Galois representations: Lecture 1

Shaunak Deo

Indian Institute of Science

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Galois groups

Galois representations are basically representations of Galois groups.
To make this notion precise, we will first describe the Galois groups of interest.

Let K be a perfect field and $\overline{K}$ be an algebraic closure of K.
If K'' and K' are finite Galois extensions of K such that $K' \subset K'' \subset \overline{K}$, then we have a surjective map $\phi_{K''/K'} : \mathrm{Gal}(K''/K) \to \mathrm{Gal}(K'/K)$ given by restriction.

So the Galois groups of finite Galois extensions of K give us an inverse system of finite groups and we define the absolute Galois group of K to be

$$G_K := \mathrm{Gal}(\overline{K}/K) = \varprojlim_{K'/K} \mathrm{Gal}(K'/K).$$

So G_K consists of $(g_{K'}) \in \prod_{K'/K} \mathrm{Gal}(K'/K)$ such that $\phi_{K''/K'}(g_{K''}) = g_{K'}$ for all finite Galois extensions $K' \subset K''$ of K.

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We equip $\prod_{K'/K} \mathrm{Gal}(K'/K)$ with the product topology induced by the discrete topology on every finite group $\mathrm{Gal}(K'/K)$.

Since $G_K \subset \prod_{K'/K} \mathrm{Gal}(K'/K)$, we endow G_K with the subspace topology. We will always view G_K as a topological group with this topology. Under this topology G_K is Hausdorff, compact and totally disconnected.

Now we focus on the case $K = \mathbb{Q}$. For every prime p, choosing an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, we get the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \xhookrightarrow{\iota_p} & \overline{\mathbb{Q}_p} \\ \uparrow & & \uparrow \\ \mathbb{Q} & \hookrightarrow & \mathbb{Q}_p \end{array}$$

This gives us an injection $i_p : G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ which depends on the choice of ι_p. If we change the embedding ι_p, the inclusion i_p changes by conjugation by an element of $G_{\mathbb{Q}}$.

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Similarly, choosing an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, we get the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \xhookrightarrow{\iota} & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{Q} & \hookrightarrow & \mathbb{R} \end{array}$$

This gives us an injection $i : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \hookrightarrow G_{\mathbb{Q}}$ which depends on the choice of ι.
So we get a complex conjugation $c \in G_{\mathbb{Q}}$ which is well defined upto conjugation.

Note that we can replace $\mathbb{Q}$ and $\mathbb{Q}_p$ above by a number field K and $K_{\mathfrak{p}}$, the completion of K at a prime ideal $\mathfrak{p}$ of the ring of integers of K, to get:

- an inclusion $G_{K_{\mathfrak{p}}} \hookrightarrow G_K$ which is well defined upto conjugation,
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Galois representations

For applications to arithmetic problems, we would like to study not only $G_{\mathbb{Q}}$ but the structure of $G_{\mathbb{Q}}$ along with the conjugacy classes of the homomorphisms $i_p : G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$ for every prime p.

For this purpose, it is natural to study the representations of $G_{\mathbb{Q}}$ as they do not distinguish between conjugacy classes.

Galois representation

A Galois representation of $G_{\mathbb{Q}}$ is a continuous homomorphism $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(R)$ for some positive integer n and a topological ring R.

For a prime p, let $I_p \subset G_{\mathbb{Q}_p}$ be the inertia group at p. Using the inclusion $i_p : G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$, we can view I_p as a subgroup of $G_{\mathbb{Q}}$.

We say that a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(R)$ is unramified at p if $\rho(I_p) = 1$.

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Restricting the ramification

Let S be a finite set of primes of $\mathbb{Q}$ along with ∞. Let $\mathbb{Q}_S$ be the maximal algebraic extension of $\mathbb{Q}$ unramified at primes outside S i.e. at all primes not belonging to S.

So $\mathbb{Q}_S$ is the compositum of all finite extensions of $\mathbb{Q}$ which are unramified at primes outside S.

It is easy to verify that $\mathbb{Q}_S$ is Galois over $\mathbb{Q}$. Let $G_{\mathbb{Q},S} := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.

