# New characterizations of Sobolev spaces

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# Introduction

- Consider $\mathbb{R}^n$ with $n \geq 1$.
- For $1 \leq p < \infty$, the homogeneous Sobolev space $\dot{W}^{1,p}$ consists of all $u \in L^1_{\text{loc}}$ modulo constants, whose distributional gradient $\nabla u \in L^p$. It is normed by

$$\|\nabla u\|_{L^p} = \Big( \int_{\mathbb{R}^n} |\nabla u|^p dx \Big)^{1/p}.$$

- The homogeneous $\dot{\text{BV}}$ space (BV = bounded variation) consists of all $u \in L^1_{\text{loc}}$ modulo constants, whose distributional gradient $\nabla u$ is a finite Radon measure (written $\nabla u \in \mathcal{M}$). It is normed by

$$\|u\|_{\dot{\text{BV}}} = \|\nabla u\|_{\mathcal{M}}.$$

- In the first part of the talk, we describe new characterizations of these spaces. In the second part, we discuss relevance of these ideas to interpolation of Besov spaces, and applications to nonlinear approximation. Finally we show some proofs.

# Besov spaces

- Define Littlewood-Paley projections by setting

$$P_j u = u * \phi_j, \quad j \in \mathbb{Z}$$

where $\phi$ is a fixed non-degenerate Schwartz function on $\mathbb{R}^n$ whose Fourier transform is compactly supported on $\{|\xi| \simeq 1\}$, and $\phi_j(x) := 2^{jn}\phi(2^j x)$.

- For $s \in \mathbb{R}$, $1 < p, q < \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is the space of tempered distributions $u$ modulo polynomials, for which

$$\|u\|_{\dot{B}^s_{p,q}} := \Big( \sum_{j \in \mathbb{Z}} [2^{js} \|P_j u\|_{L^p}]^q \Big)^{1/q} < \infty.$$

- Write $\Delta_h u(x) := u(x+h) - u(x)$ for $x, h \in \mathbb{R}^n$.
- If $0 < s < 1$, $1 < p < \infty$, then $\dot{B}^s_{p,p}$ can be identified with the space of $u \in L^1_{\text{loc}}$ modulo constants, for which

$$\Big( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\Delta_h u(x)|^p}{|h|^{sp+n}} dx dh \Big)^{\frac{1}{p}} = \Big\| \frac{\Delta_h u(x)}{|h|^s} \Big\|_{L^p(\mathbb{R}^{2n}, |h|^{-n} dx dh)} < \infty.$$

# What happens when $s = 1$?

- Often one writes, for $0 < s < 1$ and $1 \leq p < \infty$,

$$\|u\|_{\dot{W}^{s,p}} := \left\| \frac{\Delta_h u(x)}{|h|^s} \right\|_{L^p(\mathbb{R}^{2n}, |h|^{-n} dx dh)}$$
$$= \Big( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\Delta_h u(x)|^p}{|h|^{sp+n}} dx dh \Big)^{\frac{1}{p}}.$$

- We might ask what happens if we compute instead

$$\left\| \frac{\Delta_h u(x)}{|h|} \right\|_{L^p(\mathbb{R}^{2n}, |h|^{-n} dx dh)}.$$

  Heuristically $\frac{|\Delta_h u(x)|}{|h|} \simeq |\nabla u(x)|$, for $|h|$ small.
  Would we recover $\|\nabla u\|_{L^p}$ from the last $L^p(\mathbb{R}^{2n})$ norm?
- Definitely not; even for $u \in C_c^\infty(\mathbb{R}^n)$, the last $L^p(\mathbb{R}^{2n})$ norm is infinite unless $u \equiv 0$ (because $|h|^{-n}$ is not $L^1_{\text{loc}}(dh)$).
- Notation: $\mathcal{Q}_b u(x,h) := \frac{|\Delta_h u(x)|}{|h|^b}$ for $b \in \mathbb{R}$.
  Then $\|u\|_{\dot{W}^{s,p}} = \|\mathcal{Q}_{s+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dx dh)}$, and we saw computing $\|\mathcal{Q}_{1+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dx dh)}$ is not a good idea.

# The BBM formula

- A famous formula by Bourgain, Brezis and Mironescu (BBM) explores what happens to $\|u\|_{\dot{W}^{s,p}}$ as $s \to 1^-$.
- On $\mathbb{R}^n$, it says for $1 \leq p < \infty$ and $u \in C_c^2$, we have

$$\lim_{s \to 1^-} (1-s) \|u\|_{\dot{W}^{s,p}}^p = \frac{k(p,n)}{p} \|\nabla u\|_{L^p}^p \tag{1}$$

  where $k(p,n)$ is some explicit constant depending on $p$ and $n$, given by $k(p,n) := \int_{\mathbb{S}^{n-1}} |e \cdot \omega|^p d\omega$ and $e \in \mathbb{S}^{n-1}$.
- In particular, $\|\mathcal{Q}_{s+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dxdh)}$ blows up like $(1-s)^{-1/p}$ as $s \to 1^-$ unless $u$ is a constant, another indication that $\|\mathcal{Q}_{1+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dxdh)}$ is not good for computing $\|\nabla u\|_{L^p}$.
- In fact, the BBM formula says something more: it says if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, and the left side of (1) is finite, then $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ if $1 < p < \infty$, and $u \in \dot{BV}(\mathbb{R}^n)$ if $p = 1$.
- Our first main result offers an alternative point of view, that does not involve varying $s$, but involves a weak-$L^p$ norm instead of the $L^p$ norm on $\mathbb{R}^{2n}$.

