Galois representations: Lecture 4

Shaunak Deo

Indian Institute of Science

Elliptic curves and Special values of L functions,
ICTS Bangalore
6 August, 2021

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Theorem (Flach)

Suppose E is an elliptic curve over $\mathbb{Q}$, $p \geq 5$ and E has good reduction at p. Let $S = \{\ell \mid E \text{ has bad reduction at } \ell\} \cup \{p, \infty\}$ and

$$\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$$

be the representation given by the action of $G_{\mathbb{Q},S}$ on $E[p]$. Suppose:

- $\bar{\rho}_p : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}_p)$ is surjective,
- For every $q \in S$, $H^0(\mathbb{Q}_q, E[p] \otimes E[p]) = 0$,
- $p \nmid \Omega^{-1} L(\mathrm{Sym}^2(E), 2)$, where Ω is a transcendental period.

Then $R^{\mathrm{univ}}_{\bar{\rho}_p} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Corollary (Mazur)

Suppose we are in the setup as above. Then for a subset of primes p with Dirichlet density one, the Galois representation $\bar{\rho}_p$ is absolutely irreducible and unobstructed.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.

Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.

Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\text{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\text{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_\mathfrak{p}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_\mathfrak{p})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_\mathfrak{p})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_\mathfrak{p})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Let f be an eigenform of level N, weight k and K_f be the extension of $\mathbb{Q}$ obtained by attaching the Hecke eigenvalues of f to $\mathbb{Q}$.
Let S be a finite set of primes of $\mathbb{Q}$ containing primes dividing N.
Let p be a prime, $\mathfrak{p}$ be a prime of $\mathcal{O}_{K_f}$ lying above p and

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(\mathcal{O}_{K_f,\mathfrak{p}})$$

be the p-adic Galois representation attached to f.

Let $k_{\mathfrak{p}}$ be the residue field of $\mathcal{O}_{K_f,\mathfrak{p}}$ and

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbb{Q},S\cup\{p,\infty\}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$$

be the reduction of $\rho_{f,\mathfrak{p}}$ modulo the maximal ideal of $\mathcal{O}_{K_f,\mathfrak{p}}$.

Theorem (Weston)

- If $k > 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for all but finitely many primes $\mathfrak{p}$ of K_f (depending on f and S).
- If $k = 2$, then $R^{\mathrm{univ}}_{\bar{\rho}_{f,\mathfrak{p}}} \simeq W(k_{\mathfrak{p}})[[X,Y,Z]]$ for a set of primes $\mathfrak{p}$ of K_f of density 1.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1 + W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.
So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1+W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.
So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1 + W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.
So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1+W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.

So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1+W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.

So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Specializing the theorem to the case of Δ and $S = \emptyset$, we have:

Theorem (Weston)

If $p \geq 17$ and $p \neq 691$, then $R^{\text{univ}}_{\bar{\rho}_{\Delta,p}} \simeq \mathbb{Z}_p[[X, Y, Z]]$.

Example by Boston–Ullom: Let

$$\bar{\rho} : G_{\mathbb{Q},\{3,7,\infty\}} \to \mathrm{GL}_2(\mathbb{F}_3)$$

be the representation coming from the action of $G_{\mathbb{Q},\{3,7,\infty\}}$ on 3-torsion points of the elliptic curve $X_0(49)$. Then

$$R^{\text{univ}}_{\bar{\rho}} \simeq \frac{\mathbb{Z}_3[[X, Y, Z, W]]}{((1 + W)^3 - 1)}.$$

In general, if p is odd and $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ is a representation, then $\mathrm{Ad}(\bar{\rho}) = \mathrm{Ad}^0(\bar{\rho}) \oplus 1$, where $\mathrm{Ad}^0(\bar{\rho})$ is the sub-representation of $\mathrm{Ad}(\bar{\rho})$ consisting of trace 0 matrices.
So if S contains a primes ℓ such that $p \mid \ell - 1$, then $H^2(G_{\mathbb{Q},S}, 1) \neq 0$ and hence, $\bar{\rho}$ is not unobstructed.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p).
Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p). Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p).
Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p). Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p). Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformations with prescribed properties

Let G be a pro-finite group satisfying the finiteness condition (Φ_p). Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation such that $\mathrm{End}_G(\bar{\rho}) = \mathbb{F}$.