So $G_{\mathbb{Q},S}$ is the quotient of $G_{\mathbb{Q}}$ by the closed normal subgroup of $G_{\mathbb{Q}}$ generated by inertia groups I_ℓ for all primes $\ell \notin S$.

If ρ is a Galois representation of $\mathbb{Q}$ which is unramified at primes lying outside S, then ρ factors through $G_{\mathbb{Q},S}$. We will be focusing on such representations in what follows.

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So $\mathbb{Q}_S$ is the compositum of all finite extensions of $\mathbb{Q}$ which are unramified at primes outside S.

It is easy to verify that $\mathbb{Q}_S$ is Galois over $\mathbb{Q}$. Let $G_{\mathbb{Q},S} := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.

So $G_{\mathbb{Q},S}$ is the quotient of $G_{\mathbb{Q}}$ by the closed normal subgroup of $G_{\mathbb{Q}}$ generated by inertia groups I_ℓ for all primes $\ell \notin S$.

If ρ is a Galois representation of $\mathbb{Q}$ which is unramified at primes lying outside S, then ρ factors through $G_{\mathbb{Q},S}$. We will be focusing on such representations in what follows.

Recall that, for a prime ℓ, we have an exact sequence $1 \to I_\ell \to G_{\mathbb{Q}_\ell} \to G_{\mathbb{F}_\ell} \to 1$ and $G_{\mathbb{F}_\ell} \simeq \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$.

Note that the automorphism of $\overline{\mathbb{F}_\ell}$ which sends an element x of $\overline{\mathbb{F}_\ell}$ to x^ℓ is a topological generator of $G_{\mathbb{F}_\ell}$. We call this generator the Frobenius element at ℓ and denote it by Frob_ℓ.

So for every $\ell \notin S$, there is a Frobenius element $\mathrm{Frob}_\ell \in G_{\mathbb{Q},S}$ which is unique upto conjugation.

So if ρ is a Galois representation of $G_{\mathbb{Q},S}$ (or equivalently of $G_{\mathbb{Q}}$ which is unramified outside S), then for all $\ell \notin S$, $\mathrm{tr}(\rho(\mathrm{Frob}_\ell))$ and $\det(\rho(\mathrm{Frob}_\ell))$ do not depend on the choice of Frob_ℓ and hence, are well defined.

If S' is a finite set of primes of $\mathbb{Q}$, then by Chebotarev density theorem, the set $\{\mathrm{Frob}_\ell \mid \ell \notin S \cup S'\}$ is dense in $G_{\mathbb{Q},S}$.

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p-adic Cyclotomic character

Let p be a prime, $n \geq 1$ be an integer and $\zeta_{p^n} \in \overline{\mathbb{Q}}$ be a primitive p^n-th root of unity. If $\sigma \in G_{\mathbb{Q}}$, then $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{k_\sigma}$ for some $k_\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^\times$.

So this action of $G_{\mathbb{Q}}$ on p^n-th roots of unity yields a character $\omega_{p^n} : G_{\mathbb{Q}} \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ which sends σ to k_σ.

Taking the inverse limit of these characters, we get a character

$$\chi_p := \varprojlim_n \omega_{p^n} : G_{\mathbb{Q}} \to \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$$

which is called the p-adic cyclotomic character.

As the extension $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is ramified only at p for all $n \geq 1$, it follows that χ_p is ramified at p and unramified at all primes $\ell \neq p$.

So for every prime $\ell \neq p$, we have $\chi_p(\mathrm{Frob}_\ell) = \ell$. Moreover, χ_p is an odd character i.e. $\chi_p(c) = -1$.

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p-adic Galois representation attached to an Elliptic curve

Let E be an elliptic curve defined over $\mathbb{Q}$ and p be a prime. Denote the set of $\overline{\mathbb{Q}}$-valued points of E by $E(\overline{\mathbb{Q}})$. For every integer $n \geq 1$, denote the multiplication by p^n isogeny of E by $[p^n]$.

Let $E[p^n] = \{P \in E(\overline{\mathbb{Q}}) \mid [p^n]P = 0\}$. It is well known that $E[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^2$. It is easy to see that the action of $G_{\mathbb{Q}}$ on $E(\overline{\mathbb{Q}})$ preserves $E[p^n]$ for all $n \geq 1$.