## $L^p$ versus weak-$L^p$

- For $1 \le p < \infty$, if $f \in L^p(\nu)$ for some measure $\nu$, then

$$\|f\|_{L^p(\nu)}^p = \int |f|^p d\nu \ge \lambda^p \nu\{x\colon |f(x)| > \lambda\} \quad \forall \lambda > 0.$$

In particular, if $f \in L^p(\nu)$, then

$$\sup_{\lambda>0} \left( \lambda \nu\{x\colon |f(x)| > \lambda\}^{1/p} \right) < \infty$$

but the converse is not necessarily true.

- If $f$ is measurable and the supremum above is finite, then $f$ is said to be in weak-$L^p(\nu)$. Its weak-$L^p$ (quasi)-norm is defined as the above supremum, and denoted by $[f]_{L^{p,\infty}(\nu)}$.
- Example: $f(x) = |x|^{-n/p}$ is in weak-$L^p(dx)$ on $\mathbb{R}^n$, because

$$\mathcal{L}^n\{x \in \mathbb{R}^n\colon |x|^{-n/p} > \lambda\} = \mathcal{L}^n(B(0, \lambda^{-p/n})) = \lambda^{-p}\mathcal{L}^n(B(0,1)).$$

(Henceforth we write $\mathcal{L}^n$ for Lebesgue measure on $\mathbb{R}^n$.)
It is not in $L^p(dx)$, because $\int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} |x|^{-n} dx = +\infty$.

# Lorentz spaces and real interpolation

- We will also need the Lorentz spaces $L^{p,r}(\nu)$, which for $1 \leq p, r < \infty$ is defined as the space of all measurable $f$ with

$$[f]_{L^{p,r}(\nu)} := \left( r \int_0^\infty \lambda^r \nu\{x \colon |f(x)| > \lambda\}^{r/p} \frac{d\lambda}{\lambda} \right)^{1/r} < \infty.$$

- They arise as real interpolation spaces: if $1 \leq p_0 < p_1 \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for some $0 < \theta < 1$, then for $1 \leq r \leq \infty$,

$$L^{p,r}(\nu) = [L^{p_0}(\nu), L^{p_1}(\nu)]_{\theta,r}$$

  where for any Banach spaces $B_0$ and $B_1$, and $f \in B_0 + B_1$,

$$\|f\|_{[B_0,B_1]_{\theta,r}} := \left( \int_0^\infty t^{-\theta} \inf_{f = f_0 + f_1} (\|f_0\|_{B_0} + t\|f_1\|_{B_1})^r \frac{dt}{t} \right)^{1/r}$$

  for $1 \leq r < \infty$, with the obvious modification when $r = \infty$.
- It is also well-known that $[f]_{L^{p,r}(\nu)} = \|f\|_{L^p(\nu)}$ if $r = p$.

## A formula for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

### Theorem (Brezis, Van Schaftingen, Yung)

*Let $n \geq 1$, $1 \leq p < \infty$ and $u \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \simeq [\mathcal{Q}_{1+\frac{n}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)} = \left[\frac{\Delta_h u}{|h|^{1+\frac{n}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)},$$

*where $\Delta_h u(x) := u(x+h) - u(x)$. In other words, for $\lambda > 0$, denote by $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{n}{p}} u(x,h) > \lambda\right\}$ the superlevel set of $\mathcal{Q}_{1+\frac{n}{p}} u$ at height $\lambda$. Then*

$$\|\nabla u\|^p_{L^p(\mathbb{R}^n)} \simeq \sup_{\lambda > 0} \left(\lambda^p \mathcal{L}^{2n}(E_\lambda)\right).$$

*In fact, we also have* $\dfrac{k(p,n)}{n} \|\nabla u\|^p_{L^p(\mathbb{R}^n)} = \lim_{\lambda \to +\infty} \left(\lambda^p \mathcal{L}^{2n}(E_\lambda)\right).$

- The power $1 + \frac{n}{p}$ is dictated by dilation invariance: if $[\mathcal{Q}_b u]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)}$ scales like $\|\nabla u\|_{L^p}$, then $b = 1 + \frac{n}{p}$.

# A formula for $\|u\|_{L^p(\mathbb{R}^n)}$

## Theorem (Gu, Yung)

*Let $n \geq 1$, $1 \leq p < \infty$ and $u \in L^p(\mathbb{R}^n)$. Then*

$$\|u\|_{L^p(\mathbb{R}^n)} \simeq [\mathcal{Q}_{\frac{n}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)} = \left[\frac{\Delta_h u}{|h|^{\frac{n}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)}.$$

*Furthermore, if now $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{\frac{n}{p}} u(x,h) > \lambda\right\}$, then*

$$2V_n \|u\|^p_{L^p(\mathbb{R}^n)} = \lim_{\lambda \to 0^+} \left(\lambda^p \mathcal{L}^{2n}(E_\lambda)\right)$$

*where $V_n$ is the volume of the unit ball in $\mathbb{R}^n$.*

- We used this to sharpen a constant in an embedding result of Domínguez and Milman.
- Theorem holds for all $u \in L^p(\mathbb{R}^n)$, not just $u \in C_c^\infty$. Can the theorem on the previous slide hold for all $u \in \dot{W}^{1,p}$?

