modular form of tame level N.

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If $\bar{\rho}$ is unobstructed, then the morphism $\Phi : R_{\bar{\rho}}^{\text{univ}} \to \mathbb{T}(N)_{\bar{\rho}}$ is an isomorphism and both are isomorphic to $W(\mathbb{F})[[X, Y, Z]]$.

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Framed deformations

Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation and suppose G satisfies the finiteness condition (Φ_p).

Let $D_{\bar{\rho}}^{\square}$ be the functor from $\mathcal{C}$ to the category of sets which sends a CNL $W(\mathbb{F})$-algebra R to the set of lifts of $\bar{\rho}$ to R.

Theorem (Kisin)

The functor $D_{\bar{\rho}}^{\square}$ is representable.

The CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{\square}$ representing $D_{\bar{\rho}}^{\square}$ is called the universal framed deformation ring of $\bar{\rho}$.

An advantage of this point of view is availability of both local and global deformation rings of $\bar{\rho}$ and the relation between them. However if $\bar{\rho}$ is absolutely irreducible, then $R_{\bar{\rho}}^{\square}$ is bigger than $R_{\bar{\rho}}^{\mathrm{univ}}$.

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Pseudo-representations

Alternatively, one can use the theory of pseudo-representations (developed by Chenevier) in which one 'lifts characteristic polynomial' of $\bar{\rho}$ rather than the representation $\bar{\rho}$ itself.

We will now see the notion of pseudo-representation of dimension 2.

Definition

Given a topological ring R, a 2-dimensional pseudo-representation of G over R is a tuple $(t, d) : G \to R$ of continuous functions such that

- $d : G \to R^\times$ is a homomorphism of groups,
- $t(1) = 2$,
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Example: If $\rho : G \to \mathrm{GL}_2(R)$ is a representation, then $(\mathrm{tr}(\rho), \det(\rho))$ is a 2-dimensional pseudo-representation of G.

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