# HIGHER TOPOLOGICAL CHARGE EFFECTS IN QCD & BEYOND

**Fabian Rennecke**

[Pisarski, FR, 1910.14052]
[FR, 2003.13876]

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**- ICTS 09/09/2021 -**

# A SIMPLE MODEL

Consider a linear sigma model to describe $N_f = 2$ low-energy QCD in the chiral limit. Meson field:

$$\phi = (\sigma + i\eta') + (\vec{a}_0 + i\vec{\pi}\,)\,\vec{\tau}$$

Classically, the theory has chiral symmetry $SU(2)_L \times SU(2)_R \times U(1)_A$ and the effective potential $V(\phi)$ is a general function of the chiral invariants

$$\Phi_1 = \mathrm{tr}\,\phi^\dagger\phi\,, \quad \Phi_2 = \mathrm{tr}\,(\phi^\dagger\phi)^2$$

Due to the axial anomaly $U(1)_A \to \mathbb{Z}_{N_f}$ and we can write down one more invariant

$$\xi = \det\phi + \det\phi^\dagger$$

't Hooft determinant
['t Hooft (1976)]

Conventional ansatz for the effective potential: $V(\phi) = \overline{V}(\Phi_1, \Phi_2) \; - c_A\,\xi$

anomalous 2-meson correlation (in general: $2N_f$-quark correlation)

⟶ makes $\eta'$ heavy

# HIGHER ORDER ANOMALOUS CORRELATIONS

This is clearly not the most general effective potential.
Instead, it is of the form

$$V(\phi) = \overline{V}(\Phi_1, \Phi_2, \xi) \supset \xi^{Q = 1, 2, 3, \ldots}$$

$\longrightarrow$ anomalous $N_f\, Q$ -meson correlations

Simple example: consider the Lagrangian $\mathscr{L} = \mathscr{L}_{\rm cl} + \mathscr{L}_{\rm A}$

$$\mathscr{L}_{\rm cl} = \mathrm{tr}\left(\partial_\mu \phi^\dagger\right)\left(\partial_\mu \phi\right) + m^2\, \mathrm{tr}\, \phi^\dagger \phi + \lambda_1\, \mathrm{tr}\left(\phi^\dagger \phi\right)^2 + \lambda_2 \left(\mathrm{tr}\, \phi^\dagger \phi\right)^2$$

$$\mathscr{L}_{\rm A} = -\chi_1 \left( \det \phi + \det \phi^\dagger \right) - \chi_2 \left[ \left( \det \phi \right)^2 + \left( \det \phi^\dagger \right)^2 \right]$$

$Q = 1$: quadratic term

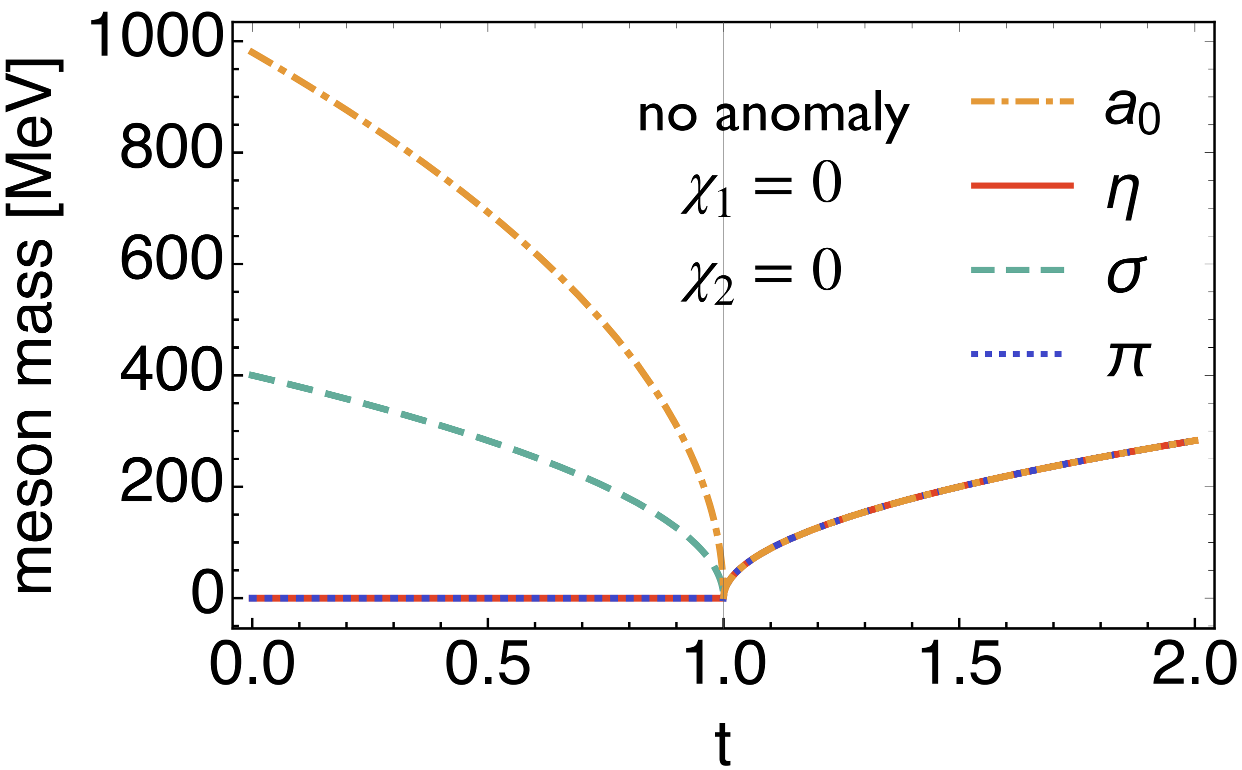
$Q = 2$: quartic term

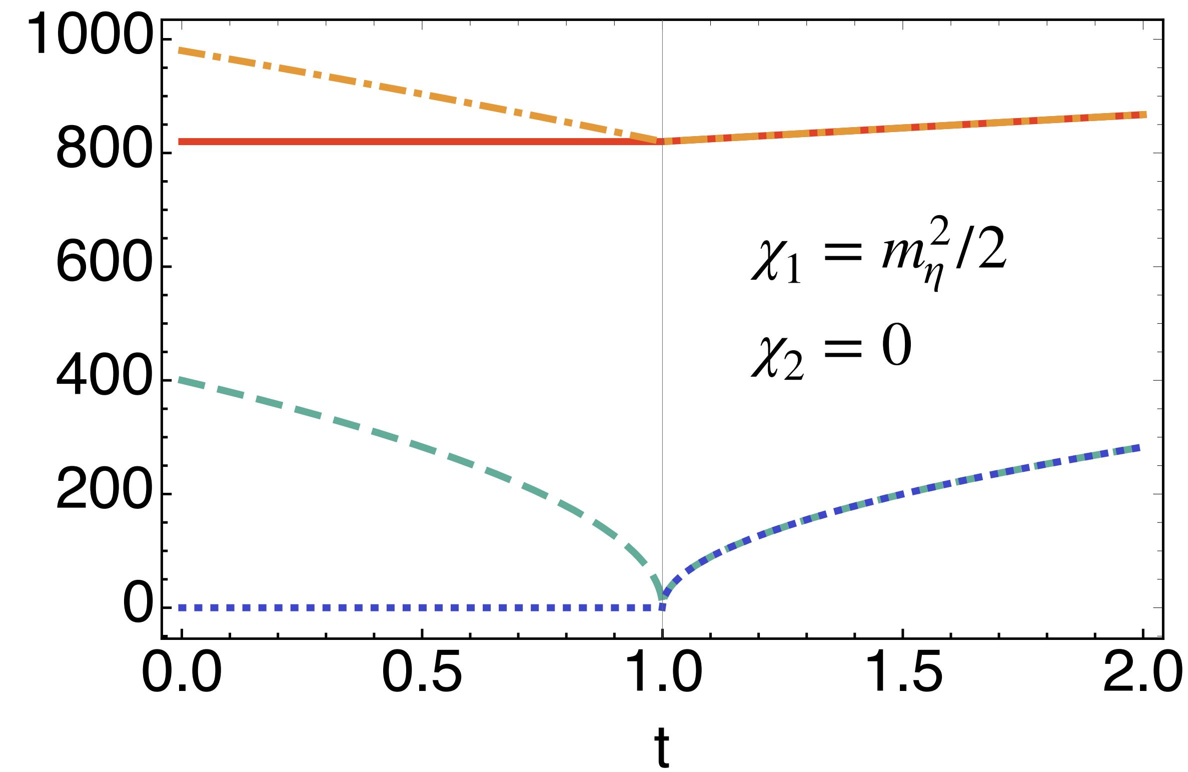
Look at the qualitative mass spectrum in mean-field approximation. Compute masses from $\mathscr{L}$ on the solution of the EoM