# A family of formulae for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

- It turns out there is a natural one-parameter family of such formulae for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$, for general $u \in \dot{W}^{1,p}$ or $u \in \dot{BV}$.
- Let $\gamma \in \mathbb{R}$. Define the measure $d\nu_\gamma = |h|^{\gamma-n} dx dh$ on $\mathbb{R}^{2n}$. (The case $\gamma = n$ corresponds to the Lebesgue measure $dxdh = \mathcal{L}^{2n}$ we used earlier.)

**Theorem (Brezis, Seeger, Van Schaftingen, Yung)**

*Let $n \geq 1$, $1 < p < \infty$ and $u \in \dot{W}^{1,p}(\mathbb{R}^n)$. Then for $\gamma \neq 0$,*

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \simeq [\mathcal{Q}_{1+\frac{\gamma}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} = \left[\frac{\Delta_h u}{|h|^{1+\frac{\gamma}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)}.$$

*Furthermore, if $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{\gamma}{p}} u(x,h) > \lambda\right\}$, then*

$$\frac{k(p,n)}{|\gamma|} \|\nabla u\|^p_{L^p(\mathbb{R}^n)} = \begin{cases} \lim_{\lambda \to +\infty} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma > 0 \\ \lim_{\lambda \to 0^+} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma < 0. \end{cases}$$

*(The case $\gamma = -p$ of the limit equality is due to Nguyen.)*

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

*Let* $n \geq 1$, $u \in \dot{BV}(\mathbb{R}^n)$. *Then for* $\gamma \in \mathbb{R} \setminus [-1, 0]$,

$$\|u\|_{\dot{BV}(\mathbb{R}^n)} = \|\nabla u\|_{\mathcal{M}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)} = \left[ \frac{\Delta_h u}{|h|^{1+\gamma}} \right]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)}.$$

*Furthermore, if* $E_\lambda := \left\{ (x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\gamma} u(x,h) > \lambda \right\}$, *then the formula*

$$\frac{k(1,n)}{|\gamma|} \|\nabla u\|_{\mathcal{M}} = \begin{cases} \lim_{\lambda \to +\infty} \Big( \lambda \nu_\gamma(E_\lambda) \Big) & \text{if } \gamma > 0 \\ \lim_{\lambda \to 0^+} \Big( \lambda \nu_\gamma(E_\lambda) \Big) & \text{if } \gamma < -1 \end{cases}$$

*holds for* $u \in \dot{W}^{1,1}$ *but can fail for* $u \in \dot{BV}$.

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

*For* $\gamma \in [-1, 0)$,

$$\sup_{u \in C_c^\infty(\mathbb{R}^n), \|\nabla u\|_{L^1(\mathbb{R}^n)}=1} [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)} = +\infty;$$

*furthermore, the formula*

$$\frac{k(1,n)}{|\gamma|} \|u\|_{\dot{BV}} = \lim_{\lambda \to 0^+} \Big( \lambda \nu_\gamma(E_\lambda) \Big)$$

*remains true for all* $u \in C_c^1(\mathbb{R}^n)$, *but fails for* $u \in \dot{W}^{1,1}(\mathbb{R}^n)$, *even though for* $u \in \dot{W}^{1,1}(\mathbb{R}^n)$ *we do have the one-sided inequality*

$$\frac{k(1,n)}{|\gamma|} \|u\|_{\dot{BV}} \leq \liminf_{\lambda \to 0^+} \Big( \lambda \nu_\gamma(E_\lambda) \Big).$$

- The case $\gamma = -1$ of the limiting formula has already been established by Brezis and Nguyen.

## Theorem (Brezis, Seeger, Van Schaftingen, Yung)

*Let* $n \geq 1$, $u \in L^1_{loc}(\mathbb{R}^n)$, $\gamma \in \mathbb{R}$. *If* $[\mathcal{Q}_{1+\frac{\gamma}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty$, *then*

$$u \in \begin{cases} \dot{W}^{1,p}(\mathbb{R}^n) & \text{if } 1 < p < \infty \\ \dot{BV}(\mathbb{R}^n) & \text{if } p = 1. \end{cases}$$

- In particular, for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 < p < \infty$ and $\gamma \neq 0$,

$$u \in \dot{W}^{1,p} \quad \Longleftrightarrow \quad \left[\frac{\Delta_h u}{|h|^{1+\frac{\gamma}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty.$$

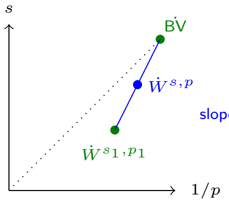
- Similarly, for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\gamma \in \mathbb{R} \setminus [-1, 0]$,

$$u \in \dot{\mathrm{BV}} \quad \Longleftrightarrow \quad \left[\frac{\Delta_h u}{|h|^{1+\gamma}}\right]_{L^{1,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty.$$

- The existence of a one-parameter family of characterizations is not just natural, but useful in applications.

