Galois representations: Lecture 2

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Ordinary families of modular forms

Let p be an odd prime and $\Lambda := \mathbb{Z}_p[[T]]$.

Note that there is a unique contiunous character $\psi : 1 + p\mathbb{Z}_p \to \Lambda^\times$ which sends $1 + p$ to $1 + T$.

Let $\mathbb{Q}_\infty$ be the extension of $\mathbb{Q}$ obtained by attaching all p-power roots of unity. So $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$. Let $\mathbb{Q}_{\mathrm{cyc}}$ be the unique subextension of $\mathbb{Q}_\infty$ such that $\mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \simeq \mathbb{Z}_p$ (cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$).

Since $\mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \simeq 1 + p\mathbb{Z}_p$, we get a character $\kappa_{\mathrm{cyc}} : G_\mathbb{Q} \to \Lambda^\times$ obtained by composing ψ with the surjective map $G_\mathbb{Q} \to \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q})$.

For an integer $k \geq 1$ and a p-power root of unity ζ, let $\phi_{k,\zeta} : \Lambda \to \overline{\mathbb{Q}_p}$ be the continuous homomorphism sending T to $\zeta(1+p)^{k-1} - 1$.

Let $\hat{\omega}_p : G_\mathbb{Q} \to \mathbb{Z}_p^\times$ be the Teichmuller lift of the mod p cyclotomic character $\omega_p : G_\mathbb{Q} \to \mathbb{F}_p^\times$.

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Let N be an integer not divisible by p and $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}_p}^\times$ be a character.

If K is a finite extension of $\mathrm{Frac}(\Lambda)$ (the fraction field of Λ), then denote the integral closure of Λ in K by R_K.

Λ-adic cuspforms

A Λ-adic cuspform F of level N and character χ is a formal q-expansion $\sum_{n=0}^{\infty} a_n(F)q^n \in R_K[[q]]$ (for some finite extension K of $\mathrm{Frac}(\Lambda)$) such that:

For every integer $k > 1$ and every p-power root of unity ζ, if $\phi : R_K \to \overline{\mathbb{Q}_p}$ is a specialization extending the specialization $\phi_{k,\zeta}$, then

$$f_\phi := \phi(F) = \sum_{n=0}^{\infty} \phi(a_n(F))q^n \in \overline{\mathbb{Q}_p}[[q]]$$

is the q-expansion of a cuspform of level Np^m for some $m \geq 0$, weight k, and nebentypus $\chi.\hat{\omega}_p^{1-k}.\eta_\phi$, for some character η_ϕ of $(\mathbb{Z}/Np^m\mathbb{Z})^\times$ of p-power order.

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Let $S(N, \chi)$ be the Λ-module of Λ-adic cuspforms of level N and character χ. It has an action of Hecke operators T_ℓ for primes $\ell \nmid Np$, U_q for primes $q \mid N$ and U_p.

Let $e := \lim_{n\to\infty} U_p^{n!}$ be the ordinary projector and let $S^\circ(N,\chi) = eS(N,\chi)$. Let $\mathbb{T}$ be the Hecke algebra generated by the Hecke operators $\{T_\ell \mid \ell \nmid Np\}$, $\{U_q \mid q \mid N\}$ and U_p acting on $S^\circ(N,\chi)$ over Λ.

Let P be a minimal prime of $\mathbb{T}$. Then $\mathbb{T}/P$ is a finite extension of Λ and it corresponds to an ordinary Λ-adic cuspform F_P which is an eigenform for all the Hecke operators. Such an eigenform is called a cuspidal Hida eigenfamily.

For every $\ell \nmid Np$, denote the T_ℓ eigenvalue of F_P by $a_\ell(F_P)$ and denote the U_p eigenvalue of F_P by $a_p(F_P)$.

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Motivation

Consider the Ramanujan Δ-function:

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n.$$

It is a modular eigenform of level 1 and weight 12.

Let p be a prime. Recall that, due to the work of Deligne, we can attach a 2-dimensional p-adic Galois representation of $G_{\mathbb{Q}}$ to Δ.