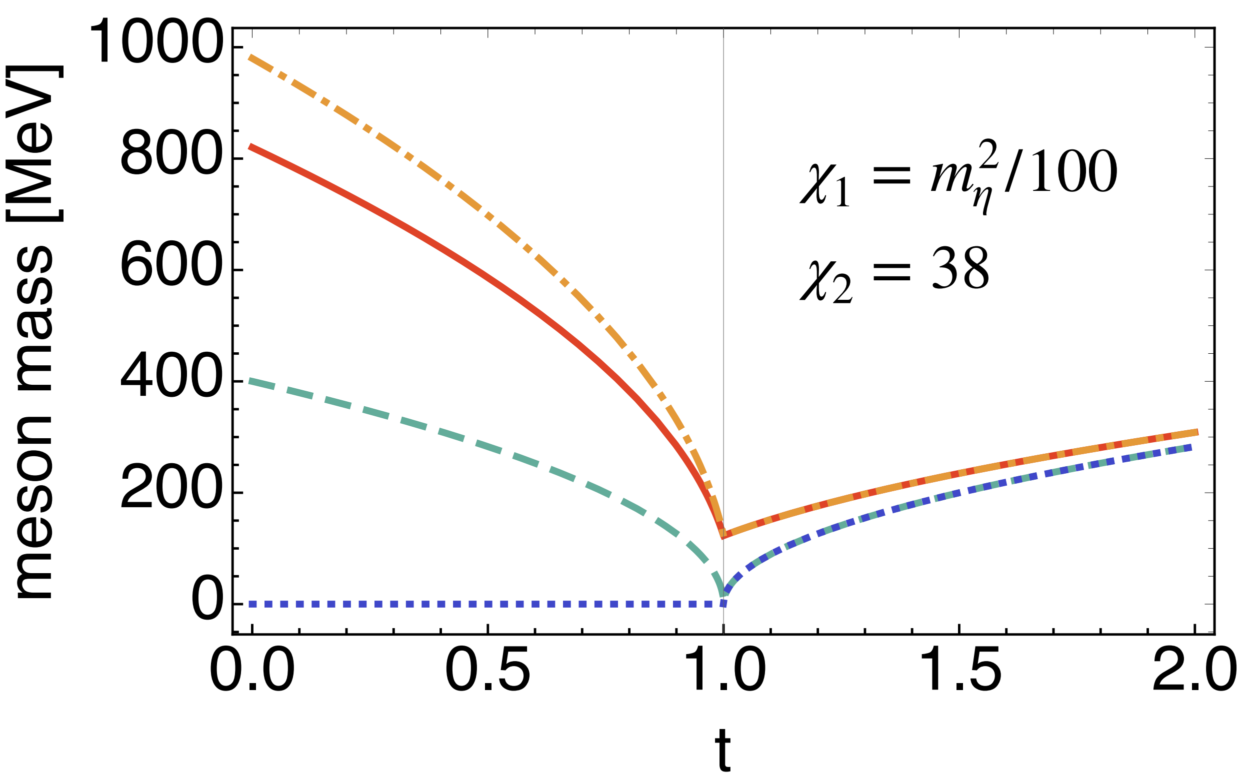
$$\frac{\delta \int d^4x\, \mathscr{L}}{\delta \phi} = 0$$

# HIGHER ORDER ANOMALOUS CORRELATIONS

How do higher order anomalous couplings affect the mass spectrum?



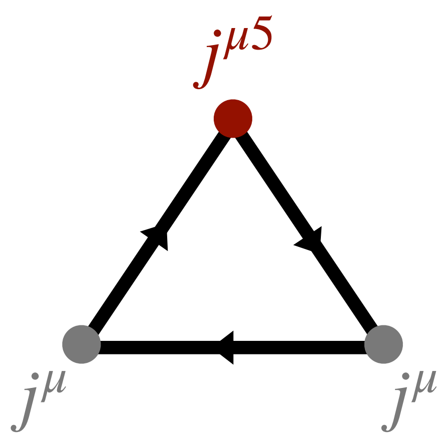




- $\pi$-$\eta$ splitting due to anomalous terms $\chi_1$ and $\chi_2$
- the larger $\chi_2$, the smaller $\chi_1$ to reproduce vacuum masses
- quartic coupling $\chi_2$ decouples from masses in the symmetric phase

# ORIGIN OF ANOMALOUS CORRELATIONS

**What is the microscopic origin of these anomalous correlations?**

Axial anomaly due to topologically nontrivial fluctuations




$\partial_\mu j^{\mu 5} \sim \mathrm{tr}\, F\tilde{F} \sim \rho =$ topological charge density

axial current $j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$

At weak coupling topological effects are described by **instantons** $A_\mu^{(Q)}$.

Typically, only instantons with topological charge $Q = \pm 1$ are taken into account

- give rise to anomalous $2N_f$-quark correlation function ('t Hooft determinant)

$$\det\left(\bar{\psi}_f \mathbb{P}_R \psi_g\right) + \det\left(\bar{\psi}_f \mathbb{P}_L \psi_g\right) \qquad \mathbb{P}_{L/R} = \frac{1 \mp \gamma^5}{2}$$

- distribution of topological charge characterized by θ-dependent free energy

$$F(\theta) \sim \Delta Z_1 \cos\theta$$

# OUTLINE

**Is there a similar story for higher order anomalous correlations?**

**What about instantons with higher topological charge?** ($|Q| > 1$: multi-instantons)

→ topic of this talk

- they give rise to anomalous $2N_f|Q|$-quark correlation functions [Pisarski, FR; 1910.14052]

$$\det\left(\bar{\psi}_f\,\mathbb{P}_R\,\psi_g\right)^{|Q|} + \det\left(\bar{\psi}_f\,\mathbb{P}_L\,\psi_g\right)^{|Q|}$$

  - outline the derivation

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

  - study implications for topological susceptibilities
  - explore possible effects on axion dark matter

# OUTLINE

**Is there a similar story for higher order anomalous correlations?**

**What about instantons with higher topological charge?** ($|Q| > 1$: multi-instantons)

⟶ topic of this talk

- they ... 14052]

**compute in a controlled limit:**

**semiclassical approximation at large T**

- derive effects induced by higher top. charge
- might be small in semiclassical limit, but must be there in general; could be significant at lower energies/temperatures

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

• study implications for topological susceptibilities

• explore possible effects on axion dark matter

# BACKGROUND: INSTANTONS

Minimize the classical action of Yang-Mills theory,

$$S = \frac{1}{2g^2}\int d^4x \,\mathrm{tr}\, F^2 \,,$$

$$F_{\mu\nu} = [D_\mu, D_\nu]$$
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$
$$D_\mu = \partial_\mu + A_\mu$$

requiring that the solution has finite action

⟶ (anti) selfdual gauge fields $F = \pm\tilde{F}$: (anti) **instantons** $A_\mu^{(Q)}$

Instantons allow for a topological classification, characterized by their topological charge

$$Q = -\frac{1}{16\pi^2}\int d^4x \,\mathrm{tr}\, F\tilde{F} \;\in \mathbb{Z}$$

Solution for $Q = 1$: $A_\mu^{(1)}(x) = U_1\,\bar{\sigma}^{\mu\nu}\,U_1^\dagger \dfrac{\rho_1^2}{(x-z_1)^2}\dfrac{(x-z_1)_\nu}{(x-z_1)^2+\rho_1^2}$

[Belavin, Polyakov, Schwartz, Tyupkin (1975)]

$$\sigma^\mu = (-i, \vec{\sigma})^\mu$$
$$\bar{\sigma}^\mu = (i, \vec{\sigma})^\mu$$
$$\bar{\sigma}^{\mu\nu} = \frac{1}{2}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)$$

Quarks on a topological background acquire zero modes with net chirality (index theorem)

$$\gamma_\mu\left(\partial_\mu + A_\mu^{(Q)}\right)\psi^{(Q)} = 0 \qquad\qquad N_f\, Q = n_L - n_R$$

# of left- and right-handed quark zero modes

# PARTITION FUNCTION IN A MULTI-INSTANTON BACKGROUND

- get the ingredients: instantons and the corresponding quark zero modes
- compute semi-classically: small fluctuations around instanton background

