Heat loss at cold boundary under turbulence: an SPDE model

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Turbulence: Problems at the Interface of Mathematics and Physics

Plan of the lecture

- Stabilization by noise: a finite dimensional result. Infinite dimensional analog?
- A solution by transport noise
- Eddy viscosity/heat diffusion coefficient and SPDE models

The work presented here is based on works in preparation, jointly with Lucio Galeati, Dejun Luo, Umberto Pappalettera and based on experience with many other collaborators and students.

Stabilization by noise: a finite dimensional result

L. Arnold, H. Crauel, V. Wihstutz, Stabilization of linear systems by noise, SIAM J. Control Optimiz. 1983: in $\mathbb{R}^d$, A, B_j matrices, W_t^j independent Brownian motions, $\sigma > 0$

$$d\mathbf{X}_t = A\mathbf{X}_t dt + \sigma \sum_{j=1}^{N} B_j \mathbf{X}_t \circ dW_t^j$$

Top Lyapunov exponent: $\lambda_\sigma := \lim_{t\to\infty} \frac{1}{t} \log |\mathbf{X}_t|$

Theorem

There exist N and skew-symmetric matrices B_j, $j = 1, ..., N$ such that

$$\lim_{\sigma\to\infty} \lambda_\sigma = \frac{TrA}{d}.$$

To understand the power of the result, consider the example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Then

$$\begin{aligned} \lambda_0 &= 1 \qquad \text{(no noise)} \\ \lim_{\sigma \to \infty} \lambda_\sigma &= \frac{1+0-4}{3} = -1 \end{aligned}$$

Conjecture, by Marek Capinski (1987, talk given in Bremen): could something similar hold for

$$d_t T = \underbrace{\kappa \Delta T}_{A} dt + \sigma \sum_j \underbrace{\mathbf{u}_j \cdot \boldsymbol{\nabla}}_{B_j} T \circ dW_t^j$$

for large σ, when $\mathrm{div}\, b_j = 0$? $T = T(t, \mathbf{x})$ is temperature (for instance).

Remark 1: $\mathrm{div}\, b_j = 0 \Rightarrow B_j$ skew-symmetric in L^2.

Remark 2: $\sigma = 0 \Rightarrow L^2$-decay is $e^{-\kappa\lambda_1 t}$, λ_1 first eigenvalue of $-\Delta$ (with the given b.c.'s).

Remark 3: Expected decay by noise: *arbitrarily large* ($\frac{Tr\Delta_N}{N} \to -\infty$).

Two faces, both of interest for applications:

1. internal mixing
2. heat loss through a cold boundary (or, conversely, heating by warm boundary)

The talk focuses on the second problem, more difficult.

The true model (in the Boussinesq approximation) would be

$$\begin{aligned} \partial_t T &= \kappa \Delta T + \mathbf{u} \cdot \nabla T \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p &= \frac{1}{\mathrm{Re}} \Delta \mathbf{u} + f + g \mathbf{e}_z T \\ \operatorname{div} \mathbf{u} &= 0 \end{aligned}$$

in a domain $D \subset \mathbb{R}^d$ with positive initial temperature and cold boundary:

$$T\big|_{t=0} > 0 \text{ in } D, \ T\big|_{\partial D} = 0$$

and the question is whether, for large Re, heat diffusion through the boundary is enhanced (Nusselt number increases).

This problem is too difficult. We initially replace it by

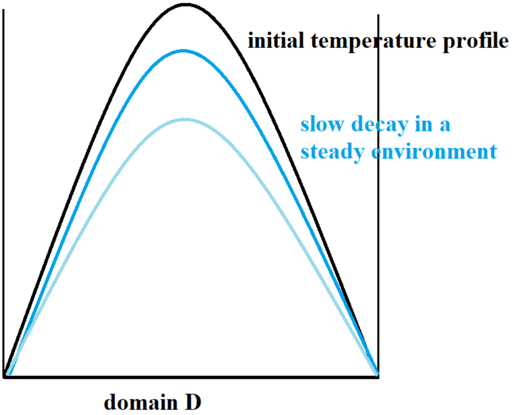
$$d_t T = \kappa \Delta T dt + \sum_j \mathbf{u}_j \cdot \boldsymbol{\nabla} T \circ dW_t^j$$

