III History of enumerative invariants of GLSMs

- Fan-Jarvis-Ruan (2013) — G finite
- Polishchuk-Vaintrob (2016) — Purely algebraic version using matrix factorizations
- Kiem-Li (2018) — Using cosection localization
- Fan-Jarvis-Ruan (2018) — "Narrow sectors" for general GLSMs
- Ciocan-Fontanine-F-Guéré-Kim-Shoemaker (2018) — Convex hybrid models
- F-Kim (2020) — General case

Thm (CF-F-G-K-S, 2018)
Enumerative invariants for convex hybrid models specialize to FJRW theory and to GW theory as defined using the cosection localized virtual cycle.

Thm (Kim-Oh, 2018) The cosection localized virtual cycle agrees with the Behrend-Fantechi virtual cycle.

Thm (F-Kim, 2020) The general GLSM invariants form a CohFT.

IV Construction

- Embedd $M_{g,r,d}$ in a smooth space

$$
\begin{tikzcd}
M_{g,r,d} \arrow[r, hook] \arrow[d] \arrow[dr, bend left, "ev"] & U_{g,r,d} \arrow[dl, dashed, "\widetilde{ev}"'] \arrow[d, dashed, "\widetilde{for}"] \\
IV /\!\!/_{\theta} G^r & \overline{M}_{g,r}
\end{tikzcd}
$$

- Find a "virtual" matrix factorization $\mathbb{K}_{g,r,d}$ on $(U_{g,r,d}, \widetilde{ev} \circ w^{\boxplus r})$ supported

- $[M]^{vir}_{g,r,d} := \underbrace{td(B_{g,r,d})}_{\text{to be defined}} \, ch\,(\mathbb{K}_{g,r,d})$

Construction of $\mathbb{K}_{g,r,d}$

LG quasimap data:

- C genus g curve
- $\underline{q} = (q_1, \dots, q_r)$ marked points
- P principal Γ-bundle
- $\varkappa : P \times_\Gamma \mathbb{C}_\chi \longrightarrow \omega_C^{\log}$
- $u : C \longrightarrow P \times_\Gamma V$

$$\begin{tikzcd} \mathcal{P} \times_{\Gamma} V/G \arrow[d] \\ \mathcal{C} \arrow[d, "\pi"] \\ \{(C, q, \mathcal{P}, \kappa)\} \end{tikzcd}$$

vector bundle on $\mathcal{C}$

universal curve

$$\mathbb{R}\pi_*(\mathcal{P} \times_{\Gamma} V/G) = [A \xrightarrow{d} B]$$

$$\mathbb{R}\pi_*\left(\mathcal{S}\times_\Gamma V / G\right) = [A \xrightarrow{d} B]$$

$$\ker d = H^0(\mathcal{S}\times_\Gamma V/G) \ni \omega$$ ← extra data

want to construct a cosection using ω

$$\begin{tikzcd}
\operatorname{tot} p^*B \arrow[d] \\
Z(\beta) \subseteq U_{g,r,d} \subseteq \operatorname{tot} A \arrow[u, bend right] \arrow[u, dotted, bend right=60, "\alpha"'] \arrow[d, "p"] \\
\{(C,q,\mathcal{S},K)\}
\end{tikzcd}$$

$$Z(\beta) \cong \{(C,q,\mathcal{S},K,u)\} = M^{2G}(V/\!\!/_\theta G) \supseteq Z(\alpha) = M_{g,r,d}$$

$$\mathbb{K} \simeq \left[\Lambda^{\text{even}} p^*B^\vee \underset{\partial}{\overset{\partial}{\rightleftarrows}} \Lambda^{\text{odd}} p^*B^\vee\right]$$

$$\partial = \lrcorner\beta + \wedge\alpha$$

$$\operatorname{Supp}(\mathbb{K}) = Z(\alpha,\beta) = M_{g,r,d}$$

Problem: In general α only exists locally on $U_{g,r,d}$

$\alpha \in \mathbb{H}^1(U_{g,r,d}, \mathcal{K}(\mathcal{B}))$

Want: To find a matrix factorization which is locally the Koszul factorization

Observation

Koszul factorizations come from sheaves of CDGAs:

- $\mathcal{X}$ DM stack
- (A, d) sheaf of CDGAs over $\mathcal{O}_{\mathcal{X}}$
- $w \in \Gamma(X, \mathcal{O}_X)$
- $\alpha \in \Gamma(X, A_{-1})$ s.t. $d(\alpha) = w$

Then

$$\begin{tikzcd} A^{\text{even}} \arrow[r, shift left, "\partial"] & A^{\text{odd}} \arrow[l, shift left, "\partial"] \end{tikzcd}$$

$$\partial = d + \cdot \alpha$$

graded Leibnitz rule $\Rightarrow \partial^2 = w$

$\therefore$ this is a matrix factorization

Idea:

To realize
$\mathcal{L}\mathcal{E}\mathcal{H}^{1}(\mathrm{tot}\, A, \mathcal{K}(B))$
we need to replace $\mathcal{K}(B)$
by a Γ-acyclic complex
but retain the CDGA structure

$$\mathcal{K}(B) \xrightarrow{\text{Godement}} G^{\bullet}\mathcal{K}(B) \xrightarrow{\text{Thom-Sullivan}} Th\, G^{\bullet}\mathcal{K}(B)$$

$G^{\bullet}\mathcal{K}(B)$: Γ-acyclic cosimplicial sheaf of CDGAs

$Th\, G^{\bullet}\mathcal{K}(B)$: Γ-acyclic sheaf of CDGAs

Then

- $\alpha \in \mathrm{Th}^{\bullet} G^{\bullet} \mathcal{K}(\mathcal{B})_{-1}$
- and $d(\alpha) = w$

$$\mathbb{K}_{g,r,d} := \left[\left(\mathrm{Th}^{\bullet} G^{\bullet} \mathcal{K}(\mathcal{B})\right)^{even} \rightleftarrows \left(\mathrm{Th}^{\bullet} G^{\bullet} \mathcal{K}(\mathcal{B})\right)^{odd} \right]$$

locally looks like

$$\left[\Lambda^{even} p^{*} \mathcal{B}^{\vee} \rightleftarrows \Lambda^{odd} p^{*} \mathcal{B}^{\vee} \right]$$