Equidistribution of unipotent random walks on homogeneous spaces – and local limit theorems –

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Joint work with Timothée Bénard

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PLAN:

A- Choquet Deny on nilpotent groups.

B- Ratner-Shah and Benoist-Quint equidistribution theorems.

C- Random ergodic theorems and Random Ratner theorem.

D- Local limit theorem on the Heisenberg group, limiting law, Lévy area, etc.

E- Non-centered local limit theorem on an arbitrary nilpotent Lie group.

1. Review of the Choquet-Deny property

Let G be a locally compact abelian group and μ a probability measure on G whose support generates a dense subgroup.

Choquet-Deny (1960) : μ-harmonic functions are constant

i.e $$\mu * f = f,\ f \in L^\infty(G) \Rightarrow f \text{ is constant.}$$

Furstenberg/Dynkin-Malyutov '60s: the same holds for discrete nilpotent groups.

Frisch-Hartman-Tamuz-Vahidi '19: in fact this characterizes nilpotent-by-finite groups among finitely generated groups.

Furstenberg's question: Does Choquet-Deny hold for all nilpotent locally compact groups?

Guivarc'h's thesis '70: YES if μ has a finite moment, i.e. $\int_G |g|^\alpha d\mu(g) < \infty$ for some $\alpha > 0$

2. Raghunathan-Dani conjectures : Ratner's theorem (1990)

Let G be a connected Lie group, Γ a lattice (= discrete subgroup of finite co-volume), $x \in G/\Gamma$ and $U = \{u_t\}_{t\in\mathbf{R}}$ a unipotent one-parameter subgroup in G.

Theorem (Ratner). *1.* $\exists H \leq G$,

$$\overline{Ux} = Hx, \qquad m_x(Hx) < \infty,$$

$m_x = H$*-invariant volume. All* U*-invariant ergodic measures are of this form.*

2. For $f \in C_c(G/\Gamma)$*, as* $T \to +\infty$,

$$\frac{1}{T}\int_0^T f(u_t x)dt \to \int_{Hx} f dm_x,$$

3. μ*-stationary measures are invariant (*μ *proba on* $U = \overline{\langle Supp\mu \rangle}$*).*

NOTE: Item 3) is a consequence of 1) via the Choquet-Deny theorem on **R :**

$\mu * \nu = \nu$, $\phi(g) := \int_{G/\Gamma} f(g^{-1}x)d\nu(x) \Rightarrow \mu * \phi = \phi$. C-D $\Rightarrow \phi$ constant, i.e. ν is U-invariant.

3. Benoist-Quint '08:

Let G be a connected Lie group, Γ a lattice, $x \in G/\Gamma$ and Λ a subgroup whose Zariski-closure is semisimple. Let μ a probability measure on $\Lambda = \overline{\langle Supp\mu \rangle}$.

Theorem (Benoist-Quint). *1.* $\exists H \leq G$,

$$\overline{\Lambda x} = Hx, m_x(Hx) < \infty,$$

$m_x = H$*-invariant volume. All* Λ*-invariant ergodic measures are of this form.*

2. For $f \in C_c(G/\Gamma)$*, as* $n \to +\infty$,

$$\frac{1}{n}\sum_1^n f(gx)d\mu^k(g) \to \int_{Hx} f dm_x,$$

3. μ*-stationary measures are* Λ*-invariant.*

Note: if all stationary measures are invariant, we say that the action is stiff (Furstenberg).

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Note: Ratner's and B-Q's theorems are equidistribution theorems valid for every point x.

By contrast ergodic theorems give convergence of averages for almost every point with respect to some invariant ergodic measure. For random walk averages one has classical ergodic theorems:

If μ is a probability measure on a group G (with $\overline{\langle Supp\mu \rangle} = G$) and (X, m) is an ergodic measure preserving G-space, then

Kakutani's random ergodic theorem (1950): for m-a.e. x, as $n \to +\infty$:

$$\frac{1}{n} \sum_{1}^{n} \mu^k * \delta_x \to m$$

Oseledets' random ergodic theorem (1964): if μ is symmetric, then for m-a.e. x, as $n \to +\infty$:

$$\mu^n * \delta_x \to m.$$

3. Benoist-Quint

Remark: In the setting of Benoist-Quint's theorem, Bénard ('21) has recently shown that the convergence also holds without Cesaro averages, provided the measure charges the identity:

$$\text{If } \mu(1) > 0, \text{ then } \mu^n * \delta_x \to m_x.$$

[based on a very general observation of Foguel ('75) that

$$\|\mu^{n+1} - \mu^n\|_{TV} \to 0$$

provided $\mu(1) > 0$ for any probability measure μ on any group.]