More precisely, the p-adic Galois representation attached to Δ by Deligne is the Galois representation

$$\rho_p : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{Z}_p)$$

such that ρ_p is absolutely irreducible, $\det(\rho_p) = \chi_p^{11}$, and for all primes $\ell \neq p$, $\mathrm{tr}(\rho_p(\mathrm{Frob}_\ell)) = \tau(\ell)$.

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So if $\bar{\rho}_p : G_{\mathbb{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_p)$ is the representation obtained by composing ρ_p with this map, then the following diagram commutes:

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Thus, ρ_p is a lift of $\bar{\rho}_p$.

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- Can we describe all lifts of $\bar{\rho}_p$ to $\mathbb{Z}_p$?
- Which of these lifts arise from modular forms (like ρ_p)?
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Thus, ρ_p is a lift of $\bar{\rho}_p$.

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Basic Setup

In his seminal paper, Mazur studied these questions in a more general setting.

In particular, let G be a profinite group i.e. G is an inverse limit of finite groups.

Let p be a prime, $\mathbb{F}$ be a finite field of characteristic p and $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation.

Let $W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$. So $W(\mathbb{F})$ is the ring of integers of the finite unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$.

Let R be a complete Noetherian local $W(\mathbb{F})$-algebra with residue field $\mathbb{F}$ (CNL $W(\mathbb{F})$-algebra in short). This means that

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Finiteness condition

We say that a profinite group G satisfies the finiteness condition (Φ_p) if for every open subgroup $H \subset G$ of finite index, there are only finite number of continuous homomorphisms from H to $\mathbb{F}_p$.

Example:

- If ℓ is a prime and K is a finite extension of $\mathbb{Q}_\ell$, then by local class field theory, G_K satisfies the condition (Φ_p).
- If K is a number field, S is a finite set of primes of K and d is a positive integer, then Hermite–Minkowski theorem states that there are only finitely many extensions F/K of degree d which are unramified outside S.

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Deformations

Two lifts ρ and ρ' of $\bar{\rho}$ to R are said to be equivalent if there exists a $M \in \mathrm{GL}_n(R)$ such that $\rho = M\rho' M^{-1}$ and $M \pmod{m_R} = Id$.

Let $D_{\bar{\rho}}(R) :=$ set of equivalence classes of lifts of $\bar{\rho}$ to R.

An element of $D_{\bar{\rho}}(R)$ is called a *deformation* of $\bar{\rho}$ to R.

If R and R' are CNL $W(\mathbb{F})$-algebras, then denote by $\mathrm{Hom}(R, R')$ the set of homomorphisms $f : R \to R'$ such that f is a homomorphism of complete local rings and it induces identity on residue fields.

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Theorem (Mazur)

If G satisfies the finiteness condition (Φ_p) and $\bar{\rho}$ is absolutely irreducible, then there exists

- a CNL $W(\mathbb{F})$-algebra $R^{\text{univ}}_{\bar{\rho}}$,
- a lift $\rho^{\text{univ}} : G \to \mathrm{GL}_n(R^{\text{univ}}_{\bar{\rho}})$ of $\bar{\rho}$

such that for any CNL $W(\mathbb{F})$-algebra R, the map

$$\mathrm{Hom}(R^{\text{univ}}_{\bar{\rho}}, R) \to D_{\bar{\rho}}(R)$$

$$\phi \mapsto [\phi \circ \rho^{\text{univ}}]$$

is a bijection.

In other words, the functor $\mathcal{F}$ from the category of CNL $W(\mathbb{F})$-algebras to the category of sets sending a CNL $W(\mathbb{F})$-algebra R to the set $D_{\bar{\rho}}(R)$ is representable.

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The ring $R_{\bar{\rho}}^{\text{univ}}$ is called the universal deformation ring of $\bar{\rho}$ and (the equivalence class of) ρ^{univ} is called the universal deformation of $\bar{\rho}$.

Ramakrishna proved Mazur's theorem for the residual representations $\bar{\rho}$'s such that $\text{End}_G(\bar{\rho}) = \mathbb{F}$.