We will now focus on deformations of $\bar{\rho}$ satisfying certain properties rather than studying all the deformations of $\bar{\rho}$.

To be precise, we will identify some properties of $\bar{\rho}$ and study the deformations of $\bar{\rho}$ retaining that property.

If the properties are 'nice' enough, then these deformations will define a representable subfunctor of the deformation functor.

So for such properties, we can define 'universal deformation ring with the prescribed property' and universal deformation satisfying the prescribed property.

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$.
  If $f \circ \rho$ has property (P), then ρ also has property (P).

Deformation condition

A *deformation condition* on deformations of $\bar{\rho}$ is a property (P) of representations $\rho : G \to \mathrm{GL}_n(R)$ over artinian CNL $W(\mathbb{F})$-algebras R, which is satisfied by $\bar{\rho}$, such that

- Given a deformation $\rho : G \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ and a homomorphism $f : R \to R'$ of artinian CNL $W(\mathbb{F})$-algebras, if ρ has property (P), then $f \circ \rho$ also has property (P).
- Let R_0, R_1 and R_2 be artinian CNL $W(\mathbb{F})$-algebras with morphisms $R_1 \to R_0$ and $R_2 \to R_0$.
  Denote the projection maps of $R_3 := R_1 \times_{R_0} R_2$ on R_1 and R_2 by p_1 and p_2, respectively.
  Then a deformation $\rho : G \to \mathrm{GL}_n(R_3)$ of $\bar{\rho}$ has property (P) if and only if both $p_1 \circ \rho$ and $p_2 \circ \rho$ have property (P).
- Let $f : R \to R'$ be an injective morphism of artinian CNL $W(\mathbb{F})$-algebras and $\rho : G \to \mathrm{GL}_n(R)$ be a deformation of $\bar{\rho}$. If $f \circ \rho$ has property (P), then ρ also has property (P).

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.

Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Let (P) be a deformation condition for $\bar{\rho}$. We define a functor $D_{\bar{\rho},P}$ from the category $\mathcal{C}^0$ of artnian CNL $W(\mathbb{F})$-algebras to the category of sets such that for an object R of $\mathcal{C}^0$

$$D_{\bar{\rho},P}(R) = \{ \text{ deformations of } \bar{\rho} \text{ to } R \text{ which have property (P) } \}.$$

We extend this functor to all of $\mathcal{C}$ by continuity: for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},P}(R) = \varprojlim_k D_{\bar{\rho},P}(R/m_R^k)$. Note that $D_{\bar{\rho},P}$ is a subfunctor of $D_{\bar{\rho}}$, the usual deformation functor of $\bar{\rho}$.

It is easy to verify, using first and second properties of a deformation condition, that the functor $D_{\bar{\rho},P}$ satisfies all the conditions of Schlessinger's criteria.
Therefore, we conclude that $D_{\bar{\rho},P}$ is representable.

In other words, there exists a CNL $W(\mathbb{F})$-algebra $R_{\bar{\rho}}^{(P)}$ and a lift $\rho_P : G \to \mathrm{GL}_n(R_{\bar{\rho}}^{(P)})$ of $\bar{\rho}$ such that for any CNL $W(\mathbb{F})$-algebra R, the map $\mathrm{Hom}(R_{\bar{\rho}}^{(P)}, R) \to D_{\bar{\rho},P}(R)$ given by $\phi \mapsto [\phi \circ \rho_P]$ is a bijection.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^\delta_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^\delta_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^{\delta}_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^\delta_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^{\delta}_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^{\times}$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^{\times} \to R^{\times}$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^{\delta}_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R^\delta_{\bar{\rho}}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Deformations with constant determinant

Let $\delta : G \to W(\mathbb{F})^\times$ be a character lifting $\det(\bar{\rho})$.

Let R be a CNL $W(\mathbb{F})$-algebra and denote by δ_R the character obtained by composing δ with the canonical homomorphism $W(\mathbb{F})^\times \to R^\times$.

We say that a lift $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ *has determinant* δ if $\det(\rho) = \delta_R$.

Denote the property of having determinant δ by (d_δ). Then it is easy to verify that the property (d_δ) is a deformation property.

So for a CNL $W(\mathbb{F})$-algebra R, $D_{\bar{\rho},d_\delta}(R)$ is the set of deformations $\rho : G \to \mathrm{GL}_n(R)$ of $\bar{\rho}$ with $\det(\rho) = \delta_R$.
Denote the universal deformation ring of $\bar{\rho}$ with determinant δ by $R_{\bar{\rho}}^{\delta}$.