4.Random Ratner's theorem

Nimish Shah ('90s), answering a question of Ratner, has generalized her equidistribution theorem to higher dimensional unipotent groups.

G a connected Lie group, Γ a lattice, $x \in G/\Gamma$ and U an arbitrary connected unipotent subgroup in G, with $\dim U = d$. Ratner proved that $\overline{Ux} = Hx$ for some closed subgroup H with $m_x(Hx) = 1$ for the H-volume m_x. And:

Theorem (Shah '94). *for* $f \in C_c(G/\Gamma)$,

$$\frac{1}{vol(B)} \int_{\exp(B)} f(ux)du \to \int_{Hx} f dm_x$$

where B *is a box* $B = \prod_1^d [0, T_i] \subset \mathcal{U} := Lie(U)$, *and each* T_i *tends to* $+\infty$.

4. Random Ratner's theorem:

Our main result is that the equidistribution continues to hold for unipotent orbits when ordinary averages are replaced by random walk averages, namely:

Theorem 1 (B+B '22) *In the setting of the Ratner-Shah theorems, if μ is an aperiodic probability measure with finite moments of all orders supported on an arbitrary unipotent subgroup U, then*

$$\frac{1}{n}\sum_{1}^{n}\mu^k * \delta_x \to m_x$$

and if μ is centered (i.e. mean 0 on $U/[U,U]$), then we even have:

$$\mu^n * \delta_x \to m_x$$

Rk: this extends my (unpublished) 2004 PhD thesis, where the same was shown for μ symmetric with finite support.

The proof will appear as a consequence of a general Local Limit Theorem we establish on arbitrary connected nilpotent Lie groups and discuss in the second part of this talk.

4. Random Ratner's theorem:

Remark: The centeredness assumption in 2. is necessary: non-centered aperiodic walks can very well have full escape of mass at infinity.

Take $\mathrm{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{Z})$, $\theta \in \mathbf{R}$ Lebesgue generic, and

$$x_0 = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \mathbf{Z}^2, u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$$\Omega_\varepsilon = \{x \in SL_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{Z}), \text{shortest vector of } x \leq \varepsilon\}$$

If μ is non-centered, say $\mathbf{E}(\mu) = 1$, then along some subsequence of n's

$$\mu^n * \delta_x(\Omega_\varepsilon) \to 1.$$

[pf: there are arbitrarily large n, m such that $n|n\theta + m| \ll \varepsilon^2$. Set $(x, y) = (n, n\theta + m) \in x_0$, then the orbit Ux_0 enters Ω_ε at time t and next leaves it at time t', where $x - ty \simeq \varepsilon$ while $x - t'y \simeq -\varepsilon$. So $t' - t \simeq 2\varepsilon/y$ while $t \simeq x/y$. The condition $|xy| \ll \varepsilon^2$ means that $t' - t \ll \sqrt{t}$. By the CLT, W.H.P. the walk at time $\frac{t+t'}{2}$ is within $\sqrt{t}$ of the mean. qed.]

5. The classical local limit theorem on R:

Let $g_1, \ldots, g_n, \ldots$ be a sequence of real valued random variables with common law μ on $\mathbf{R}$.

$$\text{set } S_n := g_1 + \ldots + g_n$$

CLT: $$\frac{S_n - nm}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2)$$ where $m = \mathbf{E}(\mu) = \mathbf{E}(g_1)$ is the mean and $\sigma^2 = \mathbf{E}((g_1 - m)^2)$ the variance.

LLT: $$\sqrt{n} \sup_{t \in \mathbf{R}} |\mu^n(I+t) - \nu_n(I+t)| \to 0$$ where $I = [a,b]$ is a fixed finite interval, and $\nu_n = \nu^n$ is the gaussian law $\mathcal{N}(nm, n\sigma^2)$.

In particular: $\mu^n(I) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \frac{|I|}{\sqrt{n}}$

The LLT gives uniform gaussian approximation in windows of bounded scale. While the CLT needs only a finite 2nd moment assumption on the law, the LLT also requires that the measure be aperiodic (i.e. not contained in a coset of a proper closed subgroup). Although cases of the LLT (e.g. when the measure is Bernoulli) go back to the 18th century, in this form (for an arbitrary aperiodic measure) it was first proved by Ch. Stone ('60s).

5. The classical local limit theorem on R:

Corollary: suppose $f(t)$ is continuous and bounded on $\mathbf{R}$ and assume that there is $\ell \in \mathbf{R}$ such that as $T \to \pm\infty$,

$$\frac{1}{|T|} \int_{[0,T]} f dt \to \ell.$$

Then $\mathbf{E}(f(S_n)) \to \ell$.