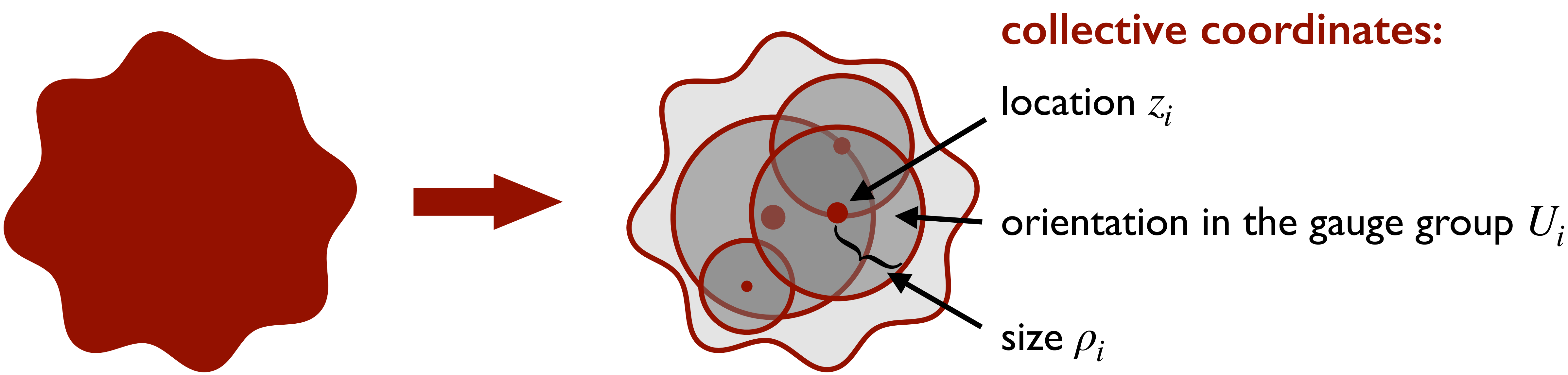
# CONSTRUCTION OF INSTANTONS

General construction of instantons with arbitrary topological charge: **ADHM**

[Atiyah, Drinfeld, Hitchin, Manin (1978)]

- reduces classical self-dual YM equations to a set of nonlinear algebraic equations
- still, explicit solutions for larger $Q$ are unknown
- use approximate solutions here

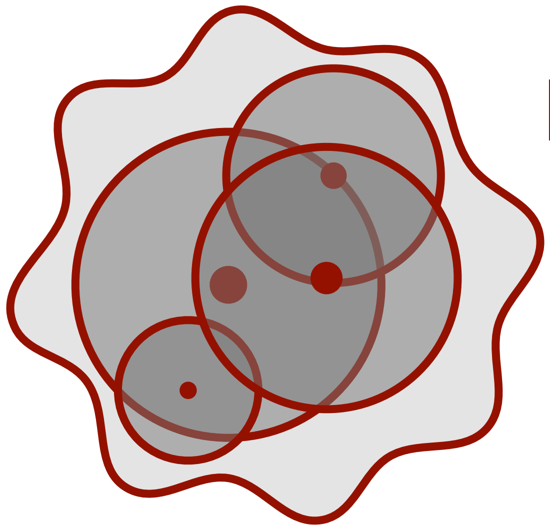
To this end, exploit that $Q$-instantons can be viewed as composition of constituent-instantons with $Q = 1$



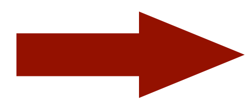

- $4N_c|Q|$ collective coordinates describe a $Q$-instanton
- arise from symmetries that yield inequivalent instanton solutions

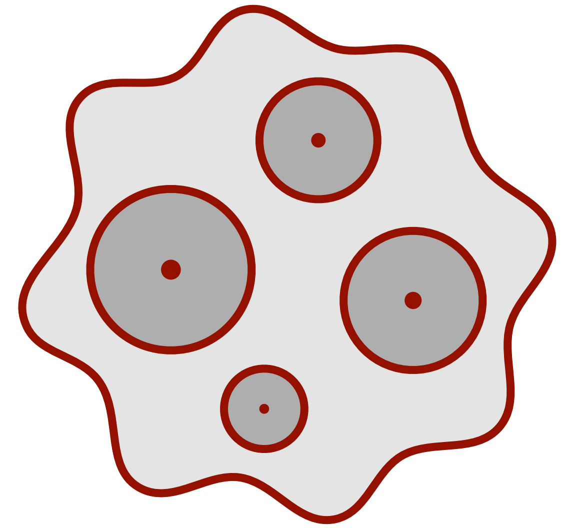
# CONSTRUCTION OF INSTANTONS

Solve ADHM by expanding in the limit of small constituent-instantons (SCI)



$|R_{ij}| \equiv |z_i - z_j| \gg \rho_i$





Results to order $\rho^4/|R|^4$:

- $Q$-instanton: $A_\mu^{(Q)}(x) = \frac{1}{\xi_0(x, \{z_i, \rho_i\})} \sum_{i=1}^{Q} A_\mu^{(1)}(x; z_i, \rho_i, U_i) + \mathcal{O}\left(\frac{\rho^4}{|R|^4}\right)$

[Christ, Weinberg, Stanton (1978)]

- $N_f\, Q$ quark zero modes: $(\, f = 1,\ldots,N_f\,,\ i = 1,\ldots,Q\,)$

$$\psi_{fi}^{(Q)}(x) = \psi_{fi}^{(1)}(x, z_i\,, \rho_i, U_i) - \sum_{j\neq i} \mathbb{X}_{ij}(x, z_i\,, \rho_i, \rho_j)\, \psi_{fj}^{(1)}(x, z_j\,, \rho_j, U_i) + \mathcal{O}\left(\frac{\rho^4}{|R|^4}\right)$$

[Pisarski, FR]

$\sim \rho_i\, U_i\, \Delta(x - z_i)\, \varphi_R$ for $|x - z_i| \gg \rho_i$

free quark propagator!

# PARTITION FUNCTION

Partition function in a $Q$-instanton background

$$Z_Q[J] = \int \mathscr{D}\Phi \exp\left\{ -S[\Phi + \bar{\Phi}^{(Q)}] + \int_x \bar{\psi} J \psi \right\}$$

$\Phi = (A, c, \bar{c}, \psi, \bar{\psi})$ $\qquad$ $\bar{\Phi}^{(Q)} = (A^{(Q)}, 0, 0, 0, 0)$ $\qquad$ source for quark-antiquark pairs (e.g. mass term)

- consider small fluctuations around topological background $A_\mu = A_\mu^{(Q)}$
- collective coordinates correspond to symmetries: resulting gauge field zero modes need to be treated exactly
- replace integral over zero modes by integral over collective coordinates:

$$Z_Q[J] = \int \left[ N \prod_{i=1}^{Q} d^4 z_i \, d\rho_i \, dU_i \right] n_Q\left(\{z_i, \rho_i, U_i\}\right) \det{}_0(J)$$

$Q$-instanton density

- gluon and ghost determinant
- quark determinant over nonzero modes
- Jacobian of coordinate change from zero modes to collective coordinates

quark zero mode determinant

$$\det \int d^4x \, \psi^{(Q)\dagger}(x) \, J(x) \, \psi^{(Q)}(x)$$

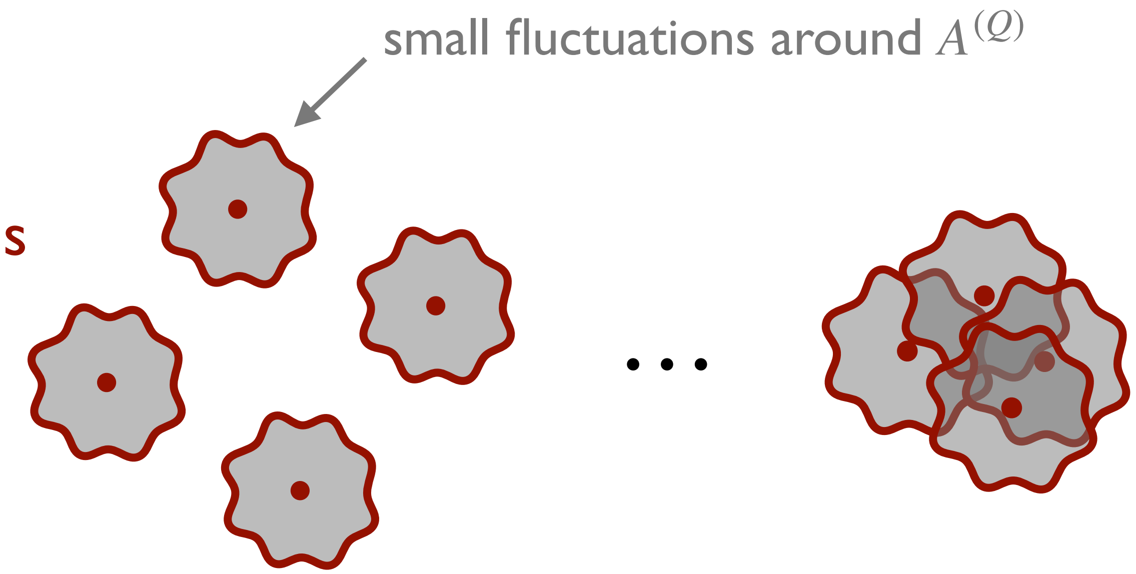
# PARTITION FUNCTION IN THE SCI LIMIT

Qualitatively different contributions to $Z_Q$ in the SCI limit due to integration over constituent-instanton locations $z_i$ :