Note that $\delta_{\mathbb{F}[\epsilon]} = \det(\bar{\rho})$.

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho}, d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$.
Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho},d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho},d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho},d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho},d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho},d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon])$ be a lift of $\bar{\rho}$ given by

$$\rho(g) = (Id + \epsilon f_\rho(g))\bar{\rho}(g)$$

with $f_\rho(g) \in M_n(\mathbb{F})$.

Then $\det(\rho) = \det(\bar{\rho})$ if and only if $\mathrm{tr}(f_\rho(g)) = 0$ for all $g \in G$. Therefore, the isomorphism $D_{\bar{\rho}}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}(\bar{\rho}))$ gives us an isomorphism

$$D_{\bar{\rho}, d_\delta}(\mathbb{F}[\epsilon]) \simeq H^1(G, \mathrm{Ad}^0(\bar{\rho}))$$

of $\mathbb{F}$-vector spaces.

By universal property, $\det(\rho^{\mathrm{univ}})$ induces a map $R^{\mathrm{univ}}_{\det(\bar{\rho})} \to R^{\mathrm{univ}}_{\bar{\rho}}$.

Now suppose $p \nmid n$. Then by Hensel's lemma, there exists an n-th root of $(\det(\rho^{\mathrm{univ}}))^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}$ i.e. a lift $\chi : G \to (R^{\mathrm{univ}}_{\bar{\rho}})^\times$ of the trivial character such that

$$\chi^n = \det(\rho^{\mathrm{univ}})^{-1}\delta_{R^{\mathrm{univ}}_{\bar{\rho}}}.$$

So $\rho^{\text{univ}} \otimes \chi$ is a deformation of $\bar{\rho}$ to $R_{\bar{\rho}}^{\text{univ}}$ with determinant δ.

Therefore, this deformation induces a map $R_{\bar{\rho}}^{\delta} \to R_{\bar{\rho}}^{\text{univ}}$.

Hence, we get a map

$$\phi : R_{\bar{\rho}}^{\delta} \widehat{\otimes}_{W(\mathbb{F})} R_{\det(\bar{\rho})}^{\text{univ}} \to R_{\bar{\rho}}^{\text{univ}}$$

which we will prove to be an isomorphism.

If

$$\chi^{\text{univ}} : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

is the universal deformation of $\det(\bar{\rho})$, then by Hensel's lemma, there exists a character

$$\eta : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

such that η is a lift of the trivial character and

$$\eta^n = \chi^{\text{univ}} \delta^{-1}_{R_{\det(\bar{\rho})}^{\text{univ}}}.$$

So $\rho^{\text{univ}} \otimes \chi$ is a deformation of $\bar{\rho}$ to $R_{\bar{\rho}}^{\text{univ}}$ with determinant δ. Therefore, this deformation induces a map $R_{\bar{\rho}}^{\delta} \to R_{\bar{\rho}}^{\text{univ}}$.

Hence, we get a map

$$\phi : R_{\bar{\rho}}^{\delta} \widehat{\otimes}_{W(\mathbb{F})} R_{\det(\bar{\rho})}^{\text{univ}} \to R_{\bar{\rho}}^{\text{univ}}$$

which we will prove to be an isomorphism.

If

$$\chi^{\text{univ}} : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

is the universal deformation of $\det(\bar{\rho})$, then by Hensel's lemma, there exists a character

$$\eta : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

such that η is a lift of the trivial character and

$$\eta^n = \chi^{\text{univ}} \delta^{-1}_{R_{\det(\bar{\rho})}^{\text{univ}}}.$$

So $\rho^{\text{univ}} \otimes \chi$ is a deformation of $\bar{\rho}$ to $R^{\text{univ}}_{\bar{\rho}}$ with determinant δ. Therefore, this deformation induces a map $R^{\delta}_{\bar{\rho}} \to R^{\text{univ}}_{\bar{\rho}}$.

Hence, we get a map

$$\phi : R^{\delta}_{\bar{\rho}} \widehat{\otimes}_{W(\mathbb{F})} R^{\text{univ}}_{\det(\bar{\rho})} \to R^{\text{univ}}_{\bar{\rho}}$$

which we will prove to be an isomorphism.