From this corollary one deduces the convergence of random averages in the setting of Ratner's theorem for one-parameter unipotent subgroups. On arbitrary nilpotent groups a similar result can be deduced from the LLT we are about to state, yielding a proof of Theorem 1.

6. The Heisenberg group:

$$U := \left\{ \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbf{R} \right\}$$

Dilation automorphisms: $D_t(x, y, z) := (tx, ty, t^2 z) \in Aut(U), D_{ts} = D_t \circ D_s$

A probability measure μ on U is said to be *centered* if its projection to the abelianization $U/[U,U]$ is centered, i.e. $\mathbf{E}_\mu(x) = \mathbf{E}_\mu(y) = 0$.

Tutubalin ('60s) proved a Central Limit Theorem in the centered case:

CLT: If μ is centered and has a finite 2nd moment, then

$$D_{\frac{1}{\sqrt{n}}}(\mu^n) \to \nu$$

The limit measure ν depends only on the 1st and 2nd moments of μ. Another proof follows from the thesis of D. Wehn's (1959), where a general *infinitesimal CLT* is proven on arbitrary Lie groups.

6. The Heisenberg group:

CLT: If μ is centered and has a finite 2nd moment, then

$$D_{\frac{1}{\sqrt{n}}}(\mu^n) \to \nu$$

The limit measure ν has a smooth density belonging to Schwartz space in x, y, z. Applying affine transformations, wlog we can renormalize the measure so that:

$$\mathbf{E}_\mu(x) = \mathbf{E}_\mu(y) = \mathbf{E}_\mu(xy) = 0,\ \mathbf{E}_\mu(z) = 0,\ \mathbf{E}_\mu(x^2) = \mathbf{E}_\mu(y^2) = 1, \mathbf{E}_\mu(|z|) < \infty.$$

The limit measure ν is part of a one-parameter semigroup $(\nu_t)_{t>0}$ of probability measures on U, where $\nu_t = u(t,x,y,z)dxdydz$, and u is the fundamental solution to the following *hypoelliptic* parabolic PDE:

$$\partial_t u = \frac{1}{2}(X^2 + Y^2)u$$

where $X = x\partial_x - \frac{1}{2}\partial_z$ and $Y = y\partial_x + \frac{1}{2}\partial_z$ are the corresponding left invariant vector fields. It *self-similar* in the sense that $\nu_t = \nu_1 \circ D_{\frac{1}{\sqrt{t}}}$.

6. The Heisenberg group:

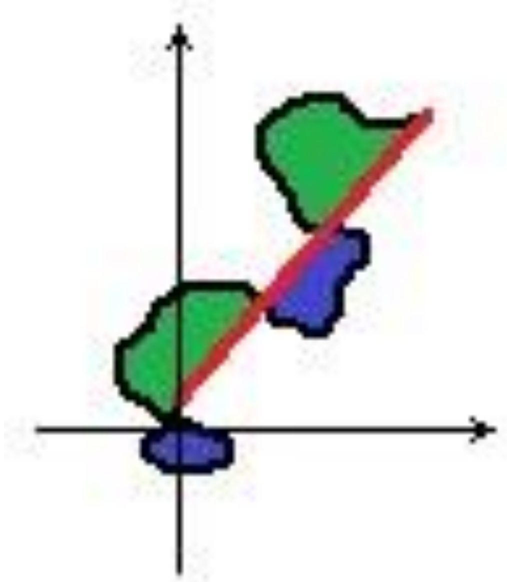
The measure ν is the distribution at time t=1 of a normalized Brownian motion on U

$$B_t = (X_t, Y_t, Z_t)$$

where $Z_t = \frac{1}{2}\int_0^t X_t dY_t - Y_t dX_t$ is the Lévy area of a standard Brownian motion (X_t, Y_t) on $\mathbf{R}^2$.

Lévy (1950) computed the density of B_1:

$$d\nu(x,y,z) = \frac{dxdydz}{2\pi^2}\int_{\mathbf{R}} \cos(2\xi z)\frac{\xi}{\sinh\xi}e^{-\frac{1}{2}(x^2+y^2)\xi/\tanh(\xi)}d\xi$$



The Lévy area is the signed area around the chord from the origin to (x,y).

The blue parts are counted positively, while the green part contribute negatively.

6. The Heisenberg group:

Proofs of the CLT typically rely on variants of Trotter's theorem on convergence of sequences of semigroups of operators. Much harder to establish is a Local Limit Theorem.

