ICTS-RRI Math Circle
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It is fairly straightforward to maximise (minimize) values of a given function. But is it possible to find the best choice of a function to maximise (minimise) a given integral?

The origins of this week's topic of discussion goes back to the 17^{th} century when Sir Isaac Newton came up with a problem: what kind of solid shape, created by revolving a plane shape about an axis, would move through a fluid with least amount of resistance?

This, along with the Brachistochrone problem, would capture the interest of many mathematicians, two of which would go on to lay the foundations of the field. Leonhard Euler became the first one to approach this field, in a geometric manner, which he discarded, two decades later, in favour of the 19 year old Lagrange's work which was more analytical in nature.

Their work would prove pivotal for many fields, like Lagrangian mechanics which tries to understand topics like planetary motion and the Lagrangian points (the position of Trojan asteroids and JWST in our solar system), and Plateau's problem of minimising surface area for a given frame (one can dip the frame in soapy water to get a solution).

Bend it like light!

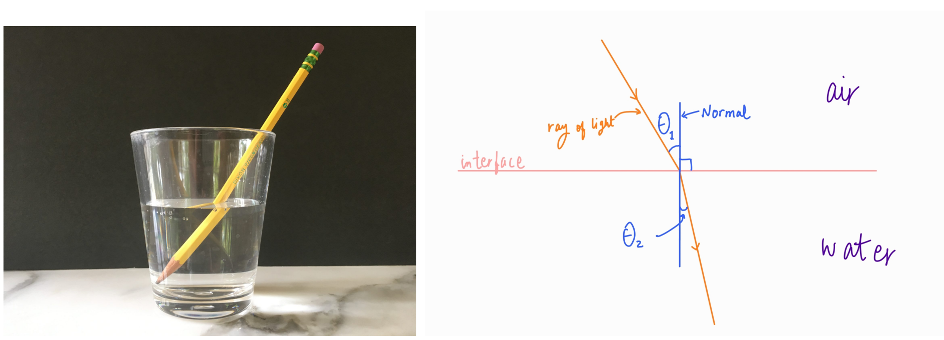

Figure 1: **Left**: How interesting that refraction makes the pencil appear broken! **Right**: This figure is the schematic showing how light bends while entering a denser medium. θ_1 and θ_2 are the angles between the ray of light and the normal at the air-water interface.

What is happening?
The speed of light is different in different media. Here, it is air and water. Light travels slower in water. When a beam of light enters the water at an angle then a consequence of this slowdown is that it bends- refracts. Below is the formula relating the angles θ_1 and θ_2 to the velocities of light in air v_{air} and in water v_{water}.

$$\frac{\sin\theta_1}{v_{\text{air}}} = \frac{\sin\theta_2}{v_{\text{water}}} \tag{1}$$

1. Can the surface act like a mirror? If so, then what are the angles?

Fascinating Brachistochrones

The great Johann Bernoulli posed a problem to all the leading mathematicians in the June of 1696. Bernoulli stated the problem as follows [1]:

> Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?

Naturally this attracted many great minds including Newton and brought us a whole host of solutions and more importantly newer avenues of mathematics. The solution to this problem was the brachistochrone [2]. We shall dissect this mystery in more depth in our session. But for now, let us get our hands dirty with the following versions of the same.

1. Consider dropping a ball vertically to a point on the ground from a height 'h'. What is the locus of the starting points of straight slides that will take the same amount of time to reach that point on the ground as the ball dropped vertically?

2. Consider a case where a ball must travel under the influence of gravity between two points at the same vertical height. You are allowed to create only one bend in the slide which is otherwise straight. What will the slide look like?

Queen Dido's Road to Royalty

Dido, also known as Elissa, was the legendary founder and first queen of the Phoenician city-state of Carthage (located in modern Tunisia), in 814 BC. She was the queen of the Phoenician city-state of Tyre (today in Lebanon) who fled tyranny to found her city in northwest Africa. Legend has it that Dido had been promised all the land she could enclose with a bull's hide. What would you do if you were Dido and this was your one shot to become a queen?

Something to think about...

Start with a two stepped bridge. The first and last steps are hinged to the edges while they are linked to each other with a hinge too. What is the shape formed by this bridge if let loose (it might not be the ideal looking bridge). What would be the shape with three steps? What about four? What about a million (a continuum of them)? Note: all the steps are identical and the ratio r of the total length of the steps and the distance between the edges is held constant and is greater than 1 (so that the bridge hangs).

References

[1] Wikipedia, "Brachistochrone curve — Wikipedia, the free encyclopedia." http://en.wikipedia.org/w/index.php?title=Brachistochrone%20curveoldid=1184078601, 2024. [Online; accessed 05-February-2024].

[2] P. Deshmukh, P. Rajauria, A. Rajans, B. Vyshakh, and S. Dutta, "The brachistochrone," *Resonance*, vol. 22, no. 9, pp. 847–866, 2017.