If

$$\chi^{\text{univ}} : G \to (R^{\text{univ}}_{\det(\bar{\rho})})^{\times}$$

is the universal deformation of $\det(\bar{\rho})$, then by Hensel's lemma, there exists a character

$$\eta : G \to (R^{\text{univ}}_{\det(\bar{\rho})})^{\times}$$

such that η is a lift of the trivial character and

$$\eta^n = \chi^{\text{univ}} \delta^{-1}_{R^{\text{univ}}_{\det(\bar{\rho})}}.$$

So $\rho^{\text{univ}} \otimes \chi$ is a deformation of $\bar{\rho}$ to $R_{\bar{\rho}}^{\text{univ}}$ with determinant δ. Therefore, this deformation induces a map $R_{\bar{\rho}}^{\delta} \to R_{\bar{\rho}}^{\text{univ}}$.

Hence, we get a map

$$\phi : R_{\bar{\rho}}^{\delta} \widehat{\otimes}_{W(\mathbb{F})} R_{\det(\bar{\rho})}^{\text{univ}} \to R_{\bar{\rho}}^{\text{univ}}$$

which we will prove to be an isomorphism.

If

$$\chi^{\text{univ}} : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

is the universal deformation of $\det(\bar{\rho})$, then by Hensel's lemma, there exists a character

$$\eta : G \to (R_{\det(\bar{\rho})}^{\text{univ}})^{\times}$$

such that η is a lift of the trivial character and

$$\eta^n = \chi^{\text{univ}} \delta^{-1}_{R_{\det(\bar{\rho})}^{\text{univ}}}.$$

If

$$\rho_\delta : G \to \mathrm{GL}_n(R^\delta_{\bar\rho})$$

is the universal deformation with determinant δ, then $\rho_\delta \otimes \eta$ is a deformation of $\bar\rho$ to $R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$.

Hence, it induces a morphism

$$\phi' : R^{\mathrm{univ}}_{\bar\rho} \to R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}.$$

As $\phi \circ \phi' \circ \rho^{\mathrm{univ}} = \rho^{\mathrm{univ}}$, it follows that $\phi \circ \phi'$ is identity.

As

$$\mathrm{Ad}(\bar\rho) = \mathrm{Ad}^0(\bar\rho) \oplus 1,$$

the dimensions of mod p tangent spaces of both $R^{\mathrm{univ}}_{\bar\rho}$ and $R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$ are same which implies that ϕ' is surjective and hence, an isomorphism.

If

$$\rho_\delta : G \to \mathrm{GL}_n(R^{\delta}_{\bar\rho})$$

is the universal deformation with determinant δ, then $\rho_\delta \otimes \eta$ is a deformation of $\bar\rho$ to $R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$.

Hence, it induces a morphism

$$\phi' : R^{\mathrm{univ}}_{\bar\rho} \to R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}.$$

As $\phi \circ \phi' \circ \rho^{\mathrm{univ}} = \rho^{\mathrm{univ}}$, it follows that $\phi \circ \phi'$ is identity.

As

$$\mathrm{Ad}(\bar\rho) = \mathrm{Ad}^0(\bar\rho) \oplus 1,$$

the dimensions of mod p tangent spaces of both $R^{\mathrm{univ}}_{\bar\rho}$ and $R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$ are same which implies that ϕ' is surjective and hence, an isomorphism.

If

$$\rho_\delta : G \to \mathrm{GL}_n(R^{\delta}_{\bar\rho})$$

is the universal deformation with determinant δ, then $\rho_\delta \otimes \eta$ is a deformation of $\bar\rho$ to $R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$.

Hence, it induces a morphism

$$\phi' : R^{\mathrm{univ}}_{\bar\rho} \to R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}.$$

As $\phi \circ \phi' \circ \rho^{\mathrm{univ}} = \rho^{\mathrm{univ}}$, it follows that $\phi \circ \phi'$ is identity.

As

$$\mathrm{Ad}(\bar\rho) = \mathrm{Ad}^0(\bar\rho) \oplus 1,$$

the dimensions of mod p tangent spaces of both $R^{\mathrm{univ}}_{\bar\rho}$ and $R^{\delta}_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$ are same which implies that ϕ' is surjective and hence, an isomorphism.