We will prove Mazur's theorem using Schlessinger's criteria.

If R and S are artinian CNL $W(\mathbb{F})$-algebras, then a homomorphism $f : R \to S$ is called *small* if f is surjective, $\ker(f)$ is principal and $m_R \ker(f) = 0$.
Example: The natural surjective map $\mathbb{F}[\epsilon] \to \mathbb{F}$, where $\mathbb{F}[\epsilon] = \mathbb{F}[x]/(x^2)$.

If R_0, R_1 and R_2 are artinian CNL $W(\mathbb{F})$-algebras with morphisms $f_1 : R_1 \to R_0$ and $f_2 : R_2 \to R_0$, then let $R_3 := R_1 \times_{R_0} R_2 = \{(a, b) \in R_1 \times R_2 \mid f_1(a) = f_2(b)\}$.
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Note that we have a map

$$D_{\bar{\rho}}(R_3) \to D_{\bar{\rho}}(R_1) \times_{D_{\bar{\rho}}(R_0)} D_{\bar{\rho}}(R_2). \tag{1}$$

Using Schlessinger's criteria, it suffices to check the following conditions:

- **H1:** If the map $R_2 \to R_0$ is small, then the map in (1) is surjective,
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Now suppose $R_2 = \mathbb{F}[\epsilon]$ and $R_0 = \mathbb{F}$. Let ϕ and ψ be two lifts of $\bar{\rho}$ to R_3.

For $i = 0, 1, 2$, denote the image of ϕ and ψ in $\mathrm{GL}_n(R_i)$ by ϕ_i and ψ_i, respectively.

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So there exists $M_i \in Id + M_n(m_{R_i})$ such that $M_i \phi_i M_i^{-1} = \psi_i$. As $R_0 = \mathbb{F}$ and both M_1 and M_2 reduce to identity in $M_n(\mathbb{F})$, we can glue M_1 and M_2 to get an $M_3 \in Id + M_n(m_{R_3})$.

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Now suppose $R_2 = \mathbb{F}[\epsilon]$ and $R_0 = \mathbb{F}$. Let ϕ and ψ be two lifts of $\bar{\rho}$ to R_3.
For $i = 0, 1, 2$, denote the image of ϕ and ψ in $\mathrm{GL}_n(R_i)$ by ϕ_i and ψ_i, respectively.

So under the map given in (1), $[\phi]$ and $[\psi]$ get mapped to $([\phi_1], [\phi_2])$ and $([\psi_1], [\psi_2])$, respectively.

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So there exists $M_i \in Id + M_n(m_{R_i})$ such that $M_i \phi_i M_i^{-1} = \psi_i$. As $R_0 = \mathbb{F}$ and both M_1 and M_2 reduce to identity in $M_n(\mathbb{F})$, we can glue M_1 and M_2 to get an $M_3 \in Id + M_n(m_{R_3})$.

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We want to prove that $D_{\bar{\rho}}(\mathbb{F}[\epsilon])$ is a finite set. Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$. Then there is a function $f_\rho : G \to M_n(\mathbb{F})$ such that $\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$.

If ρ and ρ' are two lifts of $\bar{\rho}$ to $\mathbb{F}[\epsilon]$, then $\rho - \rho' : G \to \mathrm{GL}_n(\mathbb{F})$ given by $(\rho - \rho')(g) := (Id + \epsilon(f_\rho(g) - f_{\rho'}(g)))\bar{\rho}(g)$ is also a lift of $\bar{\rho}$. Let $H := \ker(\bar{\rho})$. As H has finite index in G, there can be only finitely many lifts ρ' of $\bar{\rho}$ to $\mathbb{F}[\epsilon]$ such that $\rho|_H = \rho'|_H$.

Now $\rho : H \to Id + \epsilon M_n(\mathbb{F})$ is a homomorphism which means $f_\rho : H \to M_n(\mathbb{F})$ is a homomorphism.
As $M_n(\mathbb{F})$ is a finite p-elementary abelian group, H has finite index in G and G satisfies the finiteness property (Φ_p), there are only finitely many homomorphisms from H to $M_n(\mathbb{F})$.