# Application towards Gagliardo-Nirenberg interpolation

- Cohen, Dahmen, Daubechies and DeVore proved that for any $0 < s_1 < 1$ and any $1 < p_1 < \infty$, if
$$s_1 < \frac{1}{p_1},$$
and if $(\frac{1}{p}, s) = (1-\theta)(1,1) + \theta(\frac{1}{p_1}, s_1)$ for some $0 < \theta < 1$, then for any $u \in \dot{\mathsf{BV}} \cap \dot{W}^{s_1,p_1}$,
$$\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{\mathsf{BV}}}^{1-\theta} \|u\|_{\dot{W}^{s_1,p_1}}^{\theta}.$$




- Their proof uses bounds for coefficients of wavelet expansions of a general function in $\dot{\mathsf{BV}}(\mathbb{R}^n)$.
- We can give an alternative proof based on our theorem for $\dot{\mathsf{BV}}$.

- Indeed, let $\gamma$ be minus the slope, given by $\gamma := -\frac{1-s_1}{1-\frac{1}{p_1}} < -1$.
- Let $u \in \dot{\mathrm{B}}\mathrm{V} \cap \dot{W}^{s_1,p_1}$. Our characterization for $\dot{\mathrm{B}}\mathrm{V}$ shows that

$$\|u\|_{\dot{\mathrm{B}}\mathrm{V}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}.$$

- On the other hand, $\|u\|_{\dot{W}^{s_1,p_1}} = \|\mathcal{Q}_{s_1+\frac{\gamma}{p_1}} u\|_{L^{p_1}(\nu_\gamma)}$ because

$$\Big( \iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u|^{p_1}}{|h|^{s_1 p_1 + n}} dx dh \Big)^{\frac{1}{p_1}} = \Big( \iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u|^{p_1}}{|h|^{s_1 p_1 + \gamma}} d\nu_\gamma \Big)^{\frac{1}{p_1}}.$$

  Similarly $\|u\|_{\dot{W}^{s,p}} = \|\mathcal{Q}_{s+\frac{\gamma}{p}} u\|_{L^p(\nu_\gamma)}$.
- But our choice of $\gamma$ ensures $s + \frac{\gamma}{p} = 1 + \gamma = s_1 + \frac{\gamma}{p_1}$. Using

$$\|F\|_{L^p(\nu_\gamma)} \lesssim \|F\|_{L^{1,\infty}(\nu_\gamma)}^{1-\theta} \|F\|_{L^{p_1}(\nu_\gamma)}^{\theta}$$

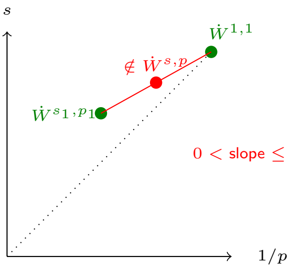
  for $F := \mathcal{Q}_{s+\frac{\gamma}{p}} u = \mathcal{Q}_{1+\gamma} u = \mathcal{Q}_{s_1+\frac{\gamma}{p_1}} u$, we obtain

$$\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{\mathrm{B}}\mathrm{V}}^{1-\theta} \|u\|_{\dot{W}^{s_1,p_1}}^{\theta}.$$

- Let's revisit the result of Cohen-Dahmen-Daubechies-DeVore.
- Suppose $0 < s_1 < 1$, $1 < p_1 < \infty$, and

$$(\frac{1}{p}, s) = (1-\theta)(1,1) + \theta(\frac{1}{p_1}, s_1) \quad \text{for some } 0 < \theta < 1.$$

- We saw if $s_1 < \frac{1}{p_1}$ then $\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{\text{BV}}}^{1-\theta} \|u\|_{\dot{W}^{s_1,p_1}}^{\theta}$.
- The previous proof made crucial use of $s_1 < \frac{1}{p_1}$, because $\|u\|_{\dot{\text{BV}}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}$ only holds when $\gamma \notin \mathbb{R} \setminus [-1, 0]$.
- In fact the result is false when $s_1 \geq \frac{1}{p_1}$ (Brezis-Mironescu).



- Let's revisit the result of Cohen-Dahmen-Daubechies-DeVore.
- Suppose $0 < s_1 < 1$, $1 < p_1 < \infty$, and

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- We saw if $s_1 < \frac{1}{p_1}$ then $\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{\text{BV}}}^{1-\theta} \|u\|_{\dot{W}^{s_1,p_1}}^{\theta}$.
- The previous proof made crucial use of $s_1 < \frac{1}{p_1}$, because $\|u\|_{\dot{\text{BV}}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}$ only holds when $\gamma \notin \mathbb{R} \setminus [-1, 0]$.
- In fact the result is false when $s_1 \geq \frac{1}{p_1}$ (Brezis-Mironescu).
- Nevertheless, the above proof can be easily adapted, to show that for any $\gamma' \in \mathbb{R} \setminus [-1, 0]$, we still have

$$[\mathcal{Q}_{s+\frac{\gamma'}{p}} u]_{L^{p,\frac{p_1}{\theta}}(\nu_{\gamma'})} \lesssim \|u\|_{\dot{\text{BV}}}^{1-\theta} \|u\|_{\dot{W}^{s_1,p_1}}^{\theta}.$$

  (See joint work with Brezis and Van Schaftingen.)
- One is then led to characterize (measurable) functions $u$ for which the left hand side is finite.