namely we take

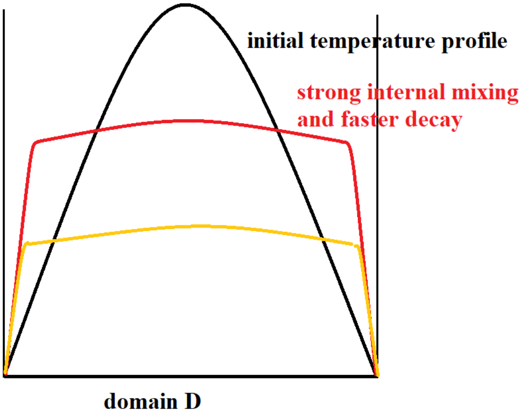
$$\mathbf{u}(t, \mathbf{x}) = \sum_j \mathbf{u}_j(\mathbf{x}) \frac{dW_t^j}{dt}$$

white noise in time, Gaussian.
In a second stage we try to connect to a more realistic model.

Heuristic role of turbulence in heat loss through a cold boundary.
No turbulence:

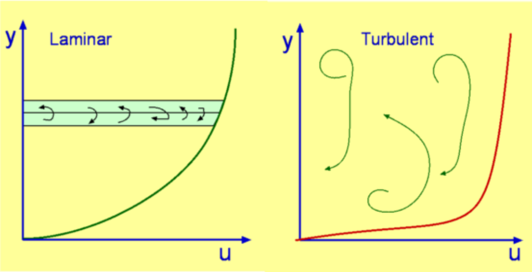


Turbulent environment: Nusselt number $\sim \frac{\partial T}{\partial n}\Big|_{\partial D}$ increases



But also the fluid is zero at the boundary

A main difficulty: also the turbulent fluid is zero at the boundary.



Therefore the mixing power of the fluid is weaker at the boundary.
However, for turbulent fluids, the boundary layer is thin compared to the laminar case.
[Picture from a page of University of Sydney]

Rigorous formulation

$D \subset \mathbb{R}^d$ bounded open with smooth boundary, J finite or countable index set, $\kappa > 0$

$$\begin{aligned} d_t T &= \kappa \Delta T dt + \sum_{j \in J} \mathbf{u}_j \cdot \nabla T \circ dW_t^j \qquad \text{in } [0, T] \times D \\ T|_{\partial D} &= 0 \\ T|_{t=0} &= T_0 \qquad \text{in } D \end{aligned}$$

with $\mathbf{u}_j$ satisfying some technical regularity assumption like $\sum_{j \in J} \|\mathbf{u}_j\|^2_{W^{1,2}(D) \cap C(\overline{D})} < \infty$ and

$$\begin{aligned} \mathbf{u}_j|_{\partial D} &= 0 \\ \operatorname{div} \mathbf{u}_j &= 0 \end{aligned}$$

Noise covariance

$$\begin{aligned} Q &: \overline{D} \times \overline{D} \to \mathbb{R}^{d \times d} \\ Q(\mathbf{x}, \mathbf{y}) &= \sum_{j \in J} \mathbf{u}_j(\mathbf{x}) \otimes \mathbf{u}_j(\mathbf{y}) \qquad \mathbf{x}, \mathbf{y} \in \overline{D} \end{aligned}$$

$$\begin{aligned} \mathbb{Q} &: L^2\left(D; \mathbb{R}^d\right) \to L^2\left(D; \mathbb{R}^d\right) \\ (\mathbb{Q}\mathbf{v})(\mathbf{x}) &= \int_D Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) \, d\mathbf{y} \end{aligned}$$