constituents far apart:
~ dilute single-instantons

$$Z_Q^{(1)} = \frac{1}{Q!} Z_1^Q$$

parametrically: $\mathscr{V}^Q$




constituents close together:
genuine multi-instanton contributions

$$\Delta Z_Q$$

parametrically: $\mathscr{V}^1$

Various nonlocal contributions of $q$-instantons ($q < Q$) and one local $Q$-instanton contribution:

$$Z_Q = \sum_{q=1}^{Q-1} Z_Q^{(q)} + \Delta Z_Q \longrightarrow$$

compute only $\Delta Z_Q$
for local contributions

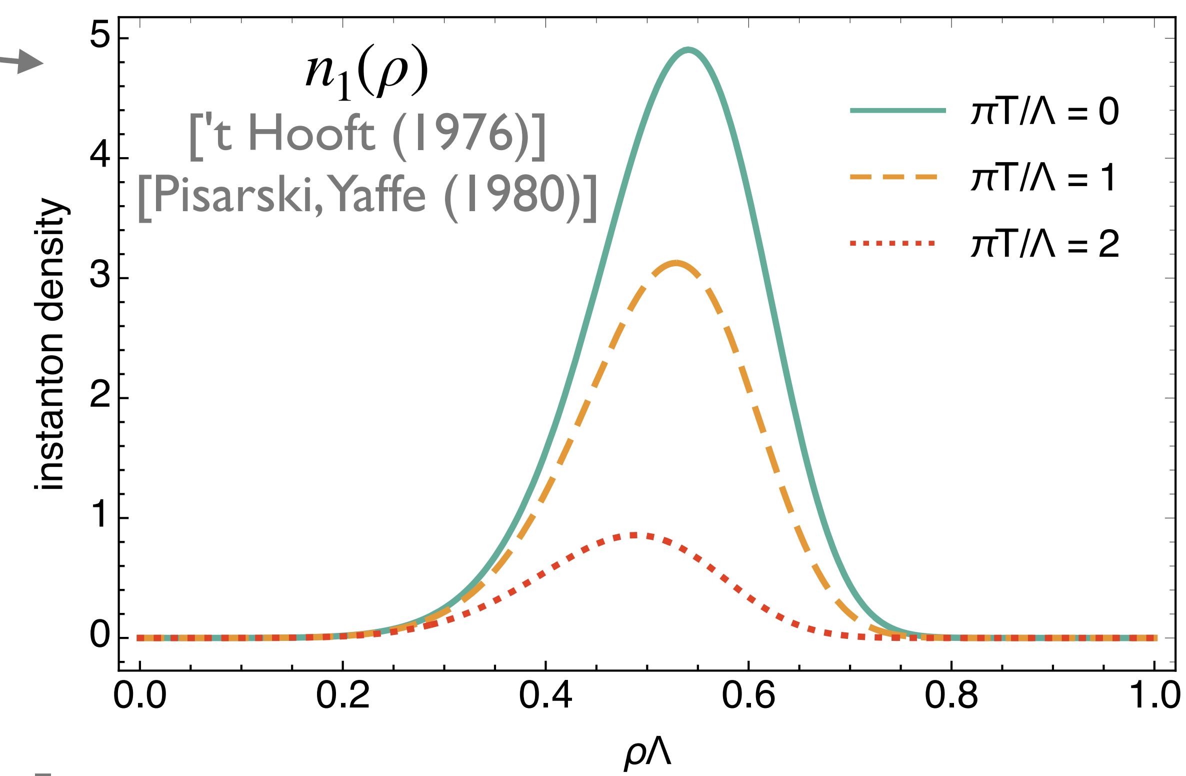
# PARTITION FUNCTION IN THE SCI LIMIT

Consider only the contribution of the quark determinant to $\Delta Z_Q$. Then:
(gauge part: accurate to order $\rho^4/|R|^4$ [Brown, Creamer & Bernard (1978)])

$$\Delta Z_Q[J] = \frac{1}{Q!}\int d^4z \left[\prod_{i=1}^{Q} d\rho_i\, n_1(\rho_i)\ \det{}_0^{(1)}(J)\right] I_Q\big(\{\rho_i\}\big) \qquad I_Q(\rho_1,\rho_2) = c_{N_f}\sum_{i=1}^{Q}\prod_{i\neq j}\rho_i^{2N_f}\rho_j^{4-2N_f}$$

Gory details in [FR, 2003.13876].
Intuitive picture:

- instanton density peaks about certain size: effective instanton size $\bar{\rho}$
- geometric overlap of $Q$ constituent-instantons: $\sim \bar{\rho}^{\,4(Q-1)}$




geometric overlap

$Z_1 = \int d^4z\, \bar{Z}_1$

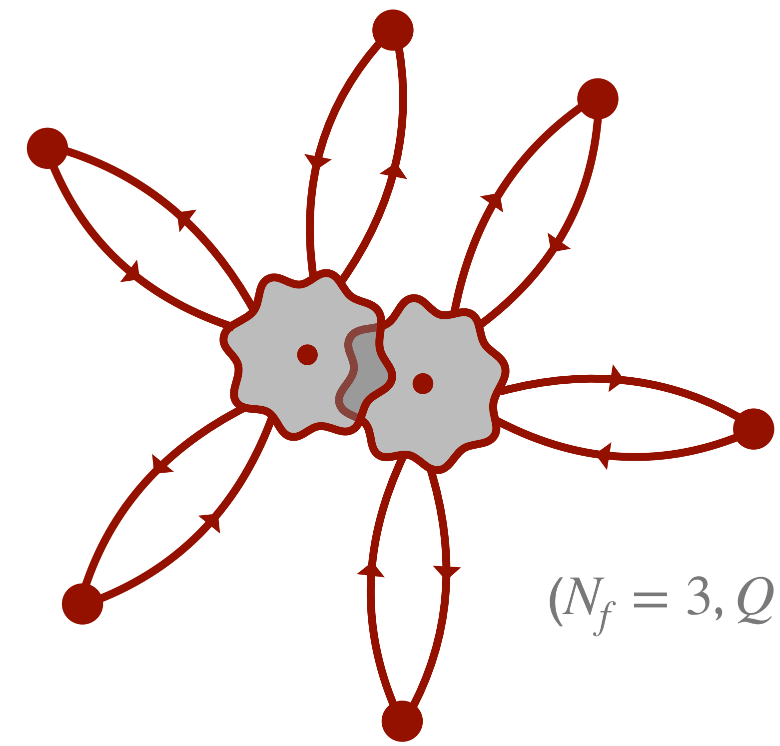
$$\Delta Z_Q \approx \mathscr{V}\,\frac{\left(\bar{c}_{N_f}\bar{\rho}^4\right)^{Q-1}}{(Q-1)!}\,\bar{Z}_1^Q \approx \mathscr{V}\,\frac{\Lambda^4}{(Q-1)!}\,\hat{Z}_1^Q \qquad \hat{Z}_1 = \bar{Z}_1/\Lambda^4$$

from integration over average multi-instanton position

# EFFECTIVE PARTITION FUNCTION

Use the large-distance form of the quark zero modes:

$$\det{}_0(J) \sim \prod_i \prod_f \int d^4x_{fi}\, \Delta(z_i - x_{fi})\, J(x_{fi})\, \Delta(x_{fi} - z_i)$$



$(N_f = 3, Q = 2)$

$\Delta Z_Q$ is identical to the effective partition function
(details in [Pisarski, FR, 1910.14052])

no instanton background!

$$\Delta Z^{\text{eff}}_{+Q}[\bar{J}] = \int \mathscr{D}\Phi\, e^{-S[\Phi] + \int_x \bar{\psi}\bar{J}\psi}\, \Delta S^{\text{eff}}_{+Q}$$

$$\Delta S^{\text{eff}}_{+Q}\Big|_{\text{singlet}} \sim \int d^4z\, \kappa_Q \det_{fg}\left[\bar{\psi}_f(z)\, \mathbb{P}_R\, \psi_g(z)\right]^Q$$

- local $2N_f Q$-quark correlation function
- for anti-instantons ($Q < 0$): $\mathbb{P}_R \to \mathbb{P}_L$

# DILUTE MULTI-INSTANTON GAS

- so far: partition function with one $Q$-instanton in the background
- but all possible gluon configurations contribute to the path integral
- assume that topological fluctuations are described by a dilute instanton gas
- reasonable at large enough temperature due to thermal screening of instanton density: (constituent-) instantons are small at large $T$, $\pi T\rho \ll 1$ [Gross, Pisarski, Yaffe (1981)]

Since the path integral involves integrations over all instanton locations,
there are genuine multi-instanton corrections to the DIGA