# A twist on diagonal Besov spaces

- For $\gamma \in \mathbb{R}$, define a measure $\mu_\gamma$ on $\mathbb{R}^n \times \mathbb{Z}$ so that

$$\int_{\mathbb{R}^n \times \mathbb{Z}} F(x,j) d\mu_\gamma = \sum_{j \in \mathbb{Z}} 2^{-j\gamma} \int_{\mathbb{R}^n} F(x,j) dx.$$

- Then for $s \in \mathbb{R}$ and $1 < p < \infty$,

$$\begin{aligned} \|u\|_{\dot{B}^s_{p,p}} &= \Big( \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} [2^{js} |P_j u(x)|]^p dx \Big)^{1/p} \\ &= \Big( \sum_{j \in \mathbb{Z}} 2^{-j\gamma} \int_{\mathbb{R}^n} [2^{j(s+\frac{\gamma}{p})} |P_j u(x)|]^p dx \Big)^{1/p} \\ &= \|2^{j(s+\frac{\gamma}{p})} P_j u(x)\|_{L^p(\mathbb{R}^n \times \mathbb{Z}, \mu_\gamma)}. \end{aligned}$$

$$\|u\|_{\dot{B}^s_{p,p}} = \|2^{j(s+\frac{\gamma}{p})} P_j u(x)\|_{L^p(\mathbb{R}^n \times \mathbb{Z}, \mu_\gamma)}$$

- For $s \in \mathbb{R}$, $1 < p < \infty$, $\gamma \in \mathbb{R}$ and $1 \leq r \leq \infty$, let $\dot{\mathcal{B}}^s_p(\gamma, r)$ be the space of all tempered distributions $u$ modulo polynomials, for which

$$\|u\|_{\dot{\mathcal{B}}^s_p(\gamma, r)} := \|2^{j(s+\frac{\gamma}{p})} P_j u(x)\|_{L^{p,r}(\mathbb{R}^n \times \mathbb{Z}, \mu_\gamma)} < \infty.$$

- Note that $\dot{\mathcal{B}}^s_p(\gamma, r) = \dot{B}^s_{p,p}$ whenever $r = p$ (independent of $\gamma$).
- But examples show that $\dot{\mathcal{B}}^s_p(\gamma, r) \neq \dot{\mathcal{B}}^s_p(\gamma', r)$ whenever $r > p$ and $\gamma \neq \gamma'$.
- Such distribution of weight into the measure has appeared also in work on radial Fourier multipliers, and Fourier restriction theorems with affine arclength measure on curves.

# Difference quotient characterization for $\dot{\mathcal{B}}^s_p(\gamma, r)$

## Theorem (Domínguez, Seeger, Street, Van Schaftingen, Yung)

*Let* $0 < s < 1$, $1 < p < \infty$, $1 \leq r \leq \infty$ *and* $\gamma \in \mathbb{R}$.

(a) *If* $u \in \dot{\mathcal{B}}^s_p(\gamma, r)$, *then the tempered distribution* $u$ *can be identified with a tempered,* $L^1_{loc}$ *function, such that*

$$[\mathcal{Q}_{s+\frac{\gamma}{p}} u]_{L^{p,r}(\nu_\gamma)} \lesssim \|u\|_{\dot{\mathcal{B}}^s_p(\gamma,r)}.$$

(b) *If* $u$ *is a measurable function on* $\mathbb{R}^n$ *with* $\mathcal{Q}_{s+\frac{\gamma}{p}} u \in L^{p,r}(\nu_\gamma)$, *then* $u$ *defines a tempered distribution in* $\dot{\mathcal{B}}^s_p(\gamma, r)$, *with*

$$\|u\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} \lesssim [\mathcal{Q}_{s+\frac{\gamma}{p}} u]_{L^{p,r}(\nu_\gamma)}.$$

- In particular, for Schwartz functions $u$, for such $s, p, \gamma, r$,

$$\|u\|_{\dot{\mathcal{B}}^s_p(\gamma,r)} := \|2^{j(s+\frac{\gamma}{p})} P_j u(x)\|_{L^{p,r}(\mathbb{R}^n \times \mathbb{Z}, \mu_\gamma)} \simeq \left[\frac{\Delta_h u}{|h|^{s+\frac{\gamma}{p}}}\right]_{L^{p,r}(\mathbb{R}^{2n}, \nu_\gamma)}.$$

- For $r = p$ (so that $\dot{\mathcal{B}}^s_p(\gamma, r) = \dot{B}^s_{p,p}$), such characterization is known if 'measurable' in (b) is replaced by $L^1_{\text{loc}}$.

