Elliptic Curves and the Special Values of L-functions

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Introduction to Elliptic Curves: Lecture 1

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Sections

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Motivation

Elliptic curves are ubiquitous in mathematics. Study of elliptic curves brings together number theory, algebra, analysis and algebraic geometry.

- Elliptic curves have been used to prove Fermat's Last Theorem.
- Elliptic curves have provided breakthrough toward resolving the Congruent Number Problem.
- Elliptic curves have been very useful in cryptography, i.e., in coding messages for secure transmission.
- Elliptic curves are also used in factorization algorithm for integers.
- The Birch and Swinnerton-Dyer Conjecture, one of the seven Millennium Problems, is a prediction about relation between the algebraic and the analytic properties of an elliptic curve.

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Weierstrass Equation for an Elliptic Curve

- An elliptic curve over a field K can be defined as *a non-singular projective curve of genus 1 with a specified point $\mathcal{O} \in E(K)$.*

- Using the Riemann-Roch Theorem, one has an equivalent description of elliptic curve as a plane cubic curve:

  An elliptic curve over a field K is a 'non-singular curve' defined by an equation of the form

  $$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in K,$$

  together with 'a point at infinity' $\mathcal{O}$.

- When the *characteristic* of the underlying field K is not 2 or 3, the Weierstrass equation above can be simplified to

  $$y^2 = x^3 + ax + b, \qquad a, \; b \in K.$$

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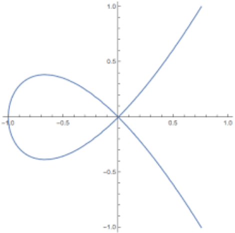

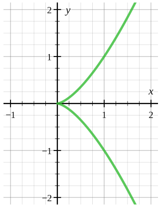

- A curve in $\mathbb{R}^2$ is called non-singular if it has a well-defined tangent at each of its points.
- For example, $y^2 = x^2(x+1)$ and $y^2 = x^3$ are singular at $(0,0)$ in the diagram above.

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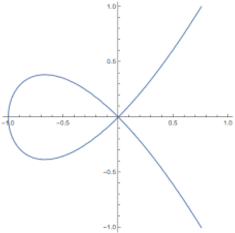

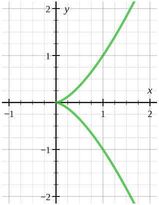

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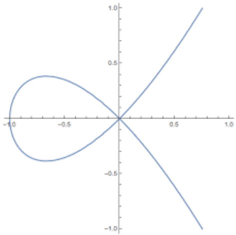

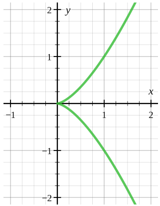

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Singular Points

- For a curve $f(x, y) = 0$ in $\mathbb{R}^2$, the slope of the tangent at a point $P = (x_0, y_0)$ is given by

$$\frac{dy}{dx}(P) = -\frac{\partial f/\partial x(P)}{\partial f/\partial y(P)}.$$

- The tangent is not well-defined at (x_0, y_0) on $f(x, y) = 0$ if

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0).$$

- Thus, (x_0, y_0) is a singular point of the cubic curve given by $f(x, y) = y^2 - (x^3 + ax + b)$ if and only if $y_0 = 0$, $3x_0^2 + a = 0$, and $x_0^3 + ax_0 + b = 0$, i.e., x_0 is a double root of $x^3 + ax + b$.

- In general, a curve over any field K (not necessarily $\mathbb{R}$) defined by a polynomial $f(x, y) \in K[x, y]$ is called non-singular at $P = (x_0, y_0)$ if the Jacobian matrix $\left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right)$ has rank 1.

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The Discriminant

- The discriminant of a cubic polynomial $g(x) \in K[x]$ is given by

$$\Delta(g) = \prod_{i>j} (\alpha_i - \alpha_j)^2 = (\alpha_1 - \alpha_2)^2 (\alpha_2 - \alpha_3)^2 (\alpha_1 - \alpha_3)^2,$$

  where α_i's are the roots of $g(x)$.

- So $g(x)$ has repeated roots if and only if $\Delta(g) = 0$.

- It can be easily computed that the discriminant of $g(x) = x^3 + ax + b$ is $\Delta(g) = -(4a^3 + 27b^2) \in K$.

