Cohomological Field Theories from Gauged Linear Sigma Models

Based on joint work with B. Kim & with Ciocan-Fontanine - Guéré - Kim - Shoemaker

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II: CohFTs

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I: GLSMs (Roughly)

A GLSM is a GIT quotient of affine space with a global function.

$$V /\!\!/_{\theta} G \xrightarrow{w} \mathbb{C}$$

where
- $V /\!\!/_{\theta} G$ is a smooth DM stack
- $Z(\partial w) \subseteq V /\!\!/_{\theta} G$ is proper

Data: (V, G, θ, w)

- V $\mathbb{C}$-vector space
- $G \subseteq Gl(V)$
- $\theta : G \to \mathbb{C}^{\times}$ character
- $w \in (\mathrm{Sym}\, V^{\vee})^{G}$ is a G-inv function

GIT Quotients

$$V^{ss}(\theta) := \left\{ v \in V \;\middle|\; \begin{array}{l} \bullet\ f(x) \in \operatorname{Sym} V \\ \bullet\ f(g \cdot x) = \theta^n(g) f(x) \\ \bullet\ f(v) \neq 0 \end{array} \right\}$$

$$V /\!\!/_\theta G := \left[V^{ss}(\theta) / G \right]$$

(quotient stack)

$$\underset{\text{open}}{\subseteq} [V/G]$$

Examples of affine GIT quotients

- Projective space
- Grassmannians
- semiprojective toric varieties
- Quiver varieties

By varying $\theta \in (V, G, \theta, w)$

GLSMs specialize to

- Complete intersections in all of the above
- Quantum Singularities ($\tilde{G}$ finite)

$$w : [V/G] \longrightarrow \mathbb{C}$$

To specialize to complete intersections in $V/\!/_\theta G$

Choose

- W a G representation
- $s \in (\operatorname{Sym} V^\vee \otimes_{\mathbb{C}} W)^G$

Gives

- $\operatorname{tot} W \underset{s}{\overset{\longleftarrow}{\longrightarrow}} V/\!/_\theta G$
- $\langle s, - \rangle : \operatorname{tot} W^\vee \to V/\!/_\theta G$

Form the GLSM

$$(V \oplus W^\vee, G, \theta, \langle s, - \rangle)$$

$$\longleftrightarrow \quad Z(s) \subseteq V/\!/_\theta G$$

GLSMs (Precise)

Data: $(V, \Gamma, \chi, w, \nu)$

- V $\mathbb{C}$-vector space
- $\Gamma \subseteq Gl(V)$ linear red. gp.
- $\chi : \Gamma \twoheadrightarrow \mathbb{C}^\times$ surjective character
- ν a $\mathbb{Q}$-character of Γ

Define (Γ extends G by "R-charge")

- $G := \ker \chi$
- $\theta := \nu|_G$

Require

- $V^{ss}(\nu) = V^{ss}(\theta) = V^{s}(\theta)$ (ν is a "good lift" of θ)
- $Z(dw)$ is proper

<u>Goal</u>: Produce an enumerative theory for GLSMs

- Specializes to Gromov-Witten theory for complete intersections in $V/\!\!/_\theta G$
- Specializes to FJRW theory for quantum singularities

(G finite) $[V/G] \xrightarrow{w} \mathbb{C}$

- Varying θ interpolates between the above

$\underline{\underline{\text{Ex}}}$ $\mathbb{C}^\times \circlearrowright \mathbb{C}^{n+2}$

$= \operatorname{Spec} \mathbb{C}[x_0, \dots, x_n, p]$

weights: $1, \dots, 1, -d$

a) $\theta_+ : \mathbb{C}^\times \xrightarrow{id} \mathbb{C}^\times$

b) $\theta_- : \mathbb{C}^\times \xrightarrow{i} \mathbb{C}^\times$

$t \longmapsto t^{-1}$

- $f(x)$ = homogeneous polynomial of degree d
- $w := p f(x)$

a) $\mathbb{C}^{n+2} /\!\!/_{\theta_+} \mathbb{C}^\times = \operatorname{tot} \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\langle -, f \rangle} \mathbb{C}$

Gives GW theory of $Z(f) \subseteq \mathbb{P}^n$

b) $\mathbb{C}^{n+2} /\!\!/_{\theta_-} \mathbb{C}^\times = [\mathbb{A}^{n+1} / \mathbb{Z}_d] \xrightarrow{f} \mathbb{C}$

Gives FJRW theory

II CohFTs

Defn

A <u>cohomological field theory</u> is:

- $\mathcal{H}$ a graded $\mathbb{C}$-vector space
- $\langle -,- \rangle$ a supercommutative pairing
- $1 \in \mathcal{H}$ (unit)
- $\Omega_{g,r}: \mathcal{H}^{\otimes r} \longrightarrow H^*(\overline{M}_{g,r})$
- satisfying natural axioms
  - permutation covariance
  - tree
  - loop
  - forgetting tails
  - metric

Ex (GW theory of Z)