If

$$\rho_\delta : G \to \mathrm{GL}_n(R^\delta_{\bar\rho})$$

is the universal deformation with determinant δ, then $\rho_\delta \otimes \eta$ is a deformation of $\bar\rho$ to $R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$.

Hence, it induces a morphism

$$\phi' : R^{\mathrm{univ}}_{\bar\rho} \to R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}.$$

As $\phi \circ \phi' \circ \rho^{\mathrm{univ}} = \rho^{\mathrm{univ}}$, it follows that $\phi \circ \phi'$ is identity.

As

$$\mathrm{Ad}(\bar\rho) = \mathrm{Ad}^0(\bar\rho) \oplus 1,$$

the dimensions of mod p tangent spaces of both $R^{\mathrm{univ}}_{\bar\rho}$ and $R^\delta_{\bar\rho}\widehat{\otimes}_{W(\mathbb{F})}R^{\mathrm{univ}}_{\det(\bar\rho)}$ are same which implies that ϕ' is surjective and hence, an isomorphism.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ. We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1.
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1.
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ. We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ. We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Minimal ramification conditions for $\ell \neq p$

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous representation such that $\mathrm{End}_{G_{\mathbb{Q},S}}(\bar{\rho}) = \mathbb{F}$. Let $\ell \neq p$ be a prime such that $\bar{\rho}$ is ramified at ℓ.
We say that $\bar{\rho}$ is minimally ramified at ℓ if one of the following conditions hold:

- **Case 1:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$,
- **Case 2:** $\bar{\rho}|_{I_\ell} \simeq \begin{pmatrix} \bar{\chi} & 0 \\ 0 & 1 \end{pmatrix}$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.
We say that ρ is minimally ramified at ℓ if one of the following conditions hold:

- $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if we are in Case 1,
- $|\rho(I_\ell)| = |\bar{\rho}(I_\ell)|$ if we are in Case 2.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Let f be a newform of level N and $\bar{\rho}$ be the residual representation of the p-adic representation ρ_f attached to f. Assume $\bar{\rho}$ is absolutely irreducible.

Suppose $\ell > 2$ is a prime such that $\ell \mid\mid N$, $p \nmid \ell^2 - 1$ and $\bar{\rho}$ is ramified at ℓ.
Then $\bar{\rho}$ is minimally ramified at ℓ and ρ_f is a deformation of $\bar{\rho}$ which is minimally ramified at ℓ.

Suppose we are in Case 2. Then ρ is minimally ramified at ℓ if and only if $\ker(\rho|_{I_\ell}) = \ker(\bar{\rho}|_{I_\ell})$.

Suppose we are in Case 1. Then $\rho|_{I_\ell} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ if and only if $(\rho(g) - 1)^2 = 0$ for all $g \in I_\ell$.

Therefore, being minimally ramified at ℓ is a deformation condition.

Denote the functor of minimally ramified at ℓ deformations of $\bar{\rho}$ by $D_{\bar{\rho},\ell}$.

So if R is a CNL $W(\mathbb{F})$-algebra, we have that $D_{\bar{\rho},\ell}(R)$ is the set of deformations ρ of $\bar{\rho}$ to R which are minimally ramified at ℓ.

Now if $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a deformation of $\bar{\rho}$, then ρ is minimally ramified at ℓ if and only if $\rho|_{I_\ell} \simeq \bar{\rho}|_{I_\ell}$.

So if f_ρ is the element in $H^1(G_{\mathbb{Q},S}, \mathrm{Ad}(\bar{\rho}))$ corresponding to ρ, then ρ is minimally ramified at ℓ if and only if $f_\rho|_{I_\ell} = 0$.

Hence, we have an isomorphism

$$D_{\bar{\rho},\ell}(\mathbb{F}[\epsilon]) \simeq \ker(H^1(G_{\mathbb{Q},S}.\mathrm{Ad}(\bar{\rho})) \to H^1(I_\ell, \mathrm{Ad}(\bar{\rho})))$$

of $\mathbb{F}$-vector spaces.

Denote the functor of minimally ramified at ℓ deformations of $\bar{\rho}$ by $D_{\bar{\rho},\ell}$.

So if R is a CNL $W(\mathbb{F})$-algebra, we have that $D_{\bar{\rho},\ell}(R)$ is the set of deformations ρ of $\bar{\rho}$ to R which are minimally ramified at ℓ.

