Galois representations: Lecture 2

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Let p be an odd prime and $\Lambda := \mathbb{Z}_p[[T]]$.

Note that there is a unique continuous character $\psi: 1+p\mathbb{Z}_p \to \Lambda^{\times}$ which sends 1+p to 1+T.

Let \mathbb{Q}_{∞} be the extension of \mathbb{Q} obtained by attaching all p-power roots of unity. So $\operatorname{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q}) \simeq \mathbb{Z}_p^{\times}$. Let $\mathbb{Q}_{\operatorname{cyc}}$ be the unique subextension of \mathbb{Q}_{∞} such that $\operatorname{Gal}(\mathbb{Q}_{\operatorname{cyc}}/\mathbb{Q}) \simeq \mathbb{Z}_p$ (cyclotomic \mathbb{Z}_p -extension of \mathbb{Q}).

Since $\operatorname{Gal}(\mathbb{Q}_{\operatorname{cyc}}/\mathbb{Q}) \simeq 1 + p\mathbb{Z}_p$, we get a character $\kappa_{\operatorname{cyc}} : G_{\mathbb{Q}} \to \Lambda^{\times}$ obtained by composing ψ with the surjective map $G_{\mathbb{Q}} \to \operatorname{Gal}(\mathbb{Q}_{\operatorname{cyc}}/\mathbb{Q})$.

For an integer $k \geq 1$ and a p-power root of unity ζ , let $\phi_{k,\zeta} : \Lambda \to \overline{\mathbb{Q}_p}$ be the continuous homomorphism sending T to $\zeta(1+p)^{k-1}-1$.

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If K is a finite extension of $\operatorname{Frac}(\Lambda)$ (the fraction field of Λ), then denote the integral closure of Λ in K by R_K .

Λ -adic cuspforms

A Λ -adic cuspform F of level N and character χ is a formal q-expansion $\sum_{n=0}^{\infty} a_n(F)q^n \in R_K[[q]]$ (for some finite extension K of $\operatorname{Frac}(\Lambda)$) such that:

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$$f_{\phi} := \phi(F) = \sum_{n=0}^{\infty} \phi(a_n(F))q^n \in \overline{\mathbb{Q}_p}[[q]]$$

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Let $e := \lim_{n \to \infty} U_p^{n!}$ be the ordinary projector and let $S^{\circ}(N, \chi) = eS(N, \chi)$. Let \mathbb{T} be the Hecke algebra generated by the Hecke operators $\{T_{\ell} \mid \ell \nmid Np\}, \{U_q \mid q \mid N\}$ and U_p acting on $S^{\circ}(N, \chi)$ over Λ .

Let P be a minimal prime of \mathbb{T} . Then \mathbb{T}/P is a finite extension of Λ and it corresponds to an ordinary Λ -adic cuspform F_P which is an eigenform for all the Hecke operators. Such an eigenform is called a cuspidal Hida eigenfamily.



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Let $S = \{q \mid q \mid N\} \cup \{p, \infty\}$. Then there exists an odd irreducible Galois representation $\rho_P : G_{\mathbb{Q},S} \to \mathrm{GL}_2(L_P)$ such that

- $\det(\rho_P) = \chi \kappa_{\rm cyc}$,
- For all primes $\ell \nmid Np$, $\operatorname{tr}(\rho_P(\operatorname{Frob}_{\ell})) = a_{\ell}(F_P)$,
- $\rho_P|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_2 is an unramified character of $G_{\mathbb{Q}_p}$ such that $\eta_2(\operatorname{Frob}_p) = a_p(F_P)$.

More generally, if L is the total fraction field of \mathbb{T} , then there exists an odd irreducible Galois representation $\rho: G_{\mathbb{O},S} \to \mathrm{GL}_2(L)$ such that

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Consider the Ramanujan Δ -function:

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

It is a modular eigenform of level 1 and weight 12.

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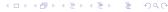
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In his seminal paper, Mazur studied these questions in a more general setting.

In particular, let G be a profinite group i.e. G is an inverse limit of finite groups.

Let p be a prime, \mathbb{F} be a finite field of characteristic p and $\bar{\rho}: G \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation.

Let $W(\mathbb{F})$ be the ring of Witt vectors of \mathbb{F} . So $W(\mathbb{F})$ is the ring of integers of the finite unramified extension of \mathbb{Q}_p with residue field \mathbb{F} .

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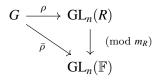
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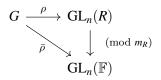


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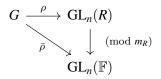
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We say that a profinite group G satisfies the finiteness condition (Φ_p) if for every open subgroup $H \subset G$ of finite index, there are only finite number of continuous homomorphisms from H to \mathbb{F}_p .

Example:

- If ℓ is a prime and K is a finite extension of \mathbb{Q}_{ℓ} , then by local class field theory, G_K satisfies the condition (Φ_n) .
- If K is a number field, S is a finite set of primes of K and d is a
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 there are only finitely many extensions F/K of degree d which
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Deformations

Two lifts ρ and ρ' of $\bar{\rho}$ to R are said to be equivalent if there exists a $M \in GL_n(R)$ such that $\rho = M\rho'M^{-1}$ and $M \pmod{m_R} = Id$.

