

The Matrix Product Ansatz from a probabilistic viewpoint

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- ▶ two positive vectors $\underline{x}, \underline{y} \in \mathbb{R}_+^B$.

We say that $\mu_N \in \mathcal{M}_1(A^N)$ is of MPA type if

$$\mu_N(\eta) := \frac{\underline{y}^T \prod_{i=1}^N M^{\eta_i} \underline{x}}{Z_N}, \quad \eta = (\eta_1, \dots, \eta_N) \in A^N,$$

where $Z_N := \underline{y}^T M^N \underline{x}$, with $M = \sum_{a \in A} M^a$, is the normalization factor.

$$\begin{aligned} A^N &\longrightarrow A^N \times B^{N+1} \\ \eta &\longrightarrow (\eta, \zeta) \end{aligned}$$

The auxiliary variables $\zeta_i \in B$ encode the matrix indices.

We construct a coupling measure

$$C_N(\eta, \zeta) := \frac{y(\zeta_1) \left(\prod_{i=1}^N M_{\zeta_i, \zeta_{i+1}}^{\eta_i} \right) x(\zeta_{N+1})}{Z_N}, \quad \forall (\eta, \zeta) \in A^N \times B^{N+1}. \quad (1)$$

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By construction we have

$$\sum_{\zeta \in B^{N+1}} C_N(\eta, \zeta) = \mu_N(\eta).$$

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On the enlarged state space $A \times B$, we define the $|A||B| \times |A||B|$ matrix

$$T_{(a,b),(a',b')} := M_{b,b'}^a, \quad (a,b), (a',b') \in A \times B.$$

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By using the Perron–Frobenius theorem, we perform a generalized **Doob transform**:

$$S = \lambda^{-1} E_\lambda^{-1} T E_\lambda,$$

where $E_\lambda = \text{diag}(\varepsilon_\lambda)$ and ε_λ is a positive eigenvector of T associated with the positive eigenvalue λ .

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S is a stochastic matrix on $A \times B$.

The coupling measure as a Markov bridge

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A **Markov bridge** has the form

$$\rho(\xi) \propto g(\xi_1) \left(\prod_{i=1}^N P_{\xi_i, \xi_{i+1}} \right) f(\xi_{N+1})$$

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$$C_N(\eta, \zeta) = \frac{\overbrace{\widehat{y}_{\eta_1, \zeta_1} \varepsilon_\lambda(\eta_1, \zeta_1)}^{\text{initial bound. weight}} \overbrace{\left(\prod_{i=1}^N S_{(\eta_i, \zeta_i), (\eta_{i+1}, \zeta_{i+1})} \right)}^{\text{Markov transitions}} \overbrace{\varepsilon_\lambda^{-1}(\eta_{N+1}, \zeta_{N+1}) \widehat{x}_{\eta_{N+1}, \zeta_{N+1}}}^{\text{final bound. weight}}}{\widetilde{Z}_N}.$$

with $\widetilde{Z}_N = \lambda^{-N} Z_N$.

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C_N is a Markov bridge on the enlarged state space $A \times B$

A probability measure $\mu_N \in \mathcal{M}^1(A^N)$ is of Matrix product ansatz type **if and only if** it is a mixture of non-homogeneous product measures where the mixing law is a Markov bridge on B^{N+1} :

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Correlations Conditionally on the hidden path ζ , the variables $\eta_1, \dots, \eta_N$ are independent. Define the local function

$$V_{b,b'} := \sum_{a \in A} a p_{b,b'}(a) \quad V_{\zeta_i, \zeta_{i+1}} = \mathbb{E}[\eta_i \mid \zeta].$$

Then, for $i \neq j$,

$$\mathbb{E}_{\mu_N}[\eta_i \eta_j] = \mathbb{E}_{\rho_{N+1}} [V_{\zeta_i, \zeta_{i+1}} V_{\zeta_j, \zeta_{j+1}}].$$

Large Deviations for MPA measures

7

The probabilistic structure underlying the MPA is reflected in the form of the large deviation rate functional.

The LDP can be derived through **two complementary approaches**:

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The LDP can be derived through **two complementary approaches**:

- ▶ Markov-process approach: exploit the **enlarged Markov structure**



contraction principle

Finite-dimensional setting: general variational representation.