Consider statistical ensemble of multi-instanton processes $(\Delta Z_1 \equiv Z_1)$.
Only count genuine multi-instanton contributions $\Delta Z_Q$ to avoid double counting:

$$\mathcal{Z} = \prod_{Q>0} \sum_{n_Q} \sum_{\bar{n}_Q} \frac{1}{n_Q!\,\bar{n}_Q!} \Delta Z_{+Q}^{n_Q} \Delta Z_{-Q}^{\bar{n}_Q} = e^{\sum_{Q>0}(\Delta Z_{+Q}+\Delta Z_{-Q})}$$

# ANOMALOUS QUARK CORRELATIONS

$\Delta S_Q^{\rm eff}$ is exponentiated in the dilute gas:

$$\mathcal{Z}^{\rm eff} = \int \mathcal{D}\Phi \, e^{-S[\Phi] + \sum_{Q>0} \left(\Delta S_{+Q}^{\rm eff} + \Delta S_{-Q}^{\rm eff}\right)} \longrightarrow$$

$\det \left(\bar{\psi}_f \, \mathbb{P}_{R/L} \, \psi_g\right)^{|Q|}$ terms
in the effective action

$\longrightarrow$ **anomalous $2N_f |Q|$ -quark correlation functions**

Bosonization of $Q = 1, 2$ terms lead to LSM from the beginning!

**Axial anomaly is also encoded in higher order correlation functions.**
**Their microscopic origin is instantons with higher topological charge.**

Use this to look for new signatures of the anomaly in quark/hadron correlations.

# $\theta$-DEPENDENCE AND TOPOLOGICAL SUSCEPTIBILITIES

cf. Maria Lombardo's talk

# Θ-DEPENDENCE FROM DILUTE INSTANTONS

Topological structure of QCD necessitates the existence of a topological Θ-term:

[Jackiw, Rebbi & Callan, Dashen, Gross (1976)]

free parameter

$$\mathscr{L} = \bar{\psi}\,\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\,FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\,F\tilde{F}$$

$\theta$-dependence in a dilute multi-instanton gas via simple substitution:

$$\Delta Z_Q \longrightarrow \Delta Z_Q\, e^{iQ\theta}$$

Resulting **free energy density**

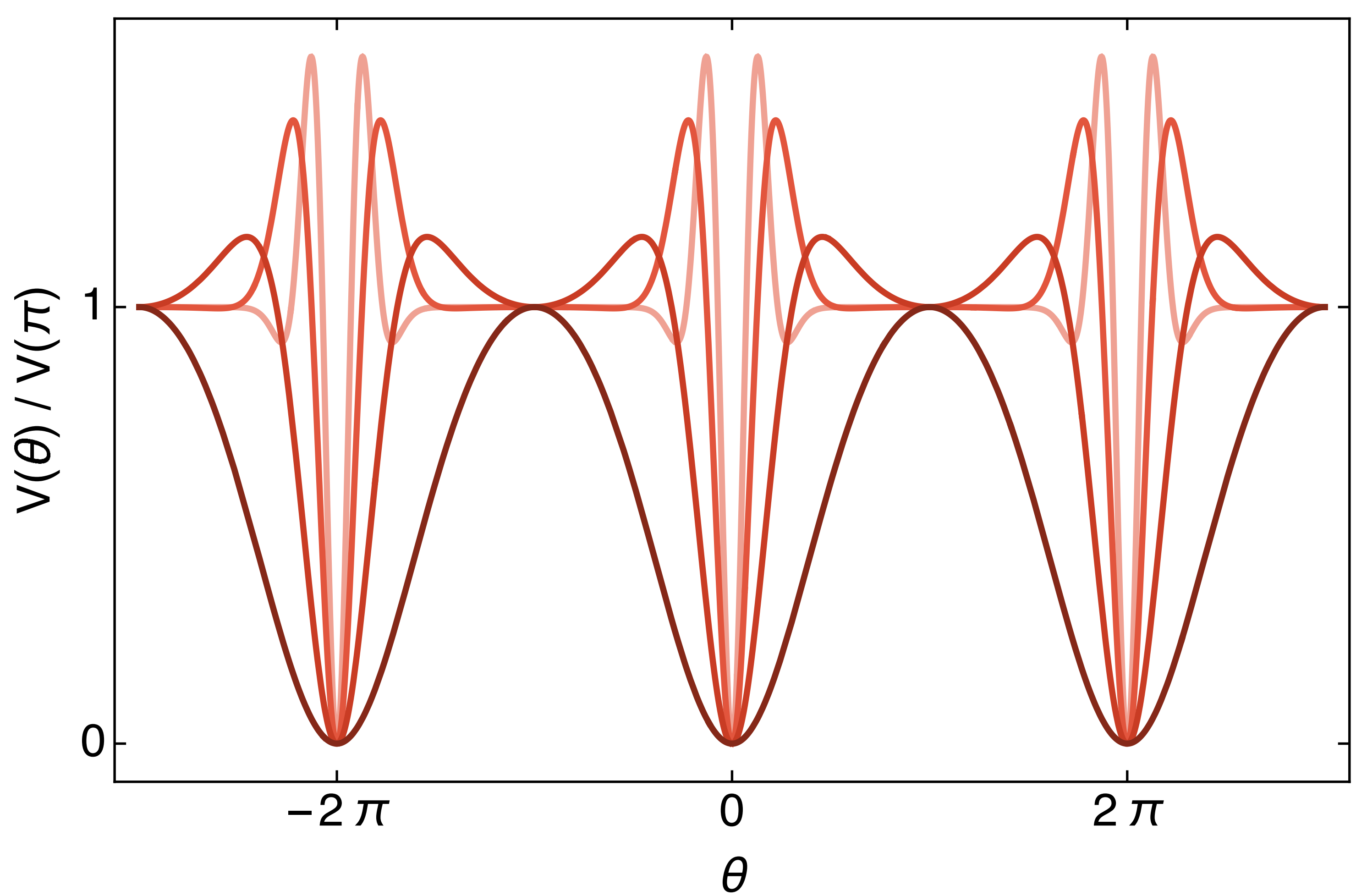
$$F(\theta) = -\frac{1}{\mathscr{V}}\ln \mathscr{Z}(\theta) = -\frac{2}{\mathscr{V}}\sum_Q \Delta Z_Q \cos(Q\theta)$$

# Θ-DEPENDENCE FROM DILUTE INSTANTONS

Use simple solution to compute the normalized free energy $\Delta F(\theta) = F(\theta) - F(0)$

$$\Delta F(\theta) = 2\Lambda^4 \sum_Q \frac{\hat{Z}_1^Q}{(Q-1)!}\left[1-\cos(Q\theta)\right] = 2\Lambda^4\,\hat{Z}_1\left[e^{\hat{Z}_1} - \cos\left(\hat{Z}_1 \sin\theta\right) e^{\hat{Z}_1 \cos\theta}\right]$$

- recover well-known single-instanton result for $\hat{Z}_1 \ll 1$: $\Delta F(\theta) = 2\Lambda^4 \hat{Z}_1\left(1-\cos\theta\right) + \mathcal{O}(\hat{Z}_1^2)$
- multi-instantons modify the simple $\cos\theta$ - behavior!




$\Delta F(\theta)/\Delta F(\pi)$ for $\hat{Z}_1 = 0.1, 1, 3, 6$ (from darkest to lightest red)

# TOPOLOGICAL SUSCEPTIBILITIES

$\Delta F(\theta)$ describes the distribution of topological charge in QCD

**topological susceptibilities:** $$\chi_{2n} = \left.\frac{\partial^{2n}\Delta F(\theta)}{\partial\theta^{2n}}\right|_{\theta=0} \sim \langle Q^{2n}\rangle_c$$

dilute single-instantons gas:

$$\left.\chi_{2n}\right|_{Q=1} = 2(-1)^{n+1}\,\bar{Z}_1 = (-1)^{n+1}\,\chi_2$$

dilute multi-instanton gas:

$$\chi_{2n} = \frac{2(-1)^{n+1}}{\mathscr{V}}\sum_Q Q^{2n}\,\Delta Z_Q$$

$\Delta Z_Q > 0$ → enhanced topological correlations from multi-instanton corrections