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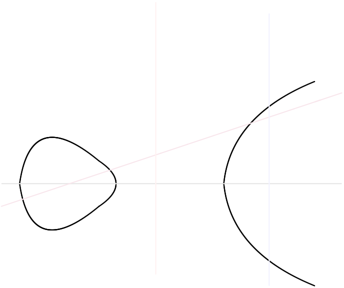
Intersection with a Straight Line: a diagram

The points with real coordinates on $E : y^2 = x^3 - 25x$ and intersection with a straight line $y = mx + c$:

$(mx + c)^2 = x^3 - 25x$



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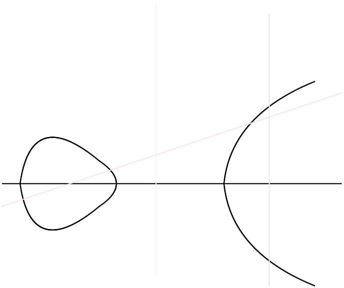

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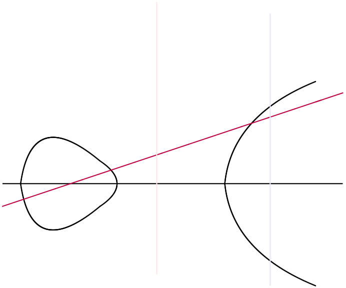

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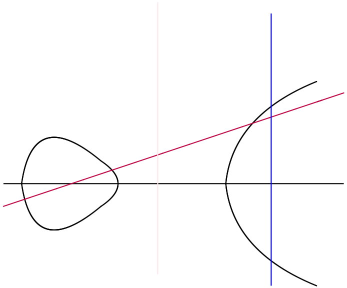

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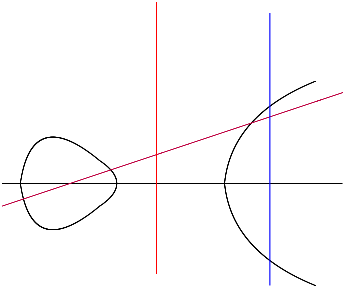

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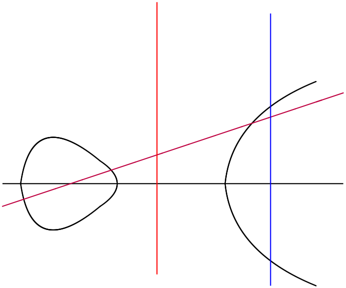

The Point at Infinity on an Elliptic Curve

- Any non-vertical straight line intersects the curve at three points.
- A non-vertical line $y = mx + c$ ($m, c \in \mathbb{R}$) will have three real points of intersection or one real and two complex points of intersection, as seen by substitution $y = mx + c$ in $y^2 = x^3 + ax + b$.
- However, the vertical lines $x = x_0$ will have either two real or two complex points of intersection. For a consistent theory, we need a third point of intersection. We adjoin an additional 'point at infinity' to the curve.
- This point can be visualized as lying on the top (and the bottom) of the xy-plane at infinity.
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The Projective Space over a Field K

- Let $\overline{K}$ denote the algebraic closure of a field K. Define a relation $\sim$ on the set of non-zero $(n+1)$-tuples

$$(a_0, a_2, \ldots, a_n) \in \overline{K}^{n+1} \setminus \{(0, 0, \ldots, 0)\} \qquad \text{by}$$

$$(a_0, a_1, a_2, \ldots, a_n) \sim (a_0', a_1', a_2', \ldots, a_n') \text{ if and only if}$$

$$a_0' = ta_0, \;\; a_1' = ta_1, \;\; \ldots, \;\; a_n' = ta_n \;\text{ for some } t \in \overline{K}^{\times}.$$

Clearly, $\sim$ is an equivalence relation.

- The projective n-space over K is defined as the set of equivalence classes

$$\mathbb{P}^n = \frac{\{(a_0, a_1, a_2, \ldots, a_n) \in \overline{K}^{n+1} \setminus (0, 0, \ldots, 0)\}}{\sim}.$$

- We denote the *affine n-space* over K by

$$\mathbb{A}^n = \{(x_1, x_2, \ldots, x_n) \; : \; x_i \in \overline{K}\}.$$

The Projective Space over a Field K

- Let $\overline{K}$ denote the algebraic closure of a field K. Define a relation $\sim$ on the set of non-zero $(n+1)$-tuples

$$(a_0, a_2, \ldots, a_n) \in \overline{K}^{n+1} \setminus \{(0, 0, \ldots, 0)\} \qquad \text{by}$$

$$(a_0, a_1, a_2, \ldots, a_n) \sim (a_0', a_1', a_2', \ldots, a_n') \text{ if and only if}$$

$$a_0' = ta_0, \;\; a_1' = ta_1, \; \ldots, \;\; a_n' = ta_n \;\text{ for some } t \in \overline{K}^{\times}.$$

Clearly, $\sim$ is an equivalence relation.

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$$\mathbb{P}^n = \frac{\{(a_0, a_1, a_2, \ldots, a_n) \in \overline{K}^{n+1} \setminus (0, 0, \ldots, 0)\}}{\sim}.$$

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- Let $[a : b : c]$ denote a point in $\mathbb{P}^2$, i.e.,

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- However, $\mathbb{P}^2$ has additional points with $c = 0$ which cannot be identified with the points in the affine plane in this way. The points $[a : b : 0]$ in $\mathbb{P}^2$ can be thought of as representation of *directions of straight lines parallel to the line $ay = bx$ in the affine plane A^2*. These additional points are known as the points at infinity in $\mathbb{P}^2$.