Now if $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a deformation of $\bar{\rho}$, then ρ is minimally ramified at ℓ if and only if $\rho|_{I_\ell} \simeq \bar{\rho}|_{I_\ell}$.

So if f_ρ is the element in $H^1(G_{\mathbb{Q},S}, \mathrm{Ad}(\bar{\rho}))$ corresponding to ρ, then ρ is minimally ramified at ℓ if and only if $f_\rho|_{I_\ell} = 0$.

Hence, we have an isomorphism

$$D_{\bar{\rho},\ell}(\mathbb{F}[\epsilon]) \simeq \ker(H^1(G_{\mathbb{Q},S}.\mathrm{Ad}(\bar{\rho})) \to H^1(I_\ell, \mathrm{Ad}(\bar{\rho})))$$

of $\mathbb{F}$-vector spaces.

Denote the functor of minimally ramified at ℓ deformations of $\bar{\rho}$ by $D_{\bar{\rho},\ell}$.

So if R is a CNL $W(\mathbb{F})$-algebra, we have that $D_{\bar{\rho},\ell}(R)$ is the set of deformations ρ of $\bar{\rho}$ to R which are minimally ramified at ℓ.

Now if $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a deformation of $\bar{\rho}$, then ρ is minimally ramified at ℓ if and only if $\rho|_{I_\ell} \simeq \bar{\rho}|_{I_\ell}$.

So if f_ρ is the element in $H^1(G_{\mathbb{Q},S}, \mathrm{Ad}(\bar{\rho}))$ corresponding to ρ, then ρ is minimally ramified at ℓ if and only if $f_\rho|_{I_\ell} = 0$.

Hence, we have an isomorphism

$$D_{\bar{\rho},\ell}(\mathbb{F}[\epsilon]) \simeq \ker(H^1(G_{\mathbb{Q},S}.\mathrm{Ad}(\bar{\rho})) \to H^1(I_\ell, \mathrm{Ad}(\bar{\rho})))$$

of $\mathbb{F}$-vector spaces.

Denote the functor of minimally ramified at ℓ deformations of $\bar{\rho}$ by $D_{\bar{\rho},\ell}$.

So if R is a CNL $W(\mathbb{F})$-algebra, we have that $D_{\bar{\rho},\ell}(R)$ is the set of deformations ρ of $\bar{\rho}$ to R which are minimally ramified at ℓ.

Now if $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a deformation of $\bar{\rho}$, then ρ is minimally ramified at ℓ if and only if $\rho|_{I_\ell} \simeq \bar{\rho}|_{I_\ell}$.

So if f_ρ is the element in $H^1(G_{\mathbb{Q},S}, \mathrm{Ad}(\bar{\rho}))$ corresponding to ρ, then ρ is minimally ramified at ℓ if and only if $f_\rho|_{I_\ell} = 0$.

Hence, we have an isomorphism

$$D_{\bar{\rho},\ell}(\mathbb{F}[\epsilon]) \simeq \ker(H^1(G_{\mathbb{Q},S}.\mathrm{Ad}(\bar{\rho})) \to H^1(I_\ell, \mathrm{Ad}(\bar{\rho})))$$

of $\mathbb{F}$-vector spaces.

Denote the functor of minimally ramified at ℓ deformations of $\bar{\rho}$ by $D_{\bar{\rho},\ell}$.

So if R is a CNL $W(\mathbb{F})$-algebra, we have that $D_{\bar{\rho},\ell}(R)$ is the set of deformations ρ of $\bar{\rho}$ to R which are minimally ramified at ℓ.

Now if $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F}[\epsilon])$ is a deformation of $\bar{\rho}$, then ρ is minimally ramified at ℓ if and only if $\rho|_{I_\ell} \simeq \bar{\rho}|_{I_\ell}$.

So if f_ρ is the element in $H^1(G_{\mathbb{Q},S}, \mathrm{Ad}(\bar{\rho}))$ corresponding to ρ, then ρ is minimally ramified at ℓ if and only if $f_\rho|_{I_\ell} = 0$.

Hence, we have an isomorphism

$$D_{\bar{\rho},\ell}(\mathbb{F}[\epsilon]) \simeq \ker(H^1(G_{\mathbb{Q},S}.\mathrm{Ad}(\bar{\rho})) \to H^1(I_\ell, \mathrm{Ad}(\bar{\rho})))$$

of $\mathbb{F}$-vector spaces.

Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

By Hensel's lemma, the characteristic polynomial of $\rho(g_0)$ has distinct roots a and b in R such that $a \equiv \bar{\eta}_1(g_0) \pmod{m_R}$ and $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if

$$\mathrm{tr}(\rho(g)(\rho(h) - \det(\rho(h)))(\rho(g_0) - b))) = 0 \text{ for all } g \in G_{\mathbb{Q},S}, h \in I_p.$$

Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

By Hensel's lemma, the characteristic polynomial of $\rho(g_0)$ has distinct roots a and b in R such that $a \equiv \bar{\eta}_1(g_0) \pmod{m_R}$ and $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if

$$\mathrm{tr}(\rho(g)(\rho(h) - \det(\rho(h)))(\rho(g_0) - b))) = 0 \text{ for all } g \in G_{\mathbb{Q},S}, h \in I_p.$$

Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

By Hensel's lemma, the characteristic polynomial of $\rho(g_0)$ has distinct roots a and b in R such that $a \equiv \bar{\eta}_1(g_0) \pmod{m_R}$ and $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if

$$\mathrm{tr}(\rho(g)(\rho(h) - \det(\rho(h)))(\rho(g_0) - b))) = 0 \text{ for all } g \in G_{\mathbb{Q},S}, h \in I_p.$$

Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

By Hensel's lemma, the characteristic polynomial of $\rho(g_0)$ has distinct roots a and b in R such that $a \equiv \bar{\eta}_1(g_0) \pmod{m_R}$ and $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if

$$\mathrm{tr}(\rho(g)(\rho(h) - \det(\rho(h)))(\rho(g_0) - b))) = 0 \text{ for all } g \in G_{\mathbb{Q},S}, h \in I_p.$$

Ordinary deformations

Let $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous, odd and absolutely irreducible representation such that $\bar{\rho}|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \bar{\eta}_1 & * \\ 0 & \bar{\eta}_2 \end{pmatrix}$, where $\bar{\eta}_1 \neq \bar{\eta}_2$ and $\bar{\eta}_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ is an unramified character.

Let g_0 be be a lift of Frob_p in $G_{\mathbb{Q}_p}$ such that $\bar{\eta}_1(g_0) \neq \bar{\eta}_2(g_0)$.

Let R be a CNL $W(\mathbb{F})$-algebra and $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R)$ be a deformation of $\bar{\rho}$.

By Hensel's lemma, the characteristic polynomial of $\rho(g_0)$ has distinct roots a and b in R such that $a \equiv \bar{\eta}_1(g_0) \pmod{m_R}$ and $b \equiv \bar{\eta}_2(g_0) \pmod{m_R}$.

We say that ρ is an ordinary deformation of $\bar{\rho}$ if

$$\mathrm{tr}(\rho(g)(\rho(h) - \det(\rho(h)))(\rho(g_0) - b))) = 0 \text{ for all } g \in G_{\mathbb{Q},S}, h \in I_p.$$

That is for all $g \in G_{\mathbb{Q},S}$ and $h \in I_p$, we have

$$\mathrm{tr}(\rho(ghg_0)) - b\,\mathrm{tr}(\rho(gh)) - \det(\rho(h))\,\mathrm{tr}(\rho(gg_0)) + b\det(\rho(h))\,\mathrm{tr}(\rho(g)) = 0.$$

We call this condition as *ordinary condition* and denote it by (ord).

It is easy to verify that ordinary condition is indeed a deformation condition.

Note that for a CNL $W(\mathbb{F})$-algebra, $D_{\bar{\rho},\mathrm{ord}}(R)$ is the set of deformations of $\bar{\rho}$ to R satisfying the ordinary condition.

Lemma

ρ is ordinary if and only if $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_1 and η_2 are characters lifting $\bar{\eta}_1$ and $\bar{\eta}_2$ and η_2 is an unramified character.

That is for all $g \in G_{\mathbb{Q},S}$ and $h \in I_p$, we have

$$\mathrm{tr}(\rho(ghg_0)) - b\,\mathrm{tr}(\rho(gh)) - \det(\rho(h))\,\mathrm{tr}(\rho(gg_0)) + b\det(\rho(h))\,\mathrm{tr}(\rho(g)) = 0.$$

We call this condition as *ordinary condition* and denote it by (ord).