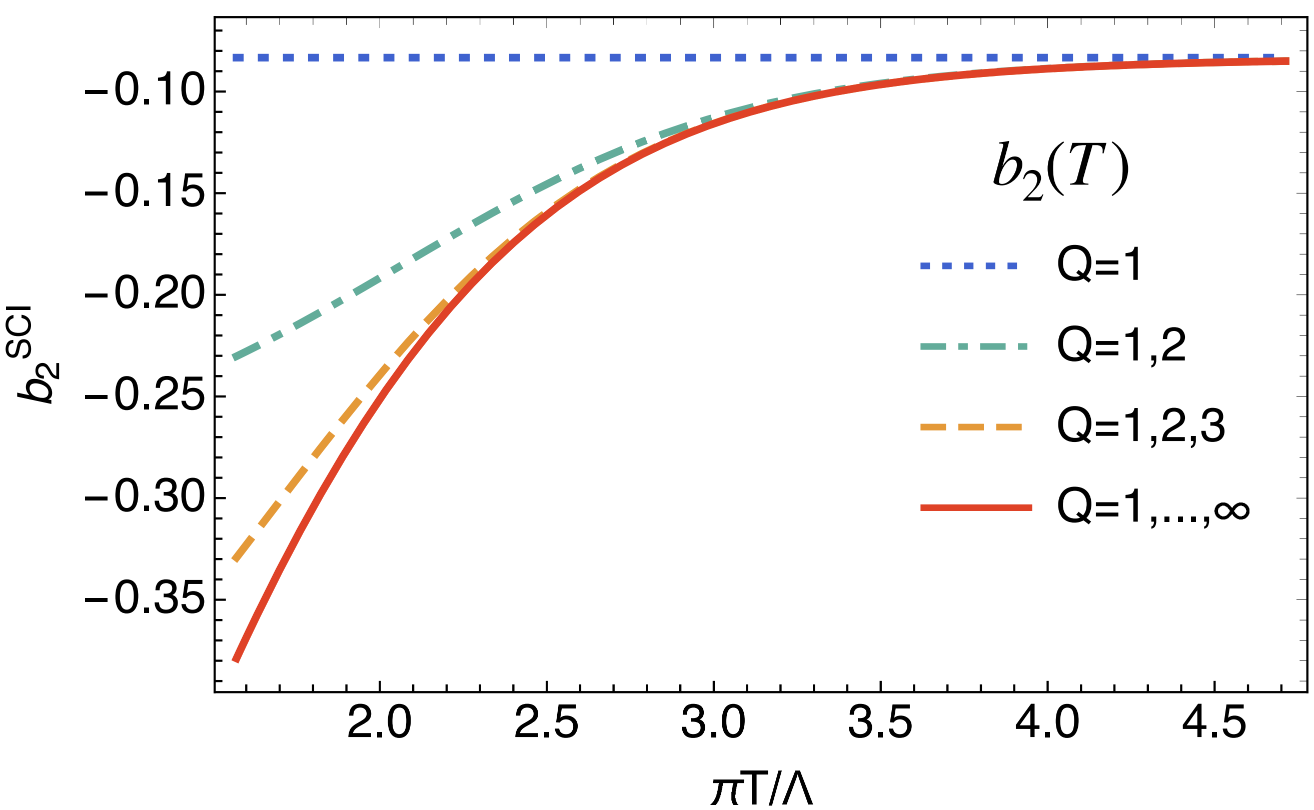
# TOPOLOGICAL SUSCEPTIBILITIES

Deviation of higher susceptibilities from $\chi_2$: anharmonicity coefficients

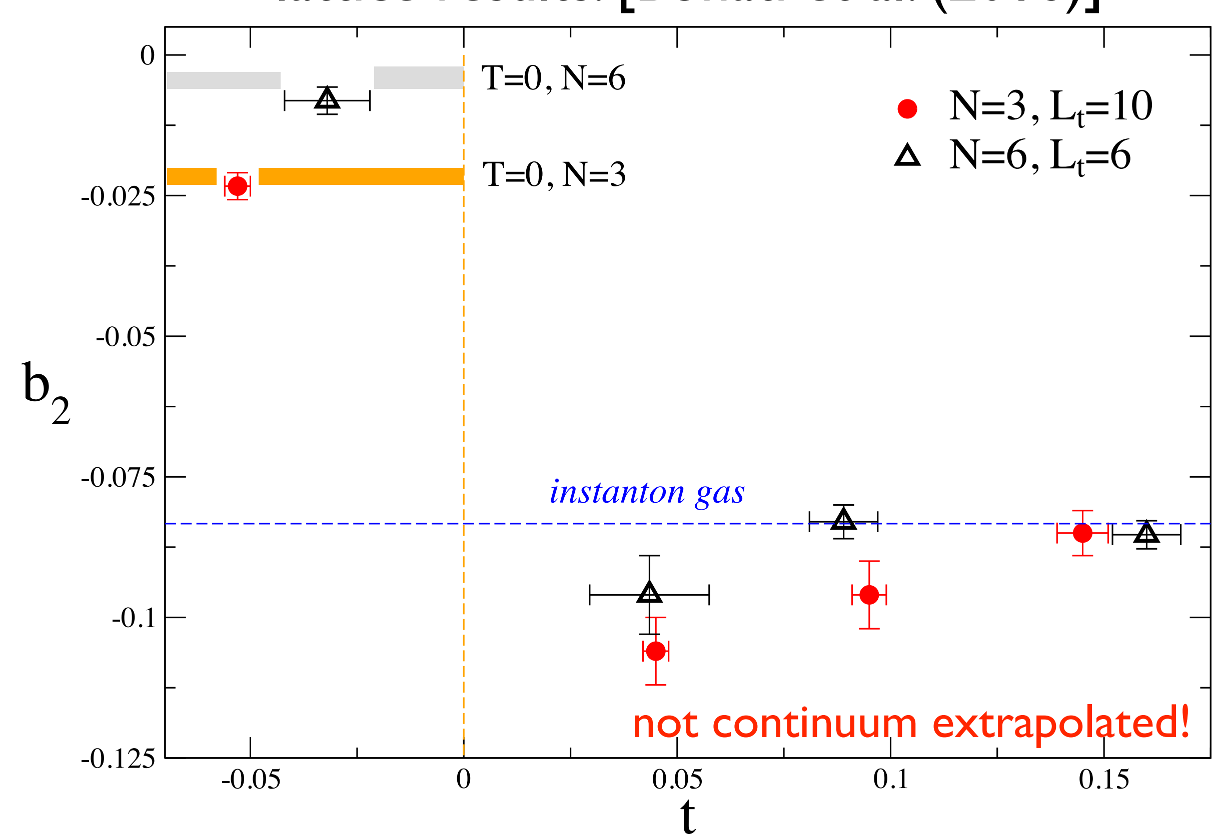
$$b_{2n} = \frac{2}{(2n+2)!}\frac{\chi_{2n+2}}{\chi_2}$$

Multi-instantons yield T-dependent corrections to constant single-instanton prediction!

Result for **quenched QCD** at large temperatures (no dynamical quarks, $m_q \sim 1/\bar{\rho}$)




lattice results: [Bonati et al. (2016)]



# AXION COSMOLOGY

# PECCEI-QUINN MECHANISM


Possible resolution of the CP problem:

**why is $\theta \approx 0$?**

(e.g. no neutron electric dipole moment)

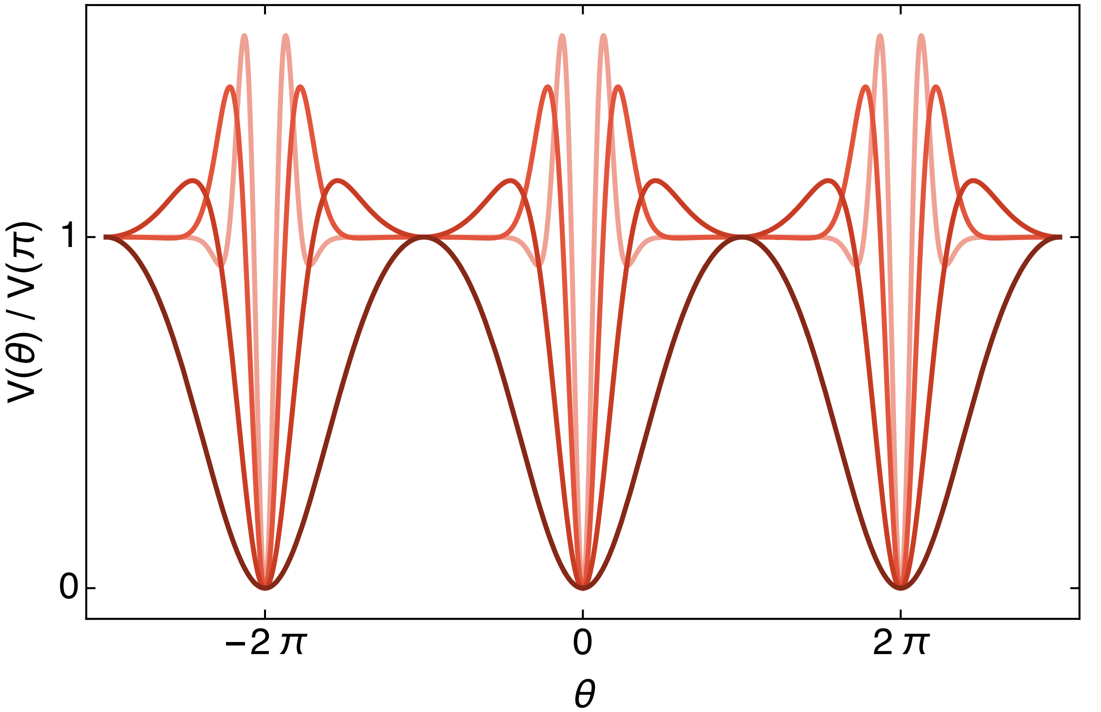
- augment SM with global axial $U(1)_{\rm PQ}$ + charged scalar that couples to quarks
- $U(1)_{\rm PQ}$ is spontaneously broken at scale $f_a$, resulting Goldstone boson: **axion**
- the axial anomaly in $U(1)_{\rm PQ}$ dictates non-derivative couplings of the axion:

$$\mathscr{L} = \bar{\psi}\gamma_\mu D_\mu \psi + \frac{1}{2}\mathrm{tr}\, FF + \frac{i\theta}{16\pi^2}\,\mathrm{tr}\, F\tilde{F} + \frac{a(x)}{f_a}\frac{i}{16\pi^2}\,\mathrm{tr}\, F\tilde{F} + \ldots$$