Let $D_{\bar{\rho}}(R) := \text{set of equivalence classes of lifts of } \bar{\rho} \text{ to } R$.

An element of $D_{\bar{\rho}}(R)$ is called a *deformation* of $\bar{\rho}$ to R.

If R and R' are CNL $W(\mathbb{F})$ -algebras, then denote by $\operatorname{Hom}(R,R')$ the set of homomorphisms $f:R\to R'$ such that f is a homomorphism of complete local rings and it induces identity on residue fields.

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Two lifts ρ and ρ' of $\bar{\rho}$ to R are said to be equivalent if there exists a $M \in GL_n(R)$ such that $\rho = M\rho'M^{-1}$ and $M \pmod{m_R} = Id$.

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Theorem (Mazur)

If G satisfies the finiteness condition (Φ_p) and $\bar{\rho}$ is absolutely irreducible, then there exists

- a CNL $W(\mathbb{F})$ -algebra $R_{\bar{\partial}}^{\text{univ}}$,
- a lift $\rho^{\text{univ}}: G \to \operatorname{GL}_n(R_{\bar{\rho}}^{\text{univ}})$ of $\bar{\rho}$

such that for any CNL $W(\mathbb{F})$ -algebra R, the map

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In other words, the functor \mathcal{F} from the category of CNL $W(\mathbb{F})$ -algebras to the category of sets sending a CNL $W(\mathbb{F})$ -algebra R to the set $D_{\bar{o}}(R)$ is representable.



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The ring $R_{\overline{\rho}}^{\mathrm{univ}}$ is called the universal deformation ring of $\overline{\rho}$ and (the equivalence class of) ρ^{univ} is called the universal deformation of $\overline{\rho}$.

Ramakrishna proved Mazur's theorem for the residual representations $\bar{\rho}$'s such that $\operatorname{End}_G(\bar{\rho}) = \mathbb{F}$.

We will prove Mazur's theorem using Schlessinger's criteria.

If R and S are artinian CNL $W(\mathbb{F})$ -algebras, then a homomorphism $f: R \to S$ is called *small* if f is surjective, $\ker(f)$ is principal and $m_R \ker(f) = 0$.

Example: The natural surjective map $\mathbb{F}[\epsilon] \to \mathbb{F}$, where $\mathbb{F}[\epsilon] = \mathbb{F}[x]/(x^2)$.

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If R_0 , R_1 and R_2 are artinian CNL $W(\mathbb{F})$ -algebras with morphisms $f_1: R_1 \to R_0$ and $f_2: R_2 \to R_0$, then let $R_3:=R_1 \times_{R_0} R_2 = \{(a,b) \in R_1 \times R_2 \mid f_1(a) = f_2(b)\}.$

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- **H1:** If the map $R_2 \to R_0$ is small, then the map in (1) is surjective.
- **H2:** If $R_0 = \mathbb{F}$ and $R_2 = \mathbb{F}[\epsilon]$, then the map in (1) is bijective,
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Suppose $R_2 \to R_0$ is small and let ϕ_1 and ϕ_2 be the lifts of $\bar{\rho}$ to R_1 and R_2 , respectively such that $([\phi_1], [\phi_2]) \in D_{\bar{\rho}}(R_1) \times_{D_{\bar{\rho}}(R_0)} D_{\bar{\rho}}(R_2)$.

So the images $\bar{\phi}_1$ and $\bar{\phi}_2$ of ϕ_1 and ϕ_2 in $GL_n(R_0)$ are equivalent i.e. there is a $\bar{M} \in Id + M_n(m_{R_0})$ such that $\bar{\phi}_1 = \bar{M}\bar{\phi}_2\bar{M}^{-1}$.

Since the map $R_2 \to R_0$ is surjective, we can lift \bar{M} to an $M \in Id + M_n(m_{R_2})$.

So the image of the representation $M\phi_2M^{-1}$ under the map $GL_n(R_2) \to GL_n(R_0)$ is same as $\bar{\phi}_1$.

Hence, we can glue ϕ_1 and $M\phi_2M^{-1}$ to get a lift ϕ_3 of $\bar{\rho}$ to R_3 i.e. $\phi_3: G \to GL_n(R_3)$ is given by $\phi_3(g) = (\phi_1(g), M\phi_2(g)M^{-1})$.

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Now suppose $R_2 = \mathbb{F}[\epsilon]$ and $R_0 = \mathbb{F}$. Let ϕ and ψ be two lifts of $\bar{\rho}$ to R_3 .

For i = 0, 1, 2, denote the image of ϕ and ψ in $GL_n(R_i)$ by ϕ_i and ψ_i , respectively.

So under the map given in (1), $[\phi]$ and $[\psi]$ get mapped to $([\phi_1], [\phi_2])$ and $([\psi_1], [\psi_2])$, respectively.

