

III History of enumerative invariants of GLSMs

- Fan-Jarvis-Ruan (2013)

G finite

Purely algebraic
version
using matrix
factorizations

- Polishchuk-Vaintrob (2016)

- Kiem-Li (2018)

Using cosection
localization

- Fan-Jarvis-Ruan (2018)

“Narrow sectors”
for general
GLSMs

- Gican-Fontanine-F-Guére -
Kim-Shoemaker (2018)

Convex hybrid
models

- F-Kim (2020)

General
case

Thm (CF-F-G-K= S, 2018)

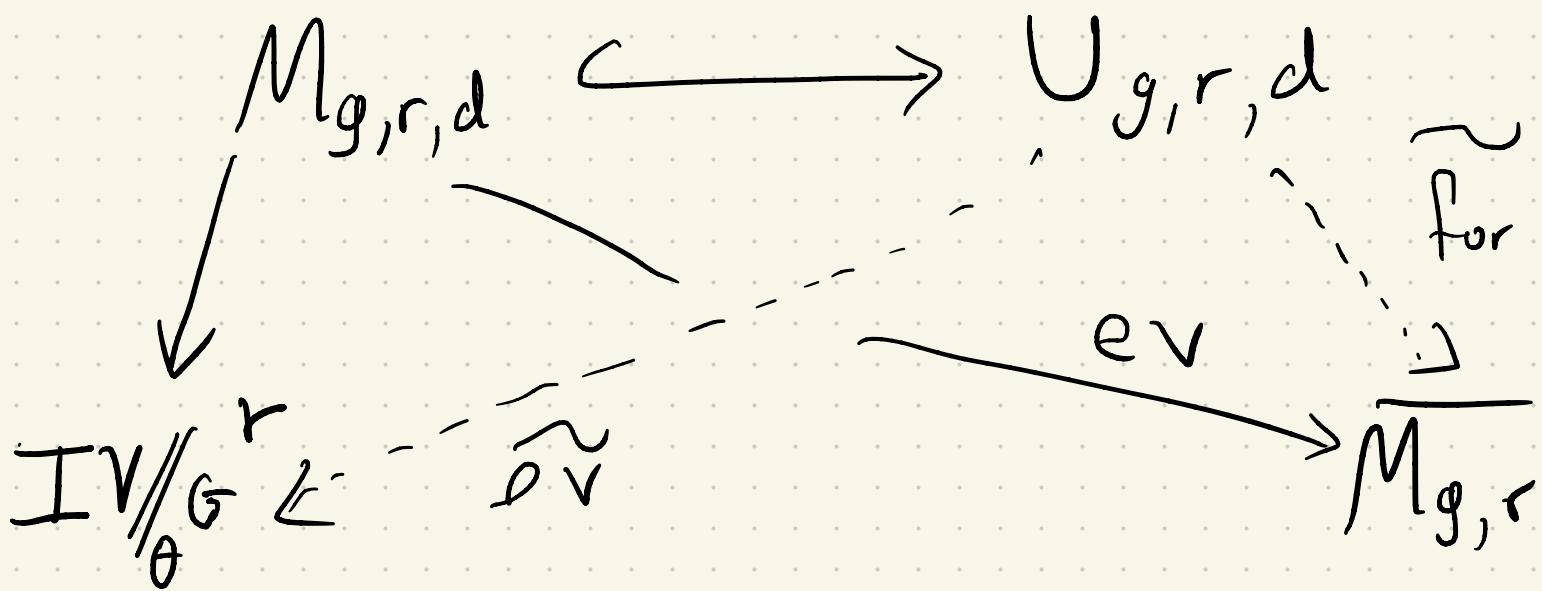
Enumerative invariants for convex hybrid models specialize to FJRW theory and to GW theory as defined using the cosection localized virtual cycle.

Thm (Kim-Oh, 2018) The cosection localized virtual cycle agrees with the Behrend-Fantechi virtual cycle.

Thm (F-Kim, 2020) The general GLSM invariants form a CohFT.