# Interpolation of diagonal Besov spaces

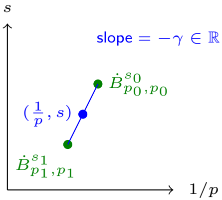
## Theorem (Domínguez, Seeger, Street, Van Schaftingen, Yung)

*Let* $1 \le p_0, p_1 \le \infty$, $p_0 \neq p_1$, *and* $s_0, s_1 \in \mathbb{R}$. *Then for* $0 < \theta < 1$ *and* $1 \le r \le \infty$,

$$[\dot{B}^{s_0}_{p_0,p_0}, \dot{B}^{s_1}_{p_1,p_1}]_{\theta,r} = \dot{\mathcal{B}}^{s}_{p}(\gamma, r),$$

*with comparable norms, where* $(\frac{1}{p}, s) = (1-\theta)(\frac{1}{p_0}, s_0) + \theta(\frac{1}{p_1}, s_1)$ *and*

$$\gamma = -\frac{s_0 - s_1}{\frac{1}{p_0} - \frac{1}{p_1}}.$$



# Interpolation of diagonal Besov spaces

**Theorem (Domínguez, Seeger, Street, Van Schaftingen, Yung)**

*Let* $1 \le p_0, p_1 \le \infty$, $p_0 \neq p_1$, *and* $s_0, s_1 \in \mathbb{R}$. *Then for* $0 < \theta < 1$ *and* $1 \le r \le \infty$,

$$[\dot{B}^{s_0}_{p_0,p_0}, \dot{B}^{s_1}_{p_1,p_1}]_{\theta,r} = \dot{\mathcal{B}}^{s}_{p}(\gamma, r),$$

*with comparable norms, where* $(\frac{1}{p}, s) = (1-\theta)(\frac{1}{p_0}, s_0) + \theta(\frac{1}{p_1}, s_1)$ *and*

$$\gamma = -\frac{s_0 - s_1}{\frac{1}{p_0} - \frac{1}{p_1}}.$$

- In particular, using our previous characterization of $\dot{\mathcal{B}}^{s}_{p}(\gamma, r)$, if in addition $0 < s_0, s_1 < 1$, then

$$\|u\|_{[\dot{B}^{s_0}_{p_0,p_0}, \dot{B}^{s_1}_{p_1,p_1}]_{\theta,r}} \simeq \left[\frac{\Delta_h u}{|h|^{s+\frac{\gamma}{p}}}\right]_{L^{p,r}(\nu_\gamma)}.$$

- The case $r = p$, namely $[\dot{B}^{s_0}_{p_0,p_0}, \dot{B}^{s_1}_{p_1,p_1}]_{\theta,p} = \dot{B}^{s}_{p,p}$, is classical.

# Applications to nonlinear approximation

- Fix a system of (smooth) wavelet basis of $L^2(\mathbb{R}^n)$.
- Let $1 < q < \infty$. For $u \in L^q(\mathbb{R}^n)$ and $k \geq 1$, let $\sigma_k(u)_{L^q}$ be the distance in $L^q(\mathbb{R}^n)$ between $u$ and its best $k$-term approximation in terms of the wavelet basis.
- For $\alpha > 0$ and $0 < r \leq \infty$, let $\mathcal{A}_r^\alpha(L^q)$ be the set of functions $u \in L^q(\mathbb{R}^n)$ for which

$$\|u\|_{\mathcal{A}_r^\alpha(L^q)} = \Big( \sum_{k=1}^{\infty} [k^\alpha \sigma_k(u)_{L^q}]^r \tfrac{1}{k} \Big)^{1/r} \quad \text{if } r < \infty$$

  with the obvious modification when $r = \infty$.
- This is useful in statistics, image processing, compression, numerical solutions to PDEs, ...
- Idea: $\|u\|_{\mathcal{A}_r^\alpha(L^q)}$ increases as $\alpha$ increases. For this norm to be finite for big $\alpha$, one needs $u$ to have very few essential terms in its wavelet expansion, making $u$ very smooth.

- DeVore, Jawerth and Popov proved that if $s > 0$ and $\frac{1}{r} = \frac{1}{q} + \frac{s}{n}$, then
$$\mathcal{A}_r^{s/n}(L^q) = \dot{B}^s_{r,r},$$
confirming the previous heuristic that functions in $\mathcal{A}_r^{\alpha}(L^q)$ gets smoother as $\alpha$ increases in this special case.
- Building upon recent work of Besoy, Haroske and Triebel, we consider the case $\frac{1}{r} \neq \frac{1}{q} + \frac{s}{n}$.

**Theorem (Domínguez, Seeger, Street, Van Schaftingen, Yung)**

*For $1 < q < \infty$, $0 < s < n(1 - \frac{1}{q})$, and $1 \le r \le \infty$, if $\frac{1}{p} = \frac{1}{q} + \frac{s}{n}$, then*
$$\mathcal{A}_r^{s/n}(L^q) = \dot{\mathcal{B}}^s_p(-n, r).$$

- The case $r = \infty$ is of special interest in applications; see e.g. Hansen and Sickel.

# Ideas of proofs: some positive results about $\dot{\text{BV}}$

- Let's remember what we said about $\dot{\text{BV}}$. Let $\gamma \in \mathbb{R} \setminus [-1, 0]$.
- We said for $n \geq 1$ and $u \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\left[\frac{u(x+h) - u(x)}{|h|^{1+\gamma}}\right]_{L^{1,\infty}(\mathbb{R}^{2n}, |h|^{\gamma - n} dx dh)} \simeq \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)}$$

  in the sense that one side is finite iff the other is, and that the two sides are comparable.
- Let's prove $\lesssim$ when $n = \gamma = 1$, which we rewrite as

$$\left[\frac{u(y) - u(x)}{|y - x|^2}\right]_{L^{1,\infty}(\mathbb{R}^2, dx dy)} \lesssim \|u'\|_{\mathcal{M}(\mathbb{R})},$$

  for the case $u \in \dot{\text{BV}}(\mathbb{R})$. First show this for $u \in C^1_c(\mathbb{R})$.
- The proof relies on the Vitali covering lemma in 1-dimension: If $X$ is a collection of intervals on $\mathbb{R}$ with $\sup_{I \in X} |I| < \infty$, then there exists a subcollection $Y \subset X$ such that all intervals from $Y$ are pairwise disjoint up to end-points, and every $I \in X$ is contained in $5J$ for some $J \in Y$.

- Goal: Show that for $u \in C_c^1(\mathbb{R})$ and $\lambda > 0$,

$$|E_\lambda| \lesssim \frac{1}{\lambda}\|u'\|_{L^1(\mathbb{R})}$$

where $E_\lambda := \left\{(x,y) \in \mathbb{R}^2 \colon \frac{|u(y)-u(x)|}{|y-x|^2} > \lambda\right\}$.
- Let $X$ be the collection of intervals $[x,y]$ where $(x,y) \in E_\lambda$.
- Vitali covering lemma applies to $X$ because if $I = [x,y] \in X$,

$$|I|^2 = |x-y|^2 < \frac{1}{\lambda}|u(x)-u(y)| \leq \frac{1}{\lambda}\int_I |u'| \leq \frac{1}{\lambda}\|u'\|_{L^1(\mathbb{R})}.$$

- We obtain a subcollection $Y \subset X$ such that all intervals from $Y$ are pairwise disjoint up to end-points, and every $I \in X$ is contained in $5J$ for some $J \in Y$.
- As a result, $E_\lambda \subset \bigcup_{I \in X} I \times I \subset \bigcup_{J \in Y} (5J) \times (5J)$, and

$$|E_\lambda| \leq \sum_{J \in Y} |5J|^2 \leq 25 \sum_{J \in Y} \frac{1}{\lambda}\int_J |u'| \leq \frac{25}{\lambda}\|u'\|_{L^1(\mathbb{R})}.$$

- We have shown that for $u \in C_c^1(\mathbb{R})$,

$$\left[\frac{u(y)-u(x)}{|y-x|^2}\right]_{L^{1,\infty}(\mathbb{R}^2,dxdy)} \lesssim \|u'\|_{\mathcal{M}(\mathbb{R})}. \tag{2}$$

- We want to show this for general $u \in \dot{\text{BV}}(\mathbb{R})$.
- But $C_c^1(\mathbb{R})$ is not dense in $\dot{\text{BV}}(\mathbb{R})$. Not even in $\dot{W}^{1,1}(\mathbb{R})$.
- The issue is that even if $u \in C^\infty(\mathbb{R})$ and $u' \in C_c(\mathbb{R})$, it is possible for $u$ to take different values at $+\infty$ and $-\infty$. Hence $u$ itself is *not* compactly supported.
- This is a problem that only occurs in 1 dim: the complement of a compact set is always connected in higher dimensions.
- Fortunately, if $\mathfrak{C}$ is the set of all $u \in C^1(\mathbb{R})$ with $u'$ compactly supported in $\mathbb{R}$ (this is slightly bigger than $C_c^1(\mathbb{R})$), then (2) still holds for $\mathfrak{C}$ via the same proof, and $\mathfrak{C}$ is dense in $\dot{W}^{1,1}(\mathbb{R})$.
- So (2) holds for all $u \in \dot{W}^{1,1}(\mathbb{R})$.
- By approximating $\dot{\text{BV}}(\mathbb{R})$ functions with $\dot{W}^{1,1}(\mathbb{R})$ functions pointwise a.e., one shows (2) holds for $u \in \dot{\text{BV}}(\mathbb{R})$ as well.

- Recap: Let $\gamma \in \mathbb{R} \setminus [-1, 0]$.
- We said for $n \geq 1$ and $u \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\left[\frac{u(x+h)-u(x)}{|h|^{1+\gamma}}\right]_{L^{1,\infty}(\mathbb{R}^{2n}, |h|^{\gamma-n} dx dh)} \simeq \|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)}$$

  in the sense that one side is finite iff the other is, and that the two sides are comparable.
- When $n = \gamma = 1$ we sketched the proof of $\lesssim$.
- Still assuming $n = \gamma = 1$, let's see why the reversed inequality holds, i.e. why the $\dot{\text{BV}}(\mathbb{R})$ norm of $u$ is controlled by

$$\left[\frac{u(x+h)-u(x)}{|h|^{2}}\right]_{L^{1,\infty}(\mathbb{R}^{2}, dx dh)}.$$

- First consider the special case $u \in C^2_c(\mathbb{R})$. Then consider the general case for $u \in L^1_{\text{loc}}(\mathbb{R})$.

- If we already know $u \in C_c^2(\mathbb{R})$, then one can show

$$\|u'\|_{L^1(\mathbb{R})} = \lim_{\lambda \to +\infty} \frac{\lambda}{2} \mathcal{L}^2 \Big\{ (x,h) \in \mathbb{R}^2 \colon \frac{|u(x+h) - u(x)|}{|h|^2} > \lambda \Big\}.$$

  This is because the set on the right side is roughly the same as

$$\Big\{ (x,h) \in \mathbb{R}^2 \colon \frac{|u'(x)|}{|h|} > \lambda \Big\}$$

  whose $\mathcal{L}^2$ measure is equal to

$$\int_{\mathbb{R}} \int_{|h| \leq \frac{|u'(x)|}{\lambda}} dh dx = \int_{\mathbb{R}} 2 \frac{|u'(x)|}{\lambda} dx.$$

  This approximation gets better as $\lambda \to +\infty$, hence the first equality holds.
- Since $\lim_{\lambda \to +\infty} \leq \sup_{\lambda > 0}$, we see that for $u \in C_c^2(\mathbb{R})$,

$$\|u'\|_{L^1(\mathbb{R})} \leq \frac{1}{2} \Big[ \frac{u(x+h) - u(x)}{|h|^2} \Big]_{L^{1,\infty}(\mathbb{R}^2, dxdh)}.$$

- On the other hand, this limiting formula

$$\|u'\|_{\mathcal{M}(\mathbb{R})} = \lim_{\lambda \to +\infty} \frac{\lambda}{2} \mathcal{L}^2 \left\{ (x,h) \in \mathbb{R}^2 \colon \frac{|u(x+h) - u(x)|}{|h|^2} > \lambda \right\}$$

  is known to *fail* for general $u \in \dot{\mathrm{BV}}(\mathbb{R})$. Thus the above proof won't work in general.
- What comes to our rescue is the BBM formula.
- If $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $A := \left[ \frac{u(x+h)-u(x)}{|h|^2} \right]_{L^{1,\infty}(\mathbb{R}^2, dxdh)} < \infty$, one can show that

$$\lim_{s \to 1^-} (1-s) \iint_{[-R,R]^2} \frac{|u(x+h) - u(x)|}{|h|^{1+s}} dxdh \lesssim A$$

  uniformly in $R$, because

$$\frac{|u(x+h) - u(x)|}{|h|^{1+s}} = \left( \frac{|u(x+h) - u(x)|}{|h|^2} \right)^{\frac{1+s}{2}} |u(x+h) - u(x)|^{\frac{1-s}{2}}$$

  and the first factor is in $L^{\frac{2}{1+s},\infty}(\mathbb{R}^2, dxdh)$ when $A < \infty$.
- The BBM formula then gives $u \in \dot{\mathrm{BV}}(\mathbb{R})$ with $\|u'\|_{\mathcal{M}(\mathbb{R})} \lesssim A$.

# Related works

- Recently, Óscar Domínguez and Mario Milman have been able to put some of the above results in an abstract framework.
- They proved that if $X$ is a $\sigma$-finite measure space, $1 \leq p < \infty$ and $\{T_t\}_{t>0}$ is a family of sublinear operators on $L^p(X)$, then for all $f \in L^p(X)$ satisfying

$$\|T_t f - f\|_{L^\infty(X)} \lesssim_f t^{1/p} \quad \text{for all } t > 0,$$

we have

$$\lim_{\lambda \to \infty} \left(\lambda |E_\lambda|^{1/p}\right) = \|f\|_{L^p(X)},$$

where

$$E_\lambda := \left\{(x,t) \in X \times (0,\infty) \colon \frac{|T_t f(x)|}{t^{1/p}} > \lambda\right\}.$$

- They found an impressive list of applications, from a characterization of $\|\Delta u\|_{L^p(\mathbb{R}^n)}$ and $\|\partial_{x_1}\partial_{x_2} u\|_{L^p(\mathbb{R}^2)}$, to relations between $\|f\|_{L^p(\mathbb{R}^n)}$ with level set estimates for spherical averages of $f$ for $p > \frac{n}{n-1}$, to ergodic theory, etc.

# Open questions

- Let $n \geq 1$, $\nu_\gamma := |h|^{\gamma-n} dx dh$, $1 \leq p < \infty$, $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, and

$$E_\lambda := \left\{ (x,h) \in \mathbb{R}^{2n} \colon \frac{|\Delta_h u(x)|}{|h|^{1+\frac{\gamma}{p}}} > \lambda \right\}.$$

1. Suppose

$$\liminf_{\lambda \to +\infty} \left( \lambda \nu_\gamma (E_\lambda)^{1/p} \right) < +\infty \quad \text{for some } \gamma > 0, \text{ or}$$

$$\liminf_{\lambda \to 0^+} \left( \lambda \nu_\gamma (E_\lambda)^{1/p} \right) < +\infty \quad \text{for some } \gamma < 0.$$

   Must it be true that $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ if $1 < p < \infty$, and $u \in \dot{\text{BV}}(\mathbb{R}^n)$ if $p = 1$?

2. Suppose $p = 1$, and $u \in \dot{\text{BV}}(\mathbb{R}^n)$. Must it be true that

$$\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} \lesssim \liminf_{\lambda \to +\infty} \left( \lambda \nu_\gamma (E_\lambda) \right) \quad \text{if } \gamma > 0?$$

$$\|\nabla u\|_{\mathcal{M}(\mathbb{R}^n)} \lesssim \liminf_{\lambda \to 0^+} \left( \lambda \nu_\gamma (E_\lambda) \right) \quad \text{if } \gamma < 0?$$

   These are true if we assume additionally $u \in \dot{W}^{1,1}(\mathbb{R}^n)$.

- See works of Nguyen, Brezis and Bourgain for positive answers to the above questions when $\gamma = -p$.
- See also Poliakovsky's work for $\gamma = n$, where $\liminf$ is replaced by $\limsup$.