Varopoulos and Alexopoulos have developed in the 1990s very versatile analytical tools to prove LLTs and heat kernel estimates on groups of polynomial growth. However these results require the driving measure μ to be absolutely continuous w.r.t Haar (and often symmetric). In 2004, I obtained the following LLT on the Heisenberg group, valid for any (compactly supported) aperiodic and centered measure:

Theorem (LLT on Heisenberg, B. '04). *Suppose μ is aperiodic on U, compactly supported, centered and normalized with $\mathbf{E}(x^2) = \mathbf{E}(y^2) = 1$, $\mathbf{E}(xy) = \mathbf{E}(z) = 0$. Then as $n \to +\infty$*

$$n^2 \sup_{x \in U} |\mu^n(xB) - \nu_n(xB)| \to 0, \text{ in particular } \mu^n(B) \sim \frac{|B|}{4n^2}$$

Here $\nu_n = \nu^n = \nu \circ D_{1/\sqrt{n}}$ is the law at time n of the Brownian motion on U. And B is a bounded Borel subset with $|\partial B| = 0$.

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By contrast, when the measure is not centered, the behavior of the walk is quite different:

Theorem 2 (non-centered LLT on Heisenberg, B+B '22). *Suppose μ is aperiodic on U, non-centered, with moments of all orders, and normalized with $\mathbf{E}(x^2) = \mathbf{E}(y^2) = 1$, $\mathbf{E}(y) = \mathbf{E}(xy) = \mathbf{E}(z) = 0$, $\mathbf{E}(x) = 1$. Then as $n \to +\infty$*

$$n^{5/2} \sup_{x \in U} |\mu^n(xBe^{nX}) - g \circ D_{\mu, \frac{1}{\sqrt{n}}}(xB)| \to 0, \text{ in particular } \mu^n(Be^{nX}) \sim \frac{|B|}{(2\pi)^{3/2} n^{5/2}}.$$

Here $D_{\mu,t}(x, y, z) = (tx, ty, t^3 z)$ and g is a standard Gaussian law $\mathcal{N}(0, I_3)$ on $\mathbf{R}^3$, and $e^X = (1, 0, 0) \in U$. Also B is a bounded Borel set with $|\partial B| = 0$.