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Proof of condition H4

Now suppose $R_1 = R_2 = R$ and $f_1 = f_2 = f$. So $R_3 = R \times_{R_0} R$. We want to prove that the map in (1) is bijective.

Since f is small, we know by condition H1 that this map is surjective. So we only need to prove that this map is injective.

Let ϕ and ψ be two lifts of $\bar{\rho}$ to R_3. For $i = 1, 2$, denote the projection of ϕ and ψ on i-th co-ordinate by ϕ_i and ψ_i, respectively.

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So we have $\bar{\psi} = \bar{M}_1\bar{\phi}\bar{M}_1^{-1} = \bar{M}_2\bar{\phi}\bar{M}_2^{-1}$. So $\bar{M}_1^{-1}\bar{M}_2 \in \mathrm{End}_G(\bar{\phi})$.

Now suppose $\mathrm{End}_G(\bar{\phi}) = R_0$ i.e. it consists of just scalars. Then $\bar{M}_1^{-1}\bar{M}_2 = \bar{r}Id$ for some $\bar{r} \in 1 + m_{R_0}$.

So if r is a lift of $\bar{r}$ in $1 + m_R$, then $rM_1\phi_1(rM_1)^{-1} = \psi_1$ and the images of rM_1 and M_2 in $Id + M_n(m_{R_0})$ is the same.

So we can glue rM_1 and M_2 to get an $M \in Id + M_n(m_{R_3})$ and it is easy to verify (using definitions of M_1 and M_2) that $M\phi M^{-1} = \psi$ which proves that $[\phi] = [\psi]$ and hence, the map in (1) is injective.

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Now suppose $\mathrm{End}_G(\bar{\phi}) = R_0$ i.e. it consists of just scalars. Then $\bar{M}_1^{-1}\bar{M}_2 = \bar{r}Id$ for some $\bar{r} \in 1 + m_{R_0}$.

So if r is a lift of $\bar{r}$ in $1 + m_R$, then $rM_1\phi_1(rM_1)^{-1} = \psi_1$ and the images of rM_1 and M_2 in $Id + M_n(m_{R_0})$ is the same.

So we can glue rM_1 and M_2 to get an $M \in Id + M_n(m_{R_3})$ and it is easy to verify (using definitions of M_1 and M_2) that $M\phi M^{-1} = \psi$ which proves that $[\phi] = [\psi]$ and hence, the map in (1) is injective.

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Let $R' \to S$ be a small map of artinian CNL $W(\mathbb{F})$-algebras, $\rho : G \to \mathrm{GL}_n(R')$ be a lift of $\bar{\rho}$ and ρ' be the image of ρ in $\mathrm{GL}_n(S)$.

We will now prove that if $\mathrm{End}_G(\rho') = S$ then $\mathrm{End}_G(\rho) = R'$.
Now let $M \in \mathrm{End}_G(\rho)$ and t be a generator of the kernel of the map $R' \to S$.

Since $\mathrm{End}_G(\rho') = S$, there exists an $r \in R'$ and $M' \in M_n(R')$ such that $M = rId + tM'$. So $tM'\rho(g) = \rho(g)tM'$.
Since $tm_{R'} = 0$ and $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$, it follows that $tM' = ts$ for some $s \in R$ which proves the claim.

Going back to our original setting, observe that the natural surjective map $R_0 \to \mathbb{F}$ factors as a sequence of small maps.

As $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$, applying our claim to every step of the sequence of small maps from R_0 to $\mathbb{F}$ gives us $\mathrm{End}_G(\bar{\phi}) = R_0$ which proves condition H4.

Let $R' \to S$ be a small map of artinian CNL $W(\mathbb{F})$-algebras, $\rho : G \to \mathrm{GL}_n(R')$ be a lift of $\bar{\rho}$ and ρ' be the image of ρ in $\mathrm{GL}_n(S)$.

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