Infinite-dimensional setting: the large deviation structure is more subtle.

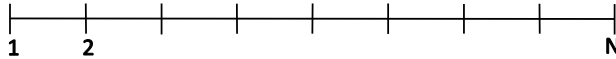
Gabrielli, I., *Large deviations and the matrix product ansatz*, preprint arXiv:2601.11820.

- ▶ Mixture approach: exploit the **mixture structure**

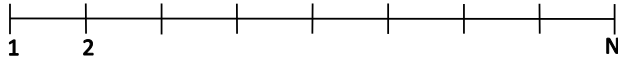
Gabrielli, Goldwurm, I., Mignosi, in preparation.

-The **model**:

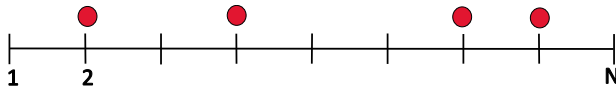
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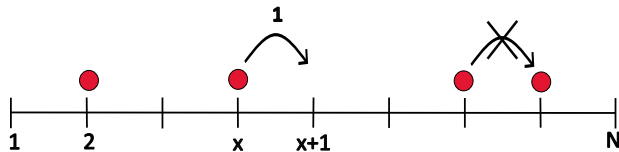
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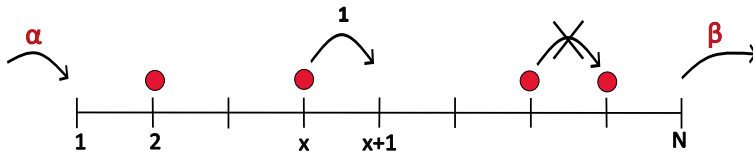
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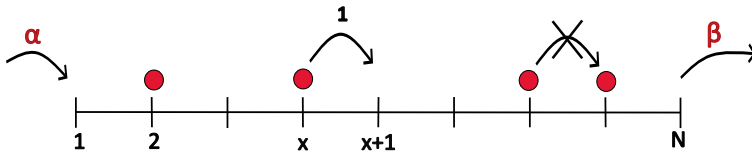
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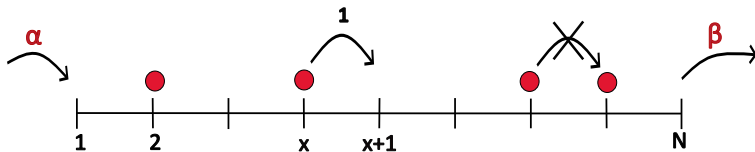


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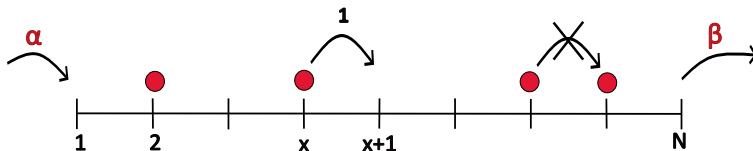
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-The **invariant measure** $\mu_N \in \mathcal{M}^1(\{0, 1\}^N)$ of the process can be represented in **MPA** form using infinite-dimensional matrices $(M^a)_{a=0,1}$ and infinite vectors $\underline{x}$ and $\underline{y}$ when they satisfy:

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$$M^1 M^0 = M^1 + M^0 \quad \underline{y}^T M^0 = \frac{1}{\alpha} \underline{y}^T \quad M^1 \underline{x} = \frac{1}{\beta} \underline{x} \quad (2)$$

Derrida, Evans, Hakim, Pasquier, *Exact solution of a 1D asymmetric exclusion model using a matrix formulation*, J. Phys. A 26 (1993), 1493–1517.

Infinite setting: LD for the boundary-driven TASEP

9

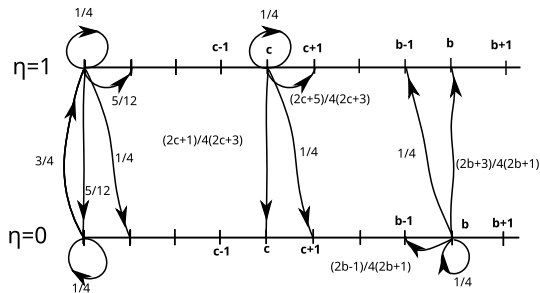
TASEP regime: $\alpha + \beta > 1$.