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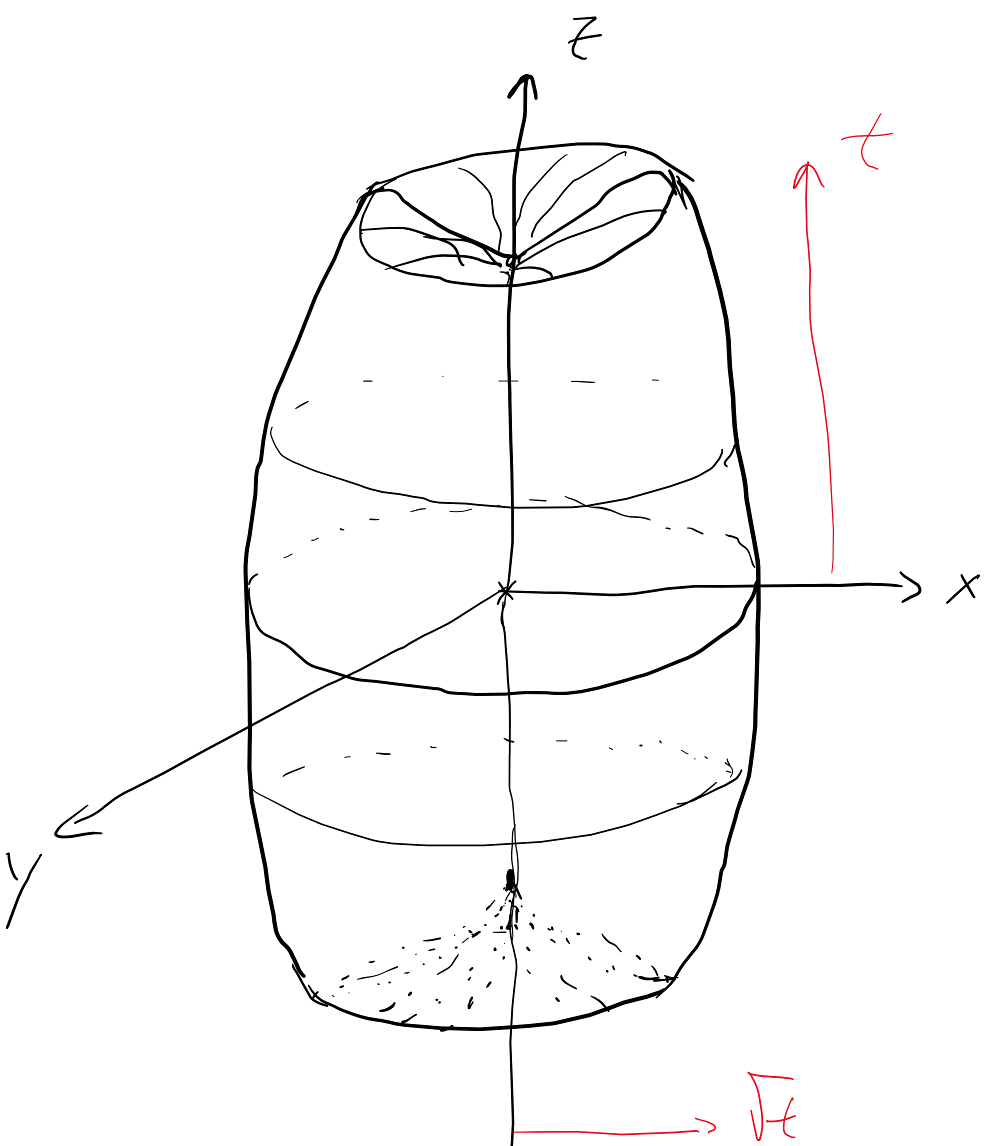
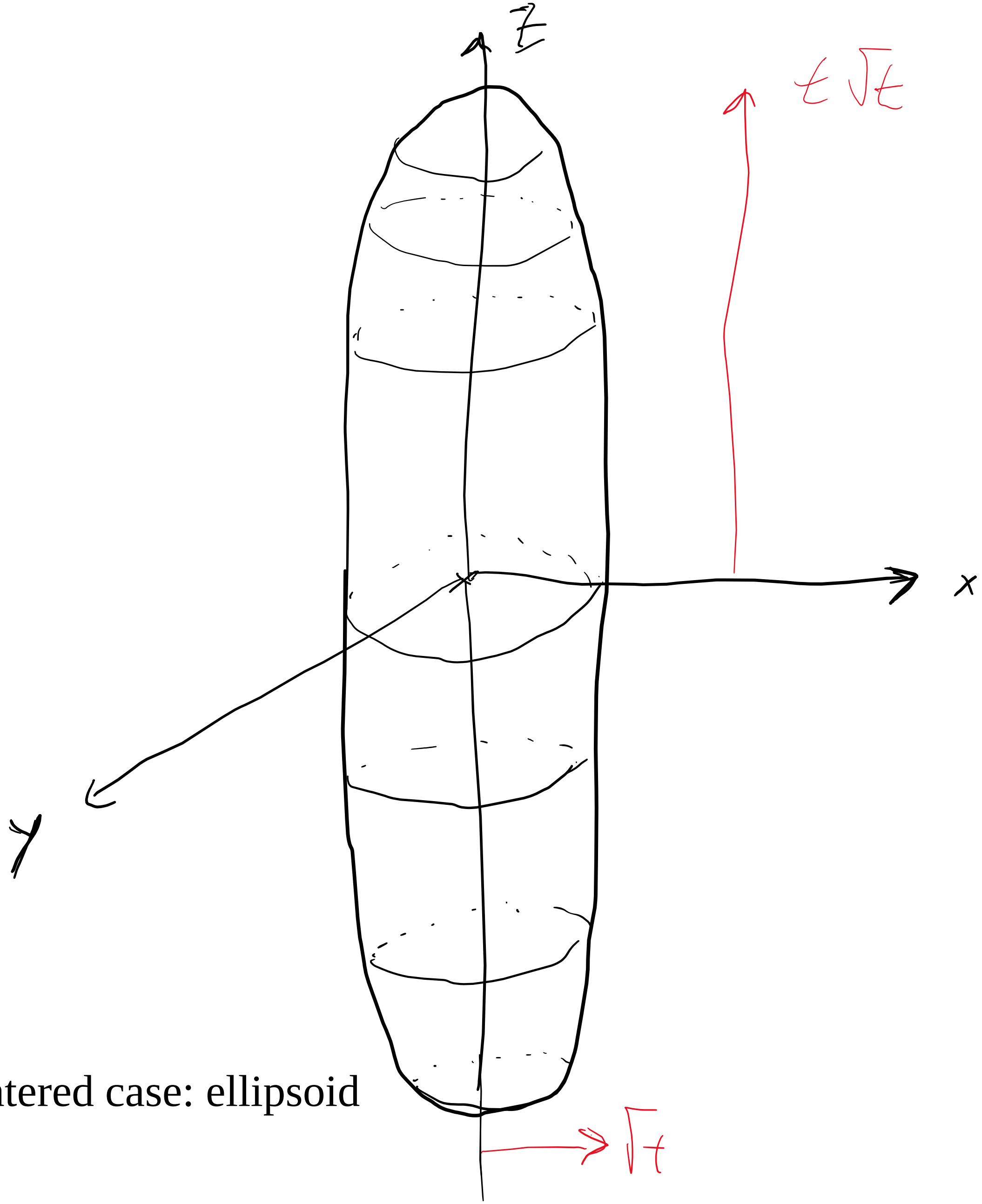
In particular we see that the existence of a non-zero drift *flattens the walk*, i.e. it typically occupies a region of volume $n^{5/2}$ rather than n^2.