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Lines in the Projective Plane

- A line in $\mathbb{P}^2$ consists of a line in $\mathbb{A}^2$ together with the point at infinity specified by its direction.

- Any two parallel lines in $\mathbb{P}^2$ intersect at the point at infinity corresponding to their common direction.

- The set of all points at infinity is itself considered to be a line L_∞, and the intersection of any other line L with L_∞ is the point at infinity corresponding to the direction of L.

- A vertical line in $\mathbb{A}^2$ contains the additional point $[0:1:0]$ in $\mathbb{P}^2$ (as its direction), therefore we can say that any two vertical lines intersect at the point $[0:1:0]$ at infinity.

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- Given an ideal I the polynomial ring $\overline{K}[Y_1, \ldots, Y_n]$, we define an affine algebraic set as

$$V_I = \{P \in \mathbb{A}^n \mid f(P) = 0 \qquad \forall f \in I\}.$$

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The Projective Closure of an Affine Variety

- Given any $f \in \overline{K}[Y_1, \ldots, Y_n]$ of degree d, we can define a homogenous polynomial
$$f^*(X_0, \ldots, X_n) = X_n^d f\left(\frac{X_0}{X_n}, \ldots, \frac{X_0}{X_n}\right) \in \overline{K}[X_0, \ldots, X_n].$$

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$$\phi_n : \mathbb{A}^n \longrightarrow \mathbb{P}^n, \qquad (y_1, \ldots, y_n) \mapsto [y_1 : \ldots : y_n : 1].$$

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Elliptic Curve as a Projective Variety

- In the projective plane, the equation of an elliptic curve is the homogenized equation $E_H : Y^2Z = X^3 + aXZ^2 + bZ^3$ using the substitution $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ in the affine equation $E : y^2 = x^3 + ax + b$.
- The affine points on E correspond to the points $[X : Y : 1]$ lying on E_H. But the elliptic curve has additional points $[X : Y : Z]$ in the projective plane given by $Z = 0$.
- Substituting $Z = 0$ in the homogenized equation, we obtain $X^3 = 0$, i.e., $X = 0$, and hence $Y \neq 0$. Therefore, an elliptic curve contains only one additional point in the projective plane, which can be taken as $[0 : 1 : 0]$.
- The point $[0 : 1 : 0]$ is known as the point at infinity on the elliptic curve and denoted by $\mathcal{O}$. It is a point of inflection for the curve, as it is a point of multiplicity 3.

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Group Structure

- The points on an elliptic curve have the structure of a group.
- Given two points P and Q on E, we can add them in a geometric way to get a third point R on E.
- For this addition of points, we have
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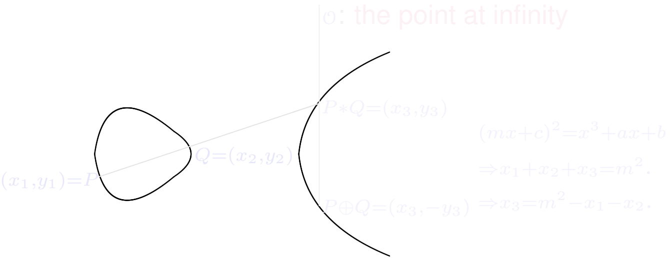

- Any two points P and Q on E can be added to obtain a third point $P \oplus Q$ on E. The x-coordinate of the sum is given by

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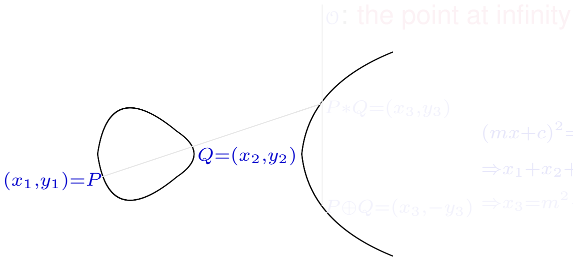

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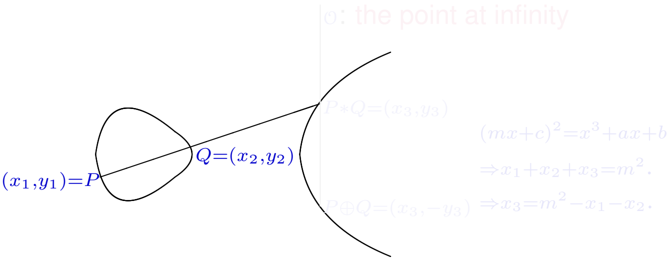

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