So the action of $G_{\mathbb{Q}}$ on $E[p^n]$ gives us a continuous homomorphism $\rho_{E,p,n} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$.

Taking the inverse limit of these Galois representations, we get a Galois representation

$$\rho_{E,p} := \varprojlim_{n} \rho_{E,p,n} : G_{\mathbb{Q}} \to \varprojlim_{n} \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z}_p)$$

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Note that $\det(\rho_{E,p}) = \chi_p$ and hence, $\rho_{E,p}$ is an odd Galois representation which is ramified at p.

The representation $\rho_{E,p}$ is absolutely irreducible.

Theorem (Neron-Ogg-Shafarevich)

Let $\ell \neq p$ be a prime. The Galois representation $\rho_{E,p}$ is unramified at ℓ if and only if E has a good reduction at ℓ.

If $\ell \neq p$ is a prime such that E has good reduction at ℓ, denote by $\bar{E}_\ell$ the reduction of E at ℓ. Then $\mathrm{tr}(\rho_{E,p}(\mathrm{Frob}_\ell)) = \ell + 1 - |\bar{E}_\ell(\mathbb{F}_\ell)|$.

More generally, if A is an abelian variety of dimension g over a number field K, then $A[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^{2g}$ and G_K acts on $A[p^n]$ giving us a Galois representation $\rho_{A,p,n} : G_K \to \mathrm{GL}_{2g}(\mathbb{Z}/p^n\mathbb{Z})$.

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The representation $\rho_{E,p}$ is absolutely irreducible.

Theorem (Neron-Ogg-Shafarevich)

Let $\ell \neq p$ be a prime. The Galois representation $\rho_{E,p}$ is unramified at ℓ if and only if E has a good reduction at ℓ.

If $\ell \neq p$ is a prime such that E has good reduction at ℓ, denote by $\bar{E}_\ell$ the reduction of E at ℓ. Then $\mathrm{tr}(\rho_{E,p}(\mathrm{Frob}_\ell)) = \ell + 1 - |\bar{E}_\ell(\mathbb{F}_\ell)|$.

More generally, if A is an abelian variety of dimension g over a number field K, then $A[p^n] \simeq (\mathbb{Z}/p^n\mathbb{Z})^{2g}$ and G_K acts on $A[p^n]$ giving us a Galois representation $\rho_{A,p,n} : G_K \to \mathrm{GL}_{2g}(\mathbb{Z}/p^n\mathbb{Z})$.

Taking the inverse limit of $\rho_{A,p,n}$, we get a p-adic Galois representation $\rho_A : G_K \to \mathrm{GL}_{2g}(\mathbb{Z}_p)$.

p-adic Galois representations attached to a modular eigenform

Let f be a modular eigenform of level $\Gamma_1(N)$ and weight $k \geq 2$.
So f is an eigenform for the Hecke operators T_ℓ and $\langle \ell \rangle$ for all primes $\ell \nmid N$.
Let ϵ_f be the nebentypus of f and for a prime $\ell \nmid N$, let $a_\ell(f)$ be the T_ℓ-eigenvalue of f.

Let K_f be the extension of $\mathbb{Q}$ generated by the set $\{a_\ell(f) \mid \ell \nmid N\}$ over $\mathbb{Q}$. So K_f is a finite extension of $\mathbb{Q}$. Denote the ring of integers of K_f by $\mathcal{O}_{K_f}$.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and $\mathcal{O}_{K_f,\mathfrak{p}}$ be the completion of $\mathcal{O}_{K_f}$ at $\mathfrak{p}$.

Since every $\ell \neq p$ is invertible in $\mathcal{O}_{K_f,\mathfrak{p}}$, we can view ϵ_f as a character taking values in $\mathcal{O}_{K_f,\mathfrak{p}}^\times$ using the recursive relation between Hecke operators.