Centered case: the apple



Non-centered case: ellipsoid

6. The Heisenberg group, proof of the LLT:

To prove the LLT, one needs to estimate the Fourier transform of the measure.

$$\widehat{\mu^n}(\xi) = \int e^{-2i\pi\xi(x)} d\mu^n(x)$$

$$\mu^n \sim \exp(\sum_1^n X_i, \sum_1^n Y_i, Z_n) \qquad Z_n = \frac{1}{2}\sum_{i<j}(X_iY_j - X_jY_i)$$

Key exponential sum cancellation estimate:

$$\forall A > 1,\ |\mathbf{E}(e^{2i\pi\xi Z_n})| = O_A(\tfrac{1}{n^A}) \text{ provided } |\xi| \gg \tfrac{1}{n^{1-\varepsilon}}$$

In my '04 thesis, I used unitary representations of the Heisenberg group, and a spectral gap estimate to establish the above key cancellation estimate.

6. The Heisenberg group, proof of the LLT:

In my ’04 thesis, I used unitary representations of the Heisenberg group, and the following spectral gap estimate to establish the above key estimate:

$$\|\pi_\lambda(\mu)\| \leq 1 - c|\lambda| \qquad \text{for the (Stone-von-Neumann) unitary irrep. } \pi_\lambda \text{ of } U\text{, with } \lambda \text{ near } 0.$$

Recently, Diaconis and Hough (2018) have given a new proof of the key estimate, based on a combinatorial “path-swapping” idea.

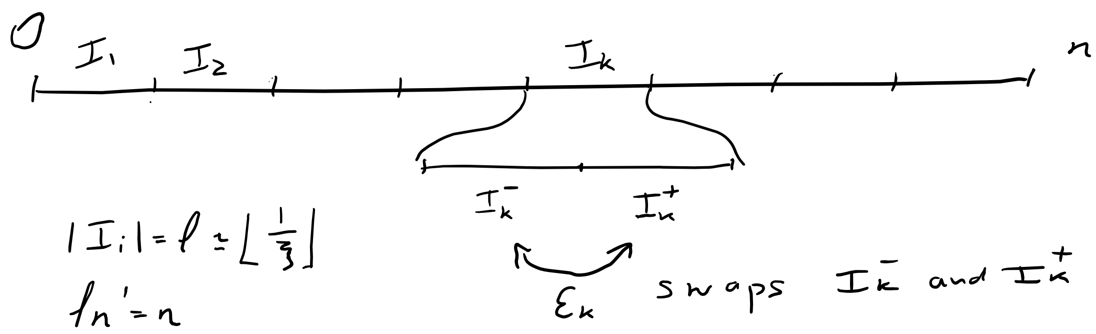
This idea turns out to be quite versatile and also works in the non-centered case.

Another crucial step in the proof, is an analogue of the gaussian replacement scheme used in classical proofs of the standard CLT for sums of real random variables (Lindeberg’s method).

The replacement scheme here is with the limiting measure:

One compares $\mu^{n-k} * \nu^k$ with $\mu^{n-k-1} * \nu^{k+1}$ and controls the difference of their Fourier transform.

6. The Heisenberg group, Diaconis-Hough "path-swap"



$Z = \sum_{i<j}(X_iY_j - X_jY_i)$

$Z = Z_k + J_k$

$\varepsilon_k(Z) = Z_k - J_k$

$J_k = X^{I_k^-}Y^{I_k^+} - X^{I_k^+}Y^{I_k^-}$
where, for any $I \subset [1,n]$
$X^I = \sum_{i\in I} X_i$

Let $e(X) = e^{2i\pi\xi X}$

Set $\pi = \prod_1^{n'}(\frac{1+\varepsilon_k}{2}) = \sum_{\sigma\in Sym(n)} \pi(\sigma)\sigma \in \mathbf{C}[Sym(n)]$