IV Construction

- Embed $M_{g,r,d}$ in a smooth space



- Find a "virtual" matrix factorization $K_{g,r,d}$ on $(U_{g,r,d}, \tilde{e}^r \circ w^{\# r})$ supported
- $[M]_{g,r,d}^{\text{vir}} := \underbrace{\text{fd}(B_{g,r,d})}_{\text{to be defined}} \text{ch } (K_{g,r,d})$

Construction of $K_{g,r,d}$

LG quasimap data:

- C genus g curve
- $q = (q_1, \dots, q_r)$ marked points
- P principal \mathbb{F} -bundle
- $\chi: P \times_{\mathbb{F}} \mathbb{G}_m \rightarrow W_C^{\log}$
- $u: C \rightarrow P \times_{\mathbb{F}} V$

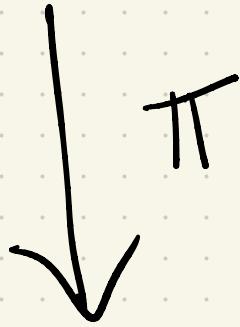
$$\mathcal{F}^{\times} \xrightarrow{\pi} V/G$$

vector
bundle
on
 \mathcal{C}



$$\mathcal{C}$$

universal
curve



$$Z(C, g, P, K)$$

$$R\pi_*(\mathcal{F}^{\times} \xrightarrow{\pi} V/G) = [A \xrightarrow{d} B]$$

$$R\pi_{*}(\mathcal{F} \times_{\mathbb{F}} V) = [A \xrightarrow{d} B]$$

$$\ker d = H^0(\mathcal{F} \times_{\mathbb{F}} V / G)$$

extra
data

want to
construct a
co-section
using ω

$$\text{tot } P^* B$$

\downarrow

$$\text{tot } A$$

$\downarrow P$

$$Z(B) \subseteq U_{g,r,d}$$

$\hookrightarrow \mathcal{Z}(C, q, \phi, K, \nu)$
 $= M^{2G}_{2r}(V/G)$

$$\mathcal{Z}(C, q, P, K)$$

$$Z(\alpha) = M_{g,r,d}$$

$$K \approx [\wedge^{\text{even}} P^* B^V \leftarrow \wedge^{\text{odd}} P^* B^V]$$

$$\partial = \wedge B + \wedge \alpha$$

$\text{Supp}(K) = Z(\alpha)$
 $= M_{g,r,d}$

Problem: In general α only exists locally on $U_{g,r,d}$

$$\alpha \in H^1(U_{g,r,d}, \mathcal{K}(B))$$

Want: To find a matrix factorization which is locally the Koszul factorization

Observation

Koszul factorizations come from sheaves of CDGAs:

- \mathcal{X} DM stack
- $\alpha(A, d)$ sheaf of CDGAs over \mathcal{X}
- $w \in \Gamma(X, \mathcal{O}_X)$
- $\alpha \in \Gamma(X, A_{-1})$ s.t. $d(\alpha) = w$

Then

$$A^{\text{even}} \xrightarrow{\partial} A^{\text{odd}}$$
$$\xleftarrow{\partial} A^{\text{odd}}$$

$$\partial = d + \cdot \alpha$$

graded Leibnitz rule $\Rightarrow \partial^2 = w$
 \therefore this is a matrix factorization

Idea:

To realize
 $\text{d} \mathcal{C}\mathcal{H}'(\text{tot } A, \mathcal{K}(B))$
we need to replace $\mathcal{K}(B)$
by a Γ -acyclic complex
but retain the CDGA structure

$$\mathcal{K}(B) \xrightarrow{\text{Godement}} G^* \mathcal{K}(B) \xrightarrow{\text{Thom-Sullivan}} \text{Th}^* G^* \mathcal{K}(B)$$

$\mathcal{K}(B)$ $G^* \mathcal{K}(B)$ $\text{Th}^* G^* \mathcal{K}(B)$

Γ -acyclic
cosimplicial
sheet of
CDGAs

Γ -acyclic
sheet
of
CDGAs

Then

- $\alpha \in \text{Th}^G K(B)$

- and $d(\alpha) = w$

$$K_{grd} := \left[(\text{Th}^G K(B))^{\text{even}} \xrightarrow{\quad} (\text{Th}^G K)^{\text{odd}} \right]$$

locally looks like

$$\left[\wedge^{\text{even}} \xrightarrow{P^* \beta^\vee} \wedge^{\text{odd}} \xleftarrow{P^* \beta^\vee} \right]$$