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Theorem (Eichler–Shimura, Deligne)

Let f be a modular eigenform of level $\Gamma_1(N)$ and weight $k \geq 2$ with nebentypus ϵ_f. Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and $S = \{q \text{ prime } \mid q \mid N\} \cup \{p, \infty\}$.
Then there exists an *odd* p-adic Galois representation

$$\rho_f : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O}_{K_{f,\mathfrak{p}}})$$

such that

- $\det(\rho_f) = \epsilon_f \chi_p^{k-1}$,
- $\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell(f)$, for all primes $\ell \nmid Np$.

Moreover, ρ_f is absolutely irreducible if and only if f is a cuspform.

Suppose f is a cuspform. Let g be the newform giving rise to f and let ρ_g be the p-adic Galois representation attached to g in the theorem above.
As level of g divides N, we can consider ρ_g as a representation of $G_{\mathbb{Q},S}$.

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By Chebotarev density theorem, the set $\{\mathrm{Frob}_\ell \mid \ell \nmid Np\}$ is dense in $G_{\mathbb{Q},S}$. Therefore, we have $\mathrm{tr}(\rho_f) = \mathrm{tr}(\rho_g)$.
Since ρ_f is absolutely irreducible, we get, by Brauer–Nesbitt theorem, that $\rho_f \simeq \rho_g$ over $\overline{\mathbb{Q}_p}$.

Let M be the level of g. So by the theorem above, we get that ρ_f is unramified at ℓ if $\ell \nmid Mp$.

Theorem (Deligne–Serre)

Let f be a modular eigenform of level $\Gamma_1(N)$ and weight 1 with nebentypus ϵ_f. Let $S = \{q \text{ prime } \mid q \mid N\} \cup \{\infty\}$. Then there exists an *odd* Galois representation

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Now for every prime $\ell \nmid N$, T_ℓ-eigenvalues of f and g are the same. This means that $\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = \mathrm{tr}(\rho_g(\mathrm{Frob}_\ell))$ for all primes $\ell \nmid Np$.

By Chebotarev density theorem, the set $\{\mathrm{Frob}_\ell \mid \ell \nmid Np\}$ is dense in $G_{\mathbb{Q},S}$. Therefore, we have $\mathrm{tr}(\rho_f) = \mathrm{tr}(\rho_g)$.
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Let M be the level of g. So by the theorem above, we get that ρ_f is unramified at ℓ if $\ell \nmid Mp$.

Theorem (Deligne–Serre)

Let f be a modular eigenform of level $\Gamma_1(N)$ and weight 1 with nebentypus ϵ_f. Let $S = \{q \text{ prime } \mid q \mid N\} \cup \{\infty\}$. Then there exists an *odd* Galois representation

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Moreover, if f is an eigenform of level $\Gamma_1(N)$ and weight 1, then ρ_f is irreducible if and only if f is a cuspform.

Suppose f is a cuspform and let g be the newform giving rise to f. From the arguments used in the case of modular eigenforms of weight $k \geq 2$, we get that $\rho_g \simeq \rho_f$.
Hence, if M is the level of g, then it follows that ρ_f is unramified at all primes $\ell \nmid M$.

Now $\mathrm{GL}_2(\mathbb{C})$ has a neighbourhood of 1 which does not contain any non-trivial subgroup. Therefore, $\ker(\rho_f)$ is an open subgroup of $G_{\mathbb{Q},S}$. Since $G_{\mathbb{Q},S}$ is compact, it follows that image of ρ_f is finite. The projective image of ρ_f is either A_4, S_4, A_5 or D_{2n} with $n \geq 3$.

Since the image of ρ_f is finite, we can view it as a representation over $\overline{\mathbb{Q}}$ (i.e. taking values in $\mathrm{GL}_2(\overline{\mathbb{Q}})$). Therefore, for every prime p, using the embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ fixed above, we can view ρ_f as a p-adic representation.

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Recall that we have the exact sequence $1 \to I_\ell \to G_{\mathbb{Q}_\ell} \to \hat{\mathbb{Z}} \to 1$.
Denote the quotient map $G_{\mathbb{Q}_\ell} \to \hat{\mathbb{Z}}$ by f.