It is easy to verify that ordinary condition is indeed a deformation condition.

Note that for a CNL $W(\mathbb{F})$-algebra, $D_{\bar{\rho},\mathrm{ord}}(R)$ is the set of deformations of $\bar{\rho}$ to R satisfying the ordinary condition.

Lemma

ρ is ordinary if and only if $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_1 and η_2 are characters lifting $\bar{\eta}_1$ and $\bar{\eta}_2$ and η_2 is an unramified character.

That is for all $g \in G_{\mathbb{Q},S}$ and $h \in I_p$, we have

$$\mathrm{tr}(\rho(ghg_0)) - b\,\mathrm{tr}(\rho(gh)) - \det(\rho(h))\,\mathrm{tr}(\rho(gg_0)) + b\det(\rho(h))\,\mathrm{tr}(\rho(g)) = 0.$$

We call this condition as *ordinary condition* and denote it by (ord).

It is easy to verify that ordinary condition is indeed a deformation condition.

Note that for a CNL $W(\mathbb{F})$-algebra, $D_{\bar{\rho},\mathrm{ord}}(R)$ is the set of deformations of $\bar{\rho}$ to R satisfying the ordinary condition.

Lemma

ρ is ordinary if and only if $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_1 and η_2 are characters lifting $\bar{\eta}_1$ and $\bar{\eta}_2$ and η_2 is an unramified character.

That is for all $g \in G_{\mathbb{Q},S}$ and $h \in I_p$, we have

$$\operatorname{tr}(\rho(ghg_0)) - b\operatorname{tr}(\rho(gh)) - \det(\rho(h))\operatorname{tr}(\rho(gg_0)) + b\det(\rho(h))\operatorname{tr}(\rho(g)) = 0.$$

We call this condition as *ordinary condition* and denote it by (ord).

It is easy to verify that ordinary condition is indeed a deformation condition.

Note that for a CNL $W(\mathbb{F})$-algebra, $D_{\bar{\rho},\mathrm{ord}}(R)$ is the set of deformations of $\bar{\rho}$ to R satisfying the ordinary condition.

Lemma

ρ is ordinary if and only if $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_1 and η_2 are characters lifting $\bar{\eta}_1$ and $\bar{\eta}_2$ and η_2 is an unramified character.

Proof: First suppose $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_i is a lift of $\bar{\eta}_i$ and η_2 is unramified at p.

So by Hensel's lemma, we see that, under this basis, $\rho(g_0) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$. Now we have

$$(\rho(h) - \det(\rho(h)))(\rho(g_0) - b) = \begin{pmatrix} 0 & b_h \\ 0 & 1 - \det(\rho(h)) \end{pmatrix} \begin{pmatrix} a - b & x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the ordinary condition is trivially satisfied.

Proof: First suppose $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_i is a lift of $\bar{\eta}_i$ and η_2 is unramified at p.

So by Hensel's lemma, we see that, under this basis, $\rho(g_0) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$. Now we have

$$(\rho(h) - \det(\rho(h)))(\rho(g_0) - b) = \begin{pmatrix} 0 & b_h \\ 0 & 1 - \det(\rho(h)) \end{pmatrix} \begin{pmatrix} a - b & x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the ordinary condition is trivially satisfied.

Proof: First suppose $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_i is a lift of $\bar{\eta}_i$ and η_2 is unramified at p.

So by Hensel's lemma, we see that, under this basis,
$\rho(g_0) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$. Now we have

$$(\rho(h) - \det(\rho(h)))(\rho(g_0) - b) = \begin{pmatrix} 0 & b_h \\ 0 & 1 - \det(\rho(h)) \end{pmatrix} \begin{pmatrix} a - b & x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the ordinary condition is trivially satisfied.

Proof: First suppose $\rho|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix}$, where η_i is a lift of $\bar{\eta}_i$ and η_2 is unramified at p.

So by Hensel's lemma, we see that, under this basis, $\rho(g_0) = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$. Now we have

$$(\rho(h) - \det(\rho(h)))(\rho(g_0) - b) = \begin{pmatrix} 0 & b_h \\ 0 & 1 - \det(\rho(h)) \end{pmatrix} \begin{pmatrix} a - b & x \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the ordinary condition is trivially satisfied.