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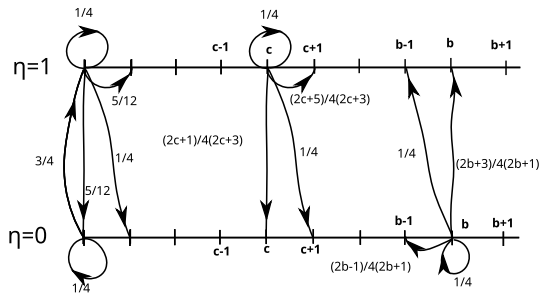
$$\mathbb{P}_N^{\tilde{\mathcal{S}}, g, f}(\eta, \zeta) \propto g(\zeta_1) \prod_{i=1}^N \tilde{\mathcal{S}}_{(\eta_i, \zeta_i), (\eta_{i+1}, \zeta_{i+1})} f(\zeta_{N+1})$$

g, f encode the boundary parameters (α, β) through endpoint weights.

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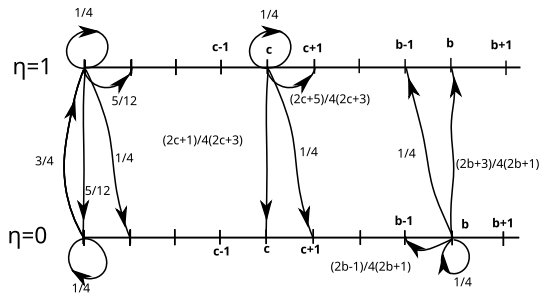
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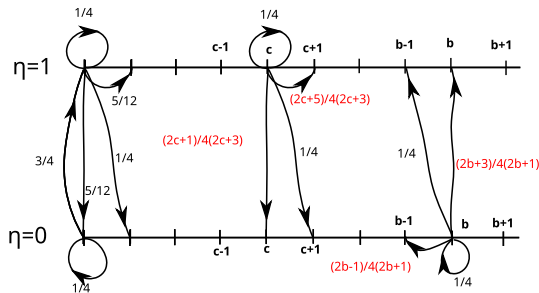
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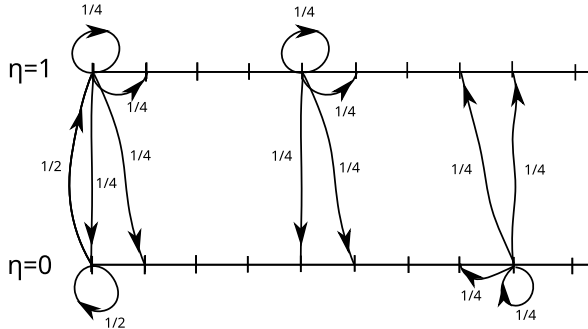
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A simpler homogeneous random walk

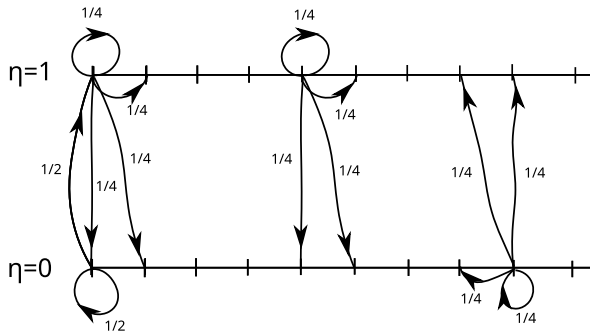
10

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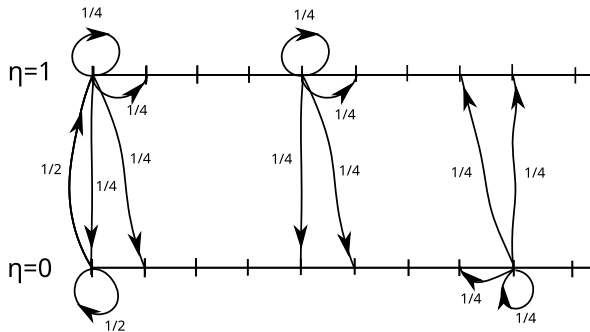
$$\mathbb{P}_N^{\tilde{S}, g, f}(\eta, \zeta) = \mathbb{P}_N^{S, g/h, fh}(\eta, \zeta) 2^{-N\hat{\nu}_N(0,0)} \frac{Z_N}{\tilde{Z}_N}$$