Let Z be a smooth variety

$M_{g,r,d}(Z)$ = Moduli of maps $\mathcal{C} \to Z$

- $\mathcal{H} := H^*(Z)$ (state space)
- $\langle v_1, v_2 \rangle := \int_Z v_1 \cup v_2$

$$
\begin{tikzcd}
 & H^*(M_{g,r,d}(Z)) \arrow[r, "\cap [M(Z)]^{vir}"] & H_*(\overline{M}_{g,r,d}(Z)) \arrow[d, "for_*"] \\
 & & H_*(\overline{M}_{g,r}) \arrow[d, "PD", "\cong"'] \\
\mathcal{H}^{\otimes r} = H^*(Z^r) \arrow[uur, "ev^*"] \arrow[rr, dashed, "\Omega_{g,r,d}"] & & H^*(\overline{M}_{g,r})
\end{tikzcd}
$$

Ex (FJRW theory)

- $\mathcal{H} := \bigoplus_{g \in G} \mathrm{Jac}(w|_{V^g})\, dw_g$

$$\mathcal{H}^{\otimes r} \overset{\Omega_{g,r}}{\dashrightarrow} H^*(\overline{M}_{g,r})$$

Observe:

$$\mathcal{H} = \bigoplus_{g \in G} \mathrm{Jac}(w|_{V^g})\, dw_g$$
$$= \bigoplus_{g \in G} H^*(V^g, K(dw|_{V^g})^\vee)$$
$$= H^*(I[V/G], K(dw)^\vee)$$

In general, given $\mathcal{X} \xrightarrow{w} \mathbb{C}$

there is a twisted Hodge complex:

$$\left[\mathcal{O}_{\mathcal{X}} \xrightarrow{\wedge dw} \Omega^1_{\mathcal{X}} \xrightarrow{\wedge dw} \cdots \xrightarrow{\wedge dw} \Omega^n_{\mathcal{X}} \right]$$

if $w = 0$

$$\left[\mathcal{O}_{\mathcal{X}} \xrightarrow{0} \Omega^1_{\mathcal{X}} \to \cdots \xrightarrow{0} \Omega^n_{\mathcal{X}} \right]$$

$$\mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, 0) = \bigoplus H^{p,q}(\mathcal{X})$$

If w is an isolated singularity in $\mathbb{A}^n$

$\wedge dw$ is regular

i.e. $(\Omega^\bullet_{\mathcal{X}}, \wedge dw) = K(dw)^\vee \simeq \mathrm{Jac}(w)dw$

$$H^*(\Omega^\bullet_{\mathcal{X}}, \wedge dw) = \mathrm{Jac}(w)dw$$

In general, given (V, G, θ, w)

define

$$\mathcal{H} := \mathbb{H}^*\left(\Omega^{\bullet}_{[V/\!/_\theta G}, \wedge dw\right)$$

$$\mathcal{H}^{\otimes r} = \mathbb{H}^*\left(\Omega^{\bullet}_{[V/\!/_\theta G]^r}, \wedge dw^{\boxplus r}\right)$$

Künneth Formula

Observation

If $\alpha \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, d\omega)$

$\beta \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, dv)$

$\Rightarrow \alpha \wedge \beta \in \mathbb{H}^*(\Omega^\bullet_{\mathcal{X}}, d(\omega+v))$

$$\left(\begin{aligned} &\overset{\text{as}}{\alpha} \wedge \beta \wedge d(\omega+v) \\ &= -\alpha \wedge d\omega \wedge \beta + \alpha \wedge \beta \wedge dv \\ &= 0 \end{aligned}\right)$$

Idea: Let

$M_{g,r,d} = M^{LG}_{g,r,d}(V, G, \theta, w)$ be the moduli space of LG quasimaps to $\mathcal{C} \longrightarrow Z(\partial w) \subseteq V /\!\!/_\theta G$

- Embedd

smooth

$$\begin{tikzcd}
M_{g,r,d} \arrow[r, hook] \arrow[d, "ev"'] \arrow[drr, "for"] & U_{g,r,d} \arrow[dl, dashed, "\widetilde{ev}"'] \arrow[dr, dashed, "\widetilde{for}"] & \\
(V /\!\!/_\theta G)^r \arrow[r, "w^{\boxplus r}"] & \mathbb{C} & \overline{M}_{g,r}
\end{tikzcd}$$

- Construct a virtual cycle in twisted Hodge cohomology of $U_{g,r,d}$

$$[M]^{vir}_{g,r,d} \in \mathbb{H}^*_{M_{g,r,d}}\left(\Omega^\bullet_{U_{g,r,d}}, 1d(\widetilde{ev} \circ w^{\boxplus r})\right)$$

supported on $M_{g,r,d}$
(which is proper by FJR 2018)

GLSM Invariants

$$
\begin{tikzcd}
\mathbb{H}^*\left(U_{g,r,d}, \wedge d\left(\widetilde{ev}\circ w^{\boxplus r}\right)\right) \arrow[r, "\wedge [M]^{vir}_{g,r,d}"] & \mathbb{H}^*_{M_{g,r,d}}\left(U_{g,r,d}\right) \arrow[d, "\widetilde{for}_*"] \\
\mathbb{H}^*\left(\Omega^\bullet_{[V/\!/_\theta G]^r}, \wedge d w^{\boxplus r}\right) \arrow[u, "\widetilde{ev}^*"] \arrow[r, dashed, "\Omega_{g,r,d}"] \arrow[d, phantom, "\shortparallel"] & H^*\left(\overline{M}_{g,r}\right) \\
\mathcal{H}^{\otimes r} &
\end{tikzcd}
$$