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So there exists $M_i \in Id + M_n(m_{R_i})$ such that $M_i \phi_i M_i^{-1} = \psi_i$. As $R_0 = \mathbb{F}$ and both M_1 and M_2 reduce to identity in $M_n(\mathbb{F})$, we can glue M_1 and M_2 to get an $M_3 \in Id + M_n(m_{R_3})$.

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So it follows by definition that $M_3\phi M_3^{-1}=\psi$ which implies $[\phi]=[\psi]$. This proves condition H2.

We want to prove that $D_{\bar{\rho}}(\mathbb{F}[\epsilon])$ is a finite set. Let $\rho: G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$. Then there is a function $f_{\rho}: G \to M_n(\mathbb{F})$ such that $\rho(g) = (Id + \epsilon f_{\rho}(g))\bar{\rho}(g)$.

If ρ and ρ' are two lifts of $\bar{\rho}$ to $\mathbb{F}[\epsilon]$, then $\rho - \rho' : G \to \operatorname{GL}_n(\mathbb{F})$ given by $(\rho - \rho')(g) := (Id + \epsilon(f_{\rho}(g) - f_{\rho'}(g)))\bar{\rho}(g)$ is also a lift of $\bar{\rho}$. Let $H := \ker(\bar{\rho})$. As H has finite index in G, there can be only finitely many lifts ρ' of $\bar{\rho}$ to $\mathbb{F}[\epsilon]$ such that $\rho|_H = \rho'|_H$.

Now $\rho: H \to Id + \epsilon M_n(\mathbb{F})$ is a homomorphism which means $f_{\rho}: H \to M_n(\mathbb{F})$ is a homomorphism.

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Since f is small, we know by condition H1 that this map is surjective. So we only need to prove that this map is injective.

Let ϕ and ψ be two lifts of $\bar{\rho}$ to R_3 . For i=1,2, denote the projection of ϕ and ψ on i-th co-ordinate by ϕ_i and ψ_i , respectively.

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So we have
$$\bar{\psi} = \bar{M}_1 \bar{\phi} \bar{M}_1^{-1} = \bar{M}_2 \bar{\phi} \bar{M}_2^{-1}$$
. So $\bar{M}_1^{-1} \bar{M}_2 \in \operatorname{End}_G(\bar{\phi})$.

Now suppose $\operatorname{End}_G(\bar{\phi})=R_0$ i.e. it consists of just scalars. Then $\bar{M}_1^{-1}\bar{M}_2=\bar{r}Id$ for some $\bar{r}\in 1+m_{R_0}$.

So if r is a lift of \bar{r} in $1 + m_R$, then $rM_1\phi_1(rM_1)^{-1} = \psi_1$ and the images of rM_1 and M_2 in $Id + M_n(m_{R_0})$ is the same.

So we can glue rM_1 and M_2 to get an $M \in Id + M_n(m_{R_3})$ and it is easy to verify (using definitions of M_1 and M_2) that $M\phi M^{-1} = \psi$ which proves that $[\phi] = [\psi]$ and hence, the map in (1) is injective.



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Denote the image of M_i in $Id + M_n(R_0)$ by \bar{M}_i and images of ϕ and ψ in $GL_n(R_0)$ by $\bar{\phi}$ and $\bar{\psi}$.

So we have $\bar{\psi} = \bar{M}_1 \bar{\phi} \bar{M}_1^{-1} = \bar{M}_2 \bar{\phi} \bar{M}_2^{-1}$. So $\bar{M}_1^{-1} \bar{M}_2 \in \text{End}_G(\bar{\phi})$.

Now suppose $\operatorname{End}_G(\bar{\phi})=R_0$ i.e. it consists of just scalars. Then $\bar{M_1}^{-1}\bar{M_2}=\bar{r}Id$ for some $\bar{r}\in 1+m_{R_0}$.

So if r is a lift of \bar{r} in $1 + m_R$, then $rM_1\phi_1(rM_1)^{-1} = \psi_1$ and the images of rM_1 and M_2 in $Id + M_n(m_{R_0})$ is the same.

So we can glue rM_1 and M_2 to get an $M \in Id + M_n(m_{R_3})$ and it is easy to verify (using definitions of M_1 and M_2) that $M\phi M^{-1} = \psi$ which proves that $[\phi] = [\psi]$ and hence, the map in (1) is injective.



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We will now prove that if $\operatorname{End}_G(\rho') = S$ then $\operatorname{End}_G(\rho) = R'$. Now let $M \in \operatorname{End}_G(\rho)$ and t be a generator of the kernel of the map $R' \to S$.

Since $\operatorname{End}_G(\rho') = S$, there exists an $r \in R'$ and $M' \in M_n(R')$ such that M = rId + tM'. So $tM'\rho(g) = \rho(g)tM'$. Since $tm_{R'} = 0$ and $\operatorname{End}_G(\bar{\rho}) = \mathbb{F}$, it follows that tM' = ts for some $s \in R$ which proves the claim

Going back to our original setting, observe that the natural surjective map $R_0 \to \mathbb{F}$ factors as a sequence of small maps.

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