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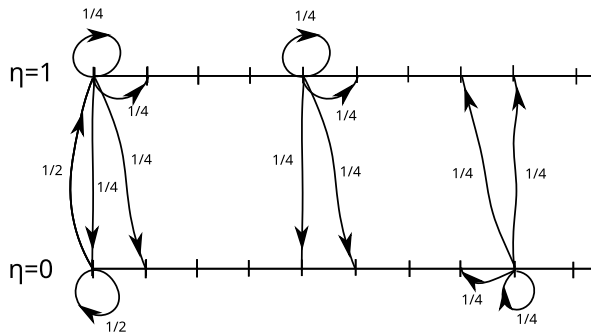
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- Fixed (η_1, ζ_1) , for $i > 1$ under $\mathbb{P}_N^S$ the increments $(\eta_i, \zeta_{i+1} - \zeta_i)$ are independent.
- For $\zeta_i > 0$ (bulk region) they are i.i.d. with law denoted by μ^I .
- For $\zeta_i = 0$ (boundary region) they are i.i.d. with law denoted by μ^B .

We consider the empirical observables:

- ▶ $\hat{z}_N(x)$ is the piecewise linear interpolation of the rescaled ζ -process,
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The triple $(\hat{z}_N, \hat{\Pi}_N^B, \hat{\Pi}_N^I)$ satisfies an LDP with rate functional:

$$\mathcal{I}^S(z, \Pi^B, \Pi^I) = \underbrace{z(0) \log \frac{\alpha}{1-\alpha} + z(1) \log \frac{\beta}{1-\beta}}_{\text{endpoint contributions}} + \underbrace{\int_0^1 \left[m(x) h(\nu^B | \mu^B) + (1 - m(x)) h(\nu^I | \mu^I) \right] dx}_{\text{entropic cost}}.$$

$h(\nu | \mu) = \sum_{a,b} \nu_{a,b} \log \frac{\nu_{a,b}}{\mu_{a,b}}$ relative entropy; $\Pi^B(x) = m(x) \nu^B(x) dx$ and $\Pi^I(x) = (1 - m(x)) \nu^I(x) dx$;
 $m(x)$ = fraction of time the walk spends at level $\zeta = 0$.

$$\mathcal{I}^{\mathcal{S}}(z, \Pi^B, \Pi') \longrightarrow \mathcal{I}^{\tilde{\mathcal{S}}}(z, \Pi^B, \Pi')$$

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By successive contractions, we pass to new macroscopic observables:

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where:

$$\Pi = \Pi^B + \Pi', \quad \rho(x) = \Pi_{1,1}(x) + \Pi_{1,0}(x), \quad \dot{z}(x) = \Pi_{1,1}(x) - \Pi_{0,-1}(x).$$

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By a final contraction over z , we obtain the rate functional for the particle density profile:

$$\mathcal{I}^{\text{TASEP}}(\rho) = \inf_z \mathcal{I}^{\tilde{S}}(z, \rho).$$

Regime: $\alpha + \beta > 1$ $a = \frac{1-\alpha}{\alpha}$, $b = \frac{1-\beta}{\beta}$ ($ab < 1$).

Let

$$F(x) := \int_0^x \rho(y) dy, \quad \mathcal{G} := \left\{ G \text{ a.c.} : G(0) = 0, 0 \leq \dot{G} \leq 1 \right\}.$$

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Then the rate functional for the empirical density profile is

$$\mathcal{I}^{\text{TASEP}}(\rho) = \inf_{G \in \mathcal{G}} \left\{ \int_0^1 \left[H(\dot{F}(y)) + H(\dot{G}(y)) \right] dy - \ln(b)(F(1) - G(1)) + \ln(ab) \min_{x \in [0,1]} (F(x) - G(x)) \right\},$$

with $H(u) := u \log u + (1 - u) \log(1 - u)$.

Remark. This final contraction recovers the large deviation functional for the TASEP of *Derrida, Lebowitz, Speer* (2003) in the variational form introduced by *Bryc, Zatitskii* (2024).



Thanks for the attention