⟶ 'dynamical' θ-angle $\bar{\theta}(x) = f_a\theta + a(x)$

axion effective potential is given by $F(\theta)$!

$$V(\bar{\theta}/f_a) \equiv \Delta F(\bar{\theta}/f_a)$$




minimum at $\bar{\theta} = 0$:

CP problem resolved!

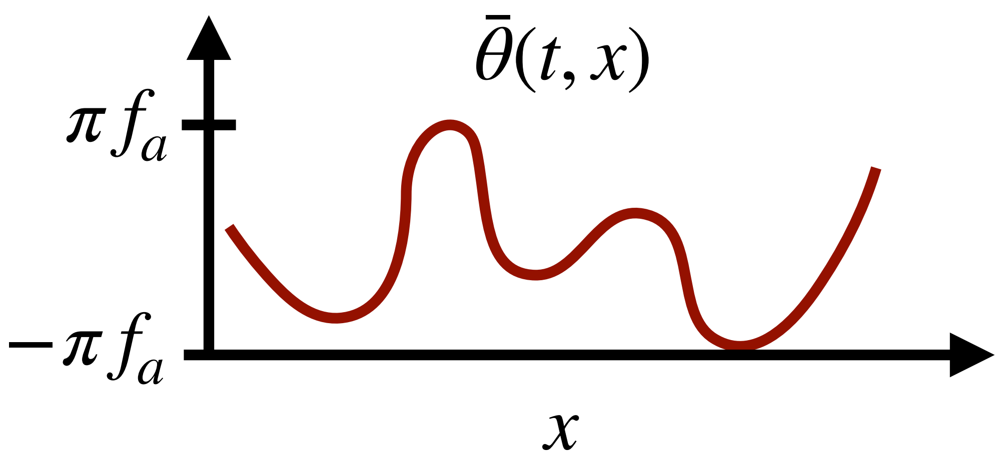
# AXION DARK MATTER

Axions are also viable dark-matter candidates.

- compatible with observations if light and weakly coupled to SM $(f_a \gtrsim 10^9\,\mathrm{GeV})$
  [Kim & Shifman,Vainshtein, Zakharov & Zhitnitsky & Dine, Fischler, Srednicki (1979+)]
- production of axion dark matter through the **vacuum realignment mechanism**
  [Preskill,Wise,Wilczek & Abbott, Sikivie & Dine, Fischler (1983)]

Assume $U(1)_{\mathrm{PQ}}$ is broken before inflation

axion is very light - fluctuates strongly



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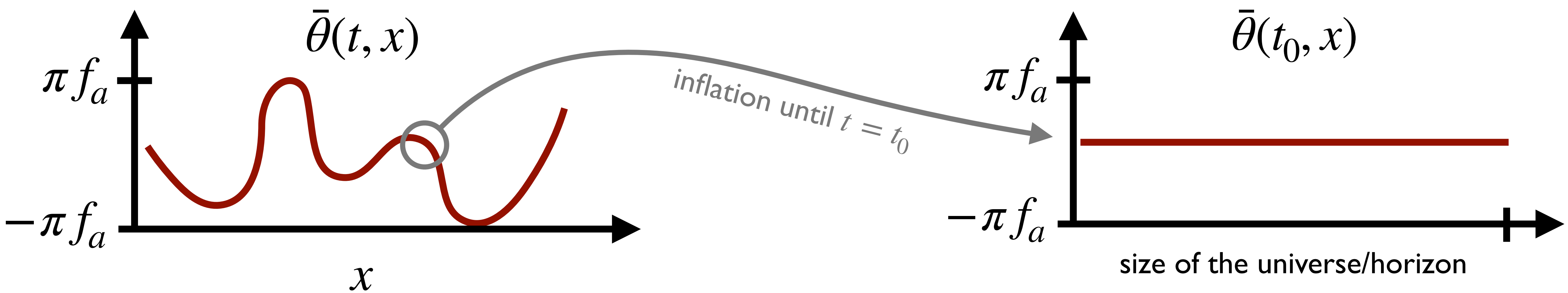
homogeneous axion field after inflation




→ **vacuum realignment: time evolution of axion dark matter from initial misalignment $\bar{\theta}(t_0)$ until today**

# AXION DARK MATTER

Time evolution of the axion governed by Einstein's equations.

- assume isotropic, homogenous, expanding & flat universe: Friedmann equations

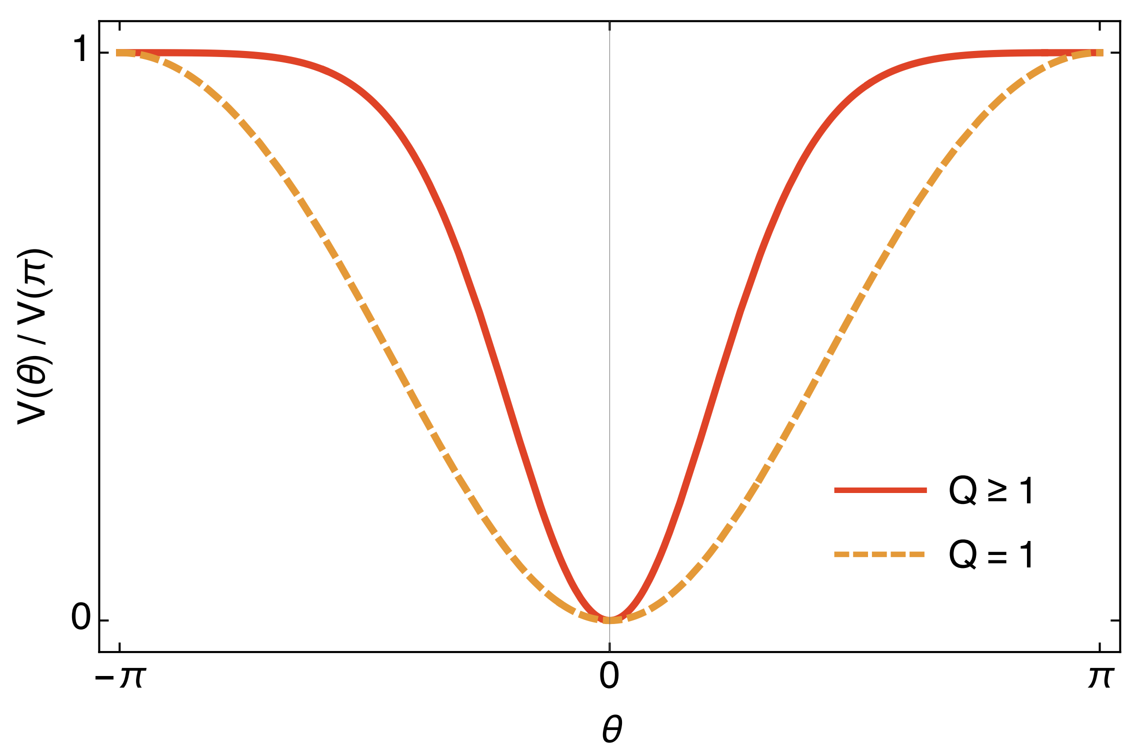
$$\frac{d^2\bar{\theta}}{dt^2} + 3H\frac{d\bar{\theta}}{dt} + \frac{dV(\bar{\theta}/f_a)}{d\bar{\theta}} = 0$$

metric: $g_{\mu\nu} = \mathrm{diag}\left(1, -a^2(t), -a^2(t), -a^2(t)\right)$

Hubble parameter: $H(t) = \frac{\dot{a}(t)}{a(t)}$

Resembles a damped harmonic oscillator

- early t: $\frac{H}{|V''|} \gg 1$: overdamping - axion is frozen in time
- late t: $\frac{H}{|V''|} \ll 1$: underdamping - oscillation around minimum




axion DM energy density

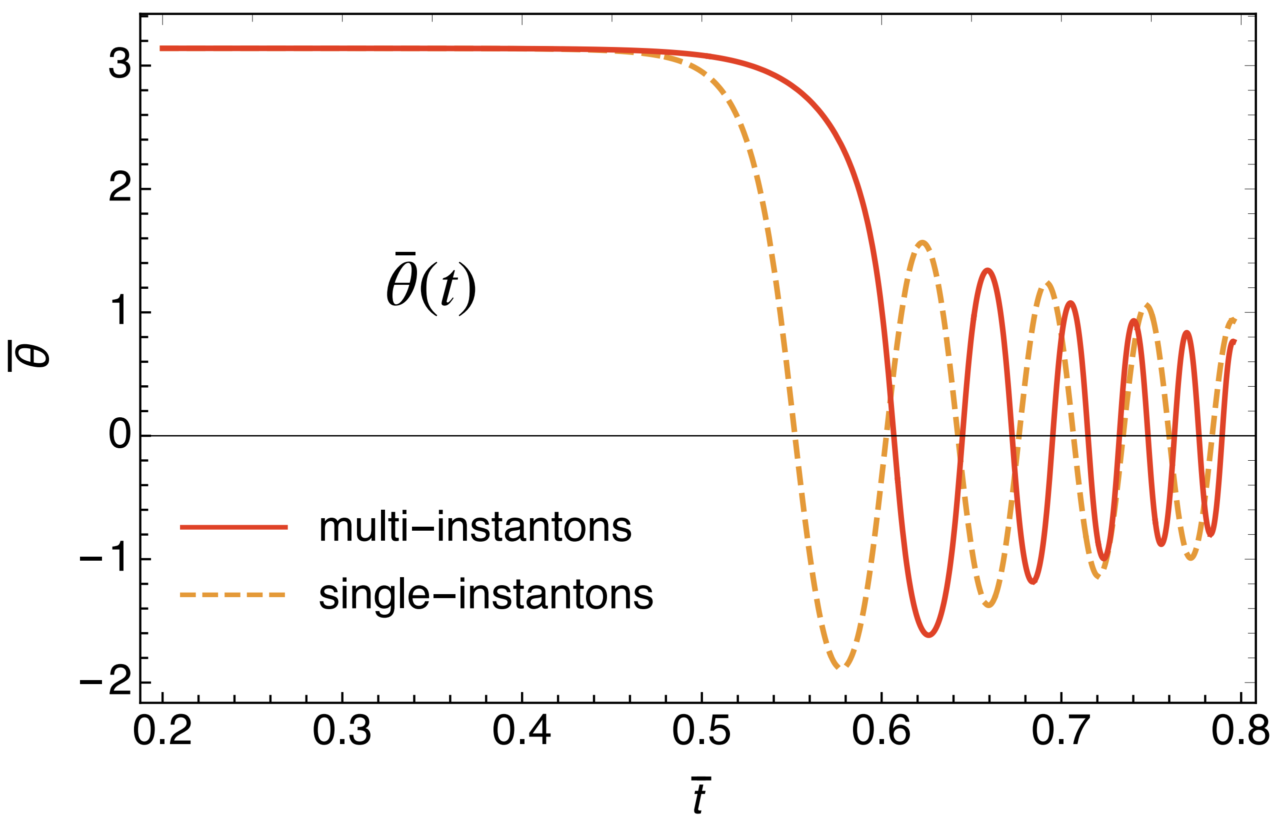
$$\rho_a = \frac{1}{2}\left(\frac{d\bar{\theta}}{dt}\right)^2 + V(\bar{\theta}/f_a)$$

# AXION DARK MATTER

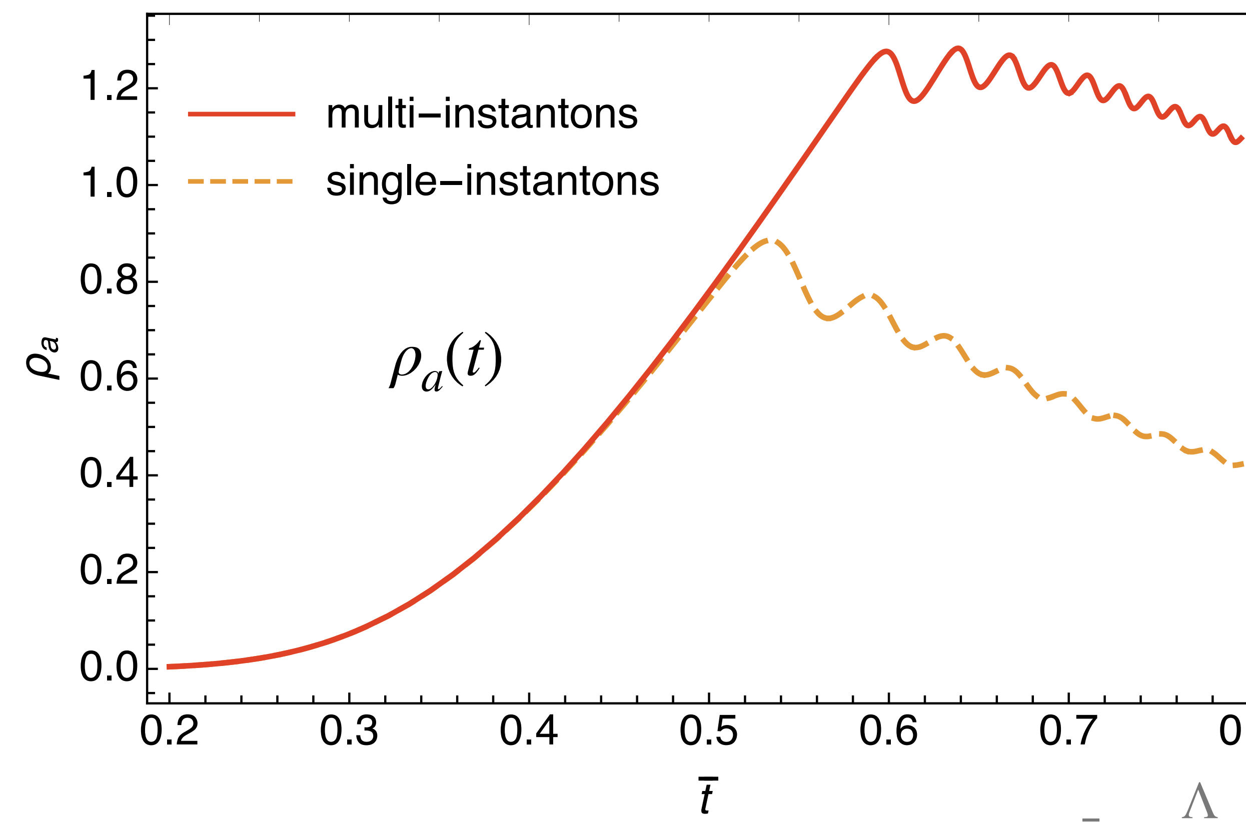
Compare single-instanton to 'full' multi-instanton solution $V(\bar{\theta}/f_a)$

(again in the quenched limit)

axion field




axion DM energy density




$\bar{t} = \frac{\Lambda}{\pi T}$

→ multi-instanton effects can lead to more axion dark matter

# DISCUSSION

Size of the effects studied here depends on the size of multi-instanton effects.

- classically: strong exponential suppression: $n_Q \sim \exp\Big( -\frac{8\pi^2}{g^2} |Q| \Big)$
- instanton density strongly suppressed by light quarks

⟶ quantum corrections are crucial for multi-instanton effects

How to assess the quantitative relevance of the effects discussed here?

- compute higher-order corrections in the SCI limit: better understanding of $\Delta Z_Q$
- account for interactions between multi-(anti-)instantons: multi-instanton liquid?

Keep in mind: semi-classical picture breaks down eventually at strong coupling!

**But: semiclassical analysis shows that all these effects must be there**

# SUMMARY

**What about instantons with higher topological charge?**

- they give rise to anomalous $2N_f|Q|$-quark correlation functions [Pisarski, FR; 1910.14052]

$$\det\left(\bar{\psi}_f \mathbb{P}_R \psi_g\right)^{|Q|} + \det\left(\bar{\psi}_f \mathbb{P}_L \psi_g\right)^{|Q|}$$

  - signatures of the axial anomaly through higher order anomalous correlations

- they yield corrections to the θ-dependence of QCD [FR; 2003.13876]

$$F(\theta) \sim -\sum_{Q=1}^{\infty} \Delta Z_Q \cos(Q\theta)$$

  - modified temperature dependence of topological susceptibilities
  - topological mechanism to increase the amount of axion dark matter