Note: the J_k's are independent and $\frac{J_k}{\ell} \sim 1$

Then we obtain the following *factorization*: $\pi e(Z) = e(Z_I)\prod_1^{n'} \frac{e(J_k)+e(-J_k)}{2}$

Hence: $|\mathbf{E}(e(Z))| = |\mathbf{E}(\pi e(Z))| \leq |\mathbf{E}\prod_1^{n'}(\frac{e(J_k)+e(-J_k)}{2})| = \prod_1^{n'}|\mathbf{E}(\cos 2\pi\xi\ell\frac{J_k}{\ell})| \leq (1-c)^{n'} \simeq e^{-n\xi} \leq e^{-n^\varepsilon}$

7. General nilpotent groups

The case of a general nilpotent Lie group is much more involved. Hough (2018) managed to handle the centered case.

The random walk $S_n = e^{x_1} \cdot \ldots \cdot e^{x_n}$ is best expressed in term of the Campbell-Hausdorff formula, which by virtue of the finite nilpotency class s, involves only terms of bounded degree.

$$\log S_n = \Pi(x_1, \ldots, x_n) = \sum_{r=1}^{s} \sum_{\substack{r_1+\cdots+r_t=r \\ 1\leq l_1<\cdots<l_t\leq n}, r_i\neq 0} L_{\mathbf{r},\mathbf{l}}(x_{l_1}^{\otimes r_1}, \ldots, x_{l_t}^{\otimes r_t})$$

where $\mathbf{r} = (r_1, \ldots, r_t)$, $\mathbf{l} = (l_1, \ldots, l_t)$ and $L_{\mathbf{r},\mathbf{l}}$ is a r-multilinear form equal to a linear combination of Lie brackets of order r involving r_i times the variable x_{l_i}.

$$\mathbf{E}(e^{2i\pi \cdot \xi \log S_n}) = \int_{\mathcal{U}^n} e^{2i\pi\xi \cdot \Pi(x_1,\ldots,x_n)} d\mu^n$$

7. General nilpotent groups

Let U be a simply connected nilpotent Lie group, μ a probability measure with finite second moment on U. Assume that its projection to $U^{ab} := U/[U,U]$ is not contained in a proper subspace subgroup.

Let $X \in \mathcal{U} := Lie(U)$ be a vector projecting to the mean of μ on U^{ab}, that is

$$X \mod [\mathcal{U},\mathcal{U}] = \mathbf{E}_\mu(\log x \mod [\mathcal{U},\mathcal{U}])$$

Define a nested sequence of ideals $(\mathcal{U}^{(i)})_{i\geq 0}$, by

$\mathcal{U}^{(0)} = \mathcal{U}^{(1)} = \mathcal{U}$ and

$\mathcal{U}^{(i+1)} = [\mathcal{U},\mathcal{U}^{(i)}] + [X_\mu,\mathcal{U}^{(i-1)}]$ for $i \geq 1$

7. General nilpotent groups: associated graded group

Define a nested sequence of ideals $(\mathcal{U}^{(i)})_{i\geq 0}$, by

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$\mathcal{U}^{(i+1)} = [\mathcal{U}, \mathcal{U}^{(i)}] + [X_\mu, \mathcal{U}^{(i-1)}]$ for $i \geq 1$

Pick supplementary subspaces $\mathfrak{m}^{(i)}$, $i \geq 1$, with

$\mathcal{U}^{(i)} = \mathfrak{m}^{(i)} \oplus \mathcal{U}^{(i+1)}$ and $X_\mu \in \mathfrak{m}^{(1)}$.

Note: one has $[\mathcal{U}^{(i)}, \mathcal{U}^{(j)}] \leq \mathcal{U}^{(i+j)}$ for $i, j \geq 1$.

One may form the associated *graded Lie algebra* $\mathfrak{gr}_{X_\mu}(\mathcal{U})$

$$\bigoplus_{i\geq 1} \mathcal{U}^{(i)}/\mathcal{U}^{(i+1)} \simeq \bigoplus_{i\geq 1} \mathfrak{m}^{(i)}$$

with $[x_i, y_j]' := [x_i, y_j] \mod \mathcal{U}^{(i+j+1)}$ for $x_i \in \mathfrak{m}^{(i)}, y_j \in \mathfrak{m}^{(j)}$

So $[\mathfrak{m}^{(i)}, \mathfrak{m}^{(j)}]' \leq \mathfrak{m}^{(i+j)}$.

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We define its μ-*homogeneous dimension* as the integer

$$d_\mu = \sum_{i\geq 1} i \dim \mathfrak{m}^{(i)}$$

and the one-parameter group of dilation automorphisms of $\mathfrak{gr}_{X_\mu}(\mathcal{U})$ as

$$D_{\mu,t}(\sum_i x_i) = \sum_i t^i x_i, \text{ where } x_i \in \mathfrak{m}^{(i)}$$

Example: If U is the Heisenberg group and $X_\mu \neq 0$, then $\mathfrak{gr}_{X_\mu}(\mathcal{U}) \simeq \mathbf{R}^3$ is abelian! Moreover $\mathfrak{m}^{(1)} = \langle x, y\rangle$, $\mathfrak{m}^{(2)} = 0$ and $\mathfrak{m}^{(3)} = \langle z\rangle$, $d_\mu = 5$ and $D_{\mu,t}(x, y, z) = (tx, ty, t^3 z)$.
On the other hand if $X_\mu = 0$, then $d_\mu = 4$, $D_{\mu,t}(x, y, z) = (tx, ty, t^2 z)$.

7. General nilpotent groups: non-centered CLT

With these notations the CLT on an arbitrary nilpotent Lie group takes the following form – and is due to Raugi '78

Theorem (non-centered CLT). *If μ has finite second moment on U, then as $n \to +\infty$*

$$D_{\mu,\frac{1}{\sqrt{n}}}(\mu^n * \delta_{e^{-nX_\mu}}) \to \nu.$$

If $\nu_t := \nu \circ D_{\mu,\sqrt{t}^{-1}}$, then $\nu_t = u(t,x)dx$, where $u(t,x)$ is the fundamental solution to the following *hypoelliptic time-dependent* PDE:

$$\partial_t u = \frac{1}{2}\sum_1^d \partial^2_{Ad(tX_\mu)X_i} u + \partial_{Ad(tX_\mu)Y} u$$

where $X_1, \ldots, X_d$ is a basis of $\mathfrak{m}^{(1)}$ in which $\mu^{(1)}$ has identity covariance matrix, and $Y = \mathbf{E}(\mu^{(2)})$, where $\mu^{(i)}$ is the linear projection of μ to $\mathfrak{m}^{(i)}$.

Beware: ν may not have full support! certain regions of U are inaccessible by the limiting diffusion process. This happens in step ≥ 3, e.g. for the free nilpotent group.

7. General nilpotent groups: non-centered LLT

Aperiodicity of the measure ensures that this PDE satisfies Hörmander's criterion. The solution is smooth, non-negative, and can be shown to satisfy a Harnack principle and the following pointwise gaussian estimates:

$$|\partial^\alpha u(t,x)| \leq C t^{-\frac{d_\mu + |\alpha|}{2}} e^{-\frac{1}{C}|D_{\frac{1}{\sqrt{t}}} x|^2}$$

Rk: $u(t,x)$ can be an ordinary gaussian: this happens iff for each i there is $X \in \mathcal{U}$ such that $ad(X)$ maps $C^i(\mathcal{U})$ onto $C^{i+1}(\mathcal{U})$ modulo $C^{i+2}(\mathcal{U})$. This is also iff $\mathfrak{gr}_{X_\mu}(\mathcal{U})$ is abelian. This is the case for all *filiform* Lie algebras (these are the n-step $(n+1)$-dimensional algebras) for generic X_μ.

Let U be a simply connected nilpotent Lie group, μ an aperiodic probability measure on U with moments of all orders and drift $X_\mu = \mathbf{E}(\mu^{(1)}) \in \mathfrak{m}^{(1)}$.

Theorem 3 (non-centered LLT on arbitrary nilpotent Lie groups). *For a bounded Borel set B in U with $|\partial B| = 0$, as $n \to +\infty$:*

$$n^{d_\mu} \sup_{x \in U} |\mu^n(xBe^{nX_\mu}) - \nu_n(xB)| \to 0 \textit{ where } \nu_n = \nu \circ D_{\mu, \frac{1}{\sqrt{n}}},\ d\nu = u(1,x)dx$$

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The limit law ν is centered on U, has same covariance matrix as μ and same mean on m^2. It is uniquely determined by this data.

The method of proof builds on the ideas of Diaconis-Hough and Hough, uses Fourier estimates and a more sophisticated variant of the path-swap idea to establish exponential sums cancellation. It also exploits a form of gaussian replacement adapted to this setting.

Power saving rates of convergence can be obtained if 1_B is replaced by a smooth function f. Also a Berry-Essen estimate (quantitative CLT) with square root error can be derived.

Consequences of the result include a new proof of the Choquet-Deny theorem for these measures, as well as the probabilistic Ratner equidistribution theorem described earlier.

Thank you!