We define the Weil group $W_{\mathbb{Q}_\ell} := \{g \in G_{\mathbb{Q}_\ell} \mid f(g) \in \mathbb{Z}\}$. We consider $W_{\mathbb{Q}_\ell}$ with topology in which I_ℓ is open and the subspace topology on I_ℓ coming from $W_{\mathbb{Q}_\ell}$ is same as the subspace topology coming from $G_{\mathbb{Q}_\ell}$. This is *not* the subspace topology on $W_{\mathbb{Q}_\ell}$ coming from $G_{\mathbb{Q}_\ell}$.

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Let L be a finite extension of $\mathbb{Q}_p$ with $p \neq \ell$. A Weil-Deligne representation of $W_{\mathbb{Q}_\ell}$ over a finite dimensional L-vector space V is a pair (r, N) such that

- $r : W_{\mathbb{Q}_\ell} \to \mathrm{GL}(V)$ is a continuous representation under the discrete topology on $\mathrm{GL}(V)$,
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If $\rho : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_n(L)$ is a continuous representation, then after conjugating if necessary, we can assume that ρ takes values in $\mathrm{GL}_n(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers of L.

Therefore if W_ℓ is the wild inertia group at ℓ, then $\rho(W_\ell)$ is finite. So there exists a finite extension K of Q_ℓ such that $\rho(I_K)$ is unipotent. Denote the composition of the map $I_\ell \to I_\ell/W_\ell \to \mathbb{Z}_p$ by ϕ_ℓ and fix a lift σ of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. So there exists an $N \in M_n(L)$ such that $\rho(g) = \exp(\phi_\ell(g)N)$ for all $g \in I_K$.

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The construction given above gives a bijection between the isomorphism classes of $\rho : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ and of n-dimensional bounded Weil-Deligne representations of $W_{\mathbb{Q}_\ell}$.

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Using the local langlands correspondence, we get a Weil-Deligne representation at every prime $\ell \mid N$ which in turn gives a p-adic Galois representation of $G_{\mathbb{Q}_\ell}$ and it turns out to be the same as $\rho_f|_{G_{\mathbb{Q}_\ell}}$.

Theorem (Carayol et al)

Let f be a newform of level $\Gamma_1(N)$ and weight $k \geq 1$ with nebentypus ϵ_f. Let ρ_f be the p-adic Galois representation attached to f as above. If $\ell \mid N$, then ρ_f is ramified at ℓ. Moreover, one of the following holds:

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p-ordinary eigenforms

Let f be a newform of level $\Gamma_1(N)$. We say that f is p-ordinary if one of the following holds:

- $p \nmid N$ and T_p-eigenvalue of f is a p-adic unit,
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Theorem (Hida, Wiles)

Let f be a p-ordinary newform of level $\Gamma_1(N)$ and weight $k \geq 2$. Let ϵ_f be the nebentypus of f and ρ_f be the p-adic Galois representation attached to f as above. Then $\rho_f|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_2 is an unramified character of $G_{\mathbb{Q}_p}$ such that

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Let f be a p-ordinary newform of level $\Gamma_1(N)$ and weight $k \geq 2$. Let ϵ_f be the nebentypus of f and ρ_f be the p-adic Galois representation attached to f as above. Then $\rho_f|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_2 is an unramified character of $G_{\mathbb{Q}_p}$ such that

- $\eta_2(\mathrm{Frob}_p)$ is the p-adic unit root of the polynomial $X^2 - a_p(f)X + \epsilon_f(p)p^{k-1}$ if $p \nmid N$,
- $\eta_2(\mathrm{Frob}_p)$ is the U_p-eigenvalue of f, if $p \mid N$.

In general, one can attach p-adic Galois representations to a wide variety of automorphic forms.

However, in more general settings, we get n-dimensional representations of the absolute Galois group of some number field K.

For instance, if F is a totally real number field, then we can attach a 2-dimensional p-adic Galois representation ρ_f of G_F to a Hilbert modular eigenform f over F.

The properties of ρ_f are similar to those of p-adic Galois representations attached to modular eigenforms. For instance, ρ_f is unramified at primes not dividing the level of f.

One can also define Hida families of Hilbert modular forms and analogues of the theorems of Hida and Wiles also hold in the setting of Hilbert modular forms.

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