Two important quantities:

$$q(\mathbf{x}) := \min_{\xi \neq \mathbf{0}} \frac{\xi^T Q(\mathbf{x}, \mathbf{x}) \xi}{\xi^T \xi}$$

$$\epsilon_Q := \left\| \mathbb{Q}^{1/2} \right\|_{L^2 \to L^2} = \sup_{\mathbf{v} \neq \mathbf{0}} \frac{\int_D \int_D \mathbf{v}(\mathbf{x})^T Q(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) \, d\mathbf{x} d\mathbf{y}}{\int_D \mathbf{v}(\mathbf{x})^T \mathbf{v}(\mathbf{x}) \, d\mathbf{x}}.$$

Noise covariance and elliptic operators

$$\left(\mathcal{L}_Q f\right)(\mathbf{x}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \partial_\beta \left(Q_{\alpha\beta}(\mathbf{x},\mathbf{x})\,\partial_\alpha f(\mathbf{x})\right)$$

Remark: $\mathcal{L}_Q$ is degenerate elliptic ($\mathbf{u}_j|_{\partial D} = 0 \Rightarrow Q|_{\partial D} = 0$)

$$\begin{aligned} A_0 f &= \kappa \Delta f \\ A_Q f &= \left(\kappa\Delta + \mathcal{L}_Q\right) f \end{aligned}$$

with domain $D(A_0) = D(A_Q) = W^{2,2}(D) \cap W_0^{1,2}(D)$,
$A_0, A_Q = D(A_0) \to L^2(D)$.

Principal eigenvalues

- λ_D = Poincaré constant of D
- $\kappa\lambda_D$ = principal eigenvalue of $-A_0$
- $\lambda_{D,\kappa,Q}$ = principal eigenvalue of $-A_Q$

$$\lambda_{D,\kappa,Q} \geq \kappa\lambda_D$$

Recall: $A_Q f = (\kappa\Delta + \mathcal{L}_Q) f$

$$(\mathcal{L}_Q f)(\mathbf{x}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \partial_\beta \left(Q_{\alpha\beta}(\mathbf{x},\mathbf{x})\, \partial_\alpha f(\mathbf{x}) \right)$$

Recall $\epsilon_Q := \left\|\mathbb{Q}^{1/2}\right\|_{L^2 \to L^2}$. Denote by e^{tA_Q} the semigroup generated by A_Q: $e^{tA_Q} T_0$ is the solution of the modified heat equation ($Q(\mathbf{x}, \mathbf{x})$ is the **eddy diffusion coefficient**)

$$\partial_t T = \operatorname{div}\left(\kappa I + Q(\mathbf{x}, \mathbf{x})\right) \nabla T.$$

Theorem

Assume $T_0 \in L^2(D)$. Then, for every $\phi \in L^\infty(D)$,

$$\mathbb{E}\left[\left(\int_D \phi(\mathbf{x})\, T(t, \mathbf{x})\, dx - \left\langle \phi, e^{tA_Q} T_0 \right\rangle_{L^2}\right)^2\right] \le \frac{\epsilon_Q}{2\kappa} \|T_0\|_{L^2}^2 \|\phi\|_\infty^2 .$$

In particular

$$\mathbb{E}\left[\left(\int_D |T(t, \mathbf{x})|\, dx\right)^2\right] \le \left(\frac{\epsilon_Q}{\kappa} + 2|D| \exp\left(-2\lambda_{D,\kappa,Q} t\right)\right) \mathbb{E}\left[\|T_0\|_{L^2}^2\right].$$

Scheme of the proof

Itô form of the Stratonovich equation:

$$d_t T = \kappa \Delta T dt + \sum_{j\in J} \underbrace{\mathbf{u}_j \cdot \nabla T \circ dW_t^j}_{\text{Stratonovich}}$$

$$d_t T = \left(\kappa \Delta T + \underbrace{\mathcal{L}_Q T}_{\text{global corrector}} \right) dt + \sum_{j\in J} \underbrace{\mathbf{u}_j \cdot \nabla T dW_t^j}_{\text{Itô}}$$

$$\left(\mathcal{L}_Q T\right)(\mathbf{x}) := \frac{1}{2} \sum_{j\in J} \mathbf{u}_j(\mathbf{x}) \cdot \nabla \left(\mathbf{u}_j(\mathbf{x}) \cdot \nabla T(\mathbf{x})\right).$$

Lemma

$\left(\mathcal{L}_Q T\right)(\mathbf{x}) = \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \partial_\beta \left(Q_{\alpha\beta}(\mathbf{x},\mathbf{x}) \partial_\alpha f(\mathbf{x})\right).$

[Based on $\operatorname{div} \mathbf{u}_j = 0$.]

Second step of the proof

$$
\begin{aligned}
d_t T &= (\kappa\Delta + \mathcal{L}_Q)\, T dt + \sum_{j\in J} \mathbf{u}_j \cdot \nabla T dW_t^j \\
\frac{\partial}{\partial t} e^{tA_Q} T_0 &= (\kappa\Delta + \mathcal{L}_Q)\, e^{tA_Q} T_0 \\
d_t \left(T_t - e^{tA_Q} T_0 \right) &= (\kappa\Delta + \mathcal{L}_Q) \left(T_t - e^{tA_Q} T_0 \right) dt + \sum_{j\in J} \mathbf{u}_j \cdot \nabla T_t dW_t^j
\end{aligned}
$$

Mild formulation:

$$
T_t - e^{tA_Q} T_0 = \sum_{j\in J} \int_0^t e^{(t-s)A_Q} \left(\mathbf{u}_j \cdot \nabla T_s \right) dW_s^j
$$

Final step of the proof

$$\left\langle \phi, T_t - e^{tA_Q} T_0 \right\rangle = \sum_{j\in J} \int_0^t \left\langle e^{(t-s)A_Q}\phi, \mathbf{u}_j \cdot \boldsymbol{\nabla} T_s \right\rangle dW_s^j$$

Maximum principle for $e^{(t-s)A_Q}\phi$ plus the inequality

$$\int_0^\infty \int_D \left|\boldsymbol{\nabla} T\left(t,\mathbf{x}\right)\right|^2 d\mathbf{x}dt \leq \frac{1}{2\kappa}\int_D T_0^2\left(\mathbf{x}\right) d\mathbf{x}$$

imply

$$\mathbb{E}\left[\left(\int_D \phi\left(\mathbf{x}\right) T\left(t,\mathbf{x}\right) d\mathbf{x} - \left\langle \phi, e^{tA_Q} T_0 \right\rangle_{L^2}\right)^2\right] \leq \frac{\epsilon_Q}{2\kappa} \left\| T_0 \right\|_{L^2}^2 \left\|\phi\right\|_\infty^2 .$$

Did we improve the decay?

Zero noise decay:

$$\left(\int_D |T(t,\mathbf{x})|\, d\mathbf{x}\right)^2 \leq \exp\left(-2\kappa\lambda_D t\right) \|T_0\|_{L^2}^2 .$$

With noise, both the same and the new estimate hold:

$$\mathbb{E}\left[\left(\int_D |T(t,\mathbf{x})|\, d\mathbf{x}\right)^2\right] \leq \left(\frac{\epsilon_Q}{\kappa} + 2\,|D| \exp\left(-2\lambda_{D,\kappa,Q}\, t\right)\right) \mathbb{E}\left[\|T_0\|_{L^2}^2\right] .$$

Decay is improved on *finite time intervals* $[0,\tau]$ if:

1. ϵ_Q is very small
2. $\lambda_{D,\kappa,Q} >> \kappa\lambda_D$

1. ϵ_Q is very small
2. $\lambda_{D,\kappa,Q} >> \kappa\lambda_D$

Under suitable assumptions on the domain D, denoted by D_δ the set

$$D_\delta = \{\mathbf{x} \in D : dist(\mathbf{x}, \partial D) > \delta\}$$

Theorem

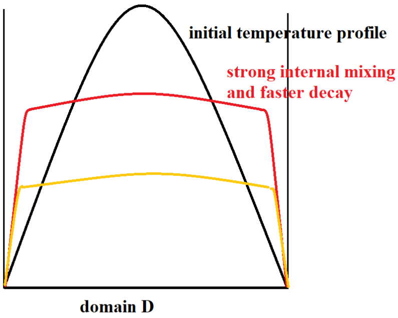
There exists a constant C_D with the following property:

$$\lambda_{D,\kappa,Q} \geq C_D \min\left(\sigma^2, \frac{\kappa}{\delta}\right)$$

for every $\sigma > 0$, $\delta > 0$ and every Q such that

$$q(\mathbf{x}) \geq \sigma^2 \text{ in } D_\delta$$

The pictures in red and yellow illustrate the shape of the (rescaled) principal eigenfunction $\phi_{\min}$ of $(\kappa\Delta + \mathcal{L}_Q)$, associated to $\lambda_{D,\kappa,Q}$



$$e^{tA_Q} T_0 = e^{-\lambda_{D,\kappa,Q} t} \langle T_0, \phi_{\min} \rangle \phi_{\min} + o\left(e^{-\lambda_{D,\kappa,Q} t} \langle T_0, \phi_{\min} \rangle \phi_{\min} \right).$$

Summary

- We have the estimates

$$\mathbb{E}\left[\left(\int_D \phi(\mathbf{x})\, T(t,\mathbf{x})\, dx - \left\langle \phi, e^{tA_Q} T_0 \right\rangle_{L^2}\right)^2\right] \leq \frac{\epsilon_Q}{2\kappa} \| T_0 \|_{L^2}^2 \, \|\phi\|_\infty^2 .$$

- The decay rate $\lambda_{D,\kappa,Q}$ improves $\kappa\lambda_D$ when the boundary layer δ is small and the value of

$$q(\mathbf{x}) := \min_{\xi \neq \mathbf{0}} \frac{\xi^T Q(\mathbf{x},\mathbf{x})\, \xi}{\xi^T \xi}$$

is large in the interior.

- The approximation is good when

$$\epsilon_Q := \left\| \mathrm{Q}^{1/2} \right\|_{L^2 \to L^2} = \sup_{\mathbf{v} \neq \mathbf{0}} \frac{\int_D \int_D \mathbf{v}(\mathbf{x})^T\, Q(\mathbf{x},\mathbf{y})\, \mathbf{v}(\mathbf{y})\, d\mathbf{x} d\mathbf{y}}{\int_D \mathbf{v}(\mathbf{x})^T\, \mathbf{v}(\mathbf{x})\, dx}$$

is small. **Can we have $q(\mathbf{x})$ large and ϵ_Q small?**

$$q(\mathbf{x}) := \min_{\xi \neq \mathbf{0}} \frac{\xi^T Q(\mathbf{x}, \mathbf{x}) \xi}{\xi^T \xi}$$

is related to the trace of the operator $\mathbb{Q}$:

$$Tr(\mathbb{Q}) = \int_D TrQ(\mathbf{x}, \mathbf{x}) \, d\mathbf{x}$$

so it is not strange that we may have:

- large $Tr(\mathbb{Q})$
- small $\left\| \mathbb{Q}^{1/2} \right\|_{L^2 \to L^2}$.

Example of noise in full space

Consider the homogeneous covariance $(Q(\mathbf{x},\mathbf{y}) = Q(\mathbf{x}-\mathbf{x})$

$$Q(\mathbf{z}) = \sigma^2 k_0^{\zeta} \int_{k_0 \leq |\mathbf{k}| \leq k_1} \frac{1}{|\mathbf{k}|^{d+\zeta}} e^{i\mathbf{k}\cdot\mathbf{z}} \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2} d\mathbf{k}.$$

- $\zeta > 0$, $k_1 = +\infty$, σ^2 large, k_0 so large that $\sigma^2 k_0^{-d}$ is small $\Rightarrow q(\mathbf{x})$ large and ϵ_Q small
  - K41 is $\zeta = \frac{4}{3}$
- $-d \leq \zeta \leq 0$, $k_0 = 1$, σ^2 small, k_1 so large that $\sigma^2 \int_{1\leq k \leq k_1} \frac{1}{k^{\zeta+1}} dk$ is large $\Rightarrow q(\mathbf{x})$ large and ϵ_Q small
  - $\zeta = -d$ is white in space, $\zeta = 0$ is the enstrophy measure.

Example of noise in bounded domain

In a bounded domain one can mimic the previous scheme by means of the fractional powers of the Stokes operator; however, it looks quite abstract. More concrete: the noise previously denoted by

$$\sum_{j \in J} \mathbf{u}_j(\mathbf{x})\, dW_t^j$$

is now specified as

$$\sum_{n=n_0}^{n_1} \sum_{i \in I_n} \Gamma_n \mathbf{v}_{n,i}(\mathbf{x})\, dW_t^{n,i}$$

where we have in mind a sequence of (small) space scales

$$r_{n_0} > \ldots > r_{n_1}$$

and, at each scale n a typical circulation Γ_n and several vortex patches $\mathbf{v}_{n,i}(\mathbf{x})$.

Vortex patch noise

We start with a noise at vorticity level:

$$\sum_{n=n_0}^{n_1} \sum_{i \in I_n} \underbrace{\Gamma_n f_{r_n}\left(\mathbf{x} - \mathbf{x}_{n,i}\right)}_{\text{vortex patch, scale } n \text{, center } \mathbf{x}_{n,i} \text{, circulation } \Gamma_n} dW_t^{n,i}$$

$$f_{r_n}(x) = r_n^{-2} f\left(\frac{\mathbf{x}}{r_n}\right), \qquad f \text{ a pdf}$$

and define $\mathbf{v}_{n,i} = \nabla^{\perp} \Delta_D^{-1} f_{r_n}\left(\cdot - \mathbf{x}_{n,i}\right)$, namely

$$\begin{aligned} \operatorname{div} \mathbf{v}_{n,i} &= 0 \\ \nabla^{\perp} \cdot \mathbf{v}_{n,i}(x) &= f_{r_n}\left(\mathbf{x} - \mathbf{x}_{n,i}\right) \\ \mathbf{v}_{n,i}\big|_{\partial D} &= 0 \end{aligned}$$

The *vortex patch noise* then is:

$$\sum_{n=n_0}^{n_1} \sum_{i \in I_n} \Gamma_n \mathbf{v}_{n,i}(\mathbf{x})\, dW_t^{n,i}.$$

Vortex patch noise

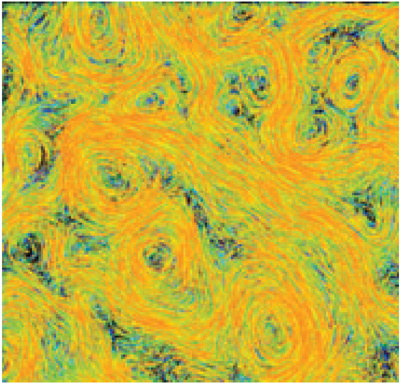
One can prove that

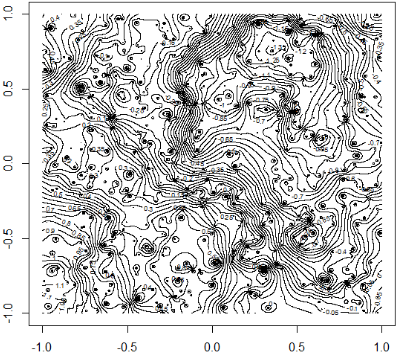
$$
\begin{aligned}
q(\mathbf{x}) &\geq \sum_{n=n_0}^{n_1} \Gamma_n^2 r_n^{-4} \\
\epsilon_Q &\leq \sum_{n=n_0}^{n_1} \Gamma_n^2 \frac{r_n^{-2}}{\log r_n^{-1}}
\end{aligned}
$$

so there is a factor r_n^{-2} which allows us to make $q(\mathbf{x})$ large and ϵ_Q small, with several combinations of multiplicative constants and choice of n_0, n_1. Kolmogorov scaling (*see F. Flandoli, Renormalized Onsager functions and merging of vortex clusters, Stochastics and Dynamics 2020*):

$$
\Gamma_n = \frac{\epsilon^{1/3} r_n^{5/3}}{\log^{1/3} r_n^{-1}}.
$$

Realizations of this turbulent velocity field are more natural than Fourier: stream function lines from experiment of Rivera-Ecke '05 versus a realization of the vortex patch noise:





Stochastic model reduction

Can we prove the previous result for a fluid model closer to reality? Let us filter the solution $\mathbf{u}$ of the Navier-Stokes equations by a mollifier:

$$\begin{aligned} \overline{\mathbf{u}} &:= \theta_\epsilon * \mathbf{u} \qquad \text{large scale} \\ \mathbf{u}_s &:= \mathbf{u} - \overline{\mathbf{u}}. \qquad \text{small scale} \end{aligned}$$

The large scale component satisfies

$$\begin{aligned} \partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} + \mathbf{u}_s) \cdot \nabla \overline{\mathbf{u}} + \nabla \overline{p} &= \nu \Delta \overline{\mathbf{u}} + R_\epsilon \\ \operatorname{div} \overline{\mathbf{u}} &= 0 \end{aligned}$$

where the remainder R_ϵ is a commutator (small under several conditions)

$$R_\epsilon = \mathbf{u} \cdot \nabla \overline{\mathbf{u}} - \theta_\epsilon * (\mathbf{u} \cdot \nabla \mathbf{u}) = [\mathbf{u} \cdot \nabla, \theta_\epsilon *] \, \mathbf{u}$$

Stochastic model reduction means that we replace

$$\begin{aligned} \partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} + \mathbf{u}_s) \cdot \nabla \overline{\mathbf{u}} + \nabla \overline{p} &= \nu \Delta \overline{\mathbf{u}} + R_\epsilon \\ \operatorname{div} \overline{\mathbf{u}} &= 0 \\ \partial_t \mathbf{u}_s &= \ldots \end{aligned}$$

by a simplified model. Example of result, (*see Flandoli-Pappalettera, 2D Euler equations with Stratonovich transport noise as a large scale stochastic model reduction, 2020*):

$$\begin{aligned} \partial_t \overline{\omega} + (\overline{\mathbf{u}} + \mathbf{u}_s) \cdot \nabla \overline{\omega} &= 0 \\ d\omega_s + \overline{\mathbf{u}} \cdot \nabla \overline{\omega} dt &= -\frac{1}{\tau} \omega_s dt + \sum_{j \in J} \boldsymbol{\nabla}^\perp \cdot \mathbf{u}_j dW_t^j \end{aligned}$$

converges to

$$d\overline{\omega} + \overline{\mathbf{u}} \cdot \nabla \overline{\omega} dt = \sum_{j \in J} \mathbf{u}_j \cdot \boldsymbol{\nabla} \overline{\omega} \circ dW_t^j .$$

Conclusion

Eddy viscosity/heat diffusion coefficient

$\uparrow$

SPDEs with transport Stratonovich noise

$\uparrow$

stochastic model reduction

Literature

The talk reports works in progress, based on:

1. *L. Galeati, On the convergence of stochastic transport equations to a deterministic parabolic one,* SPDEs: Analysis and Computation '20
2. *F. Flandoli, D. Luo, Convergence of transport noise to Ornstein-Uhlenbeck for 2D Euler equations under the enstrophy measure, Ann. Probab. '20*
3. *F. Flandoli, L. Galeati, D. Luo, Scaling limit of stochastic 2D Euler equations with transport noises to the deterministic Navier-Stokes equations, J. Evolution Equations '20*
4. *F. Flandoli, U. Pappalettera, 2D Euler equations with Stratonovich transport noise as a large scale stochastic model reduction '20.*

Thank you!

Eddy viscosity/heat diffusion coefficient

$\uparrow$

SPDEs with transport Stratonovich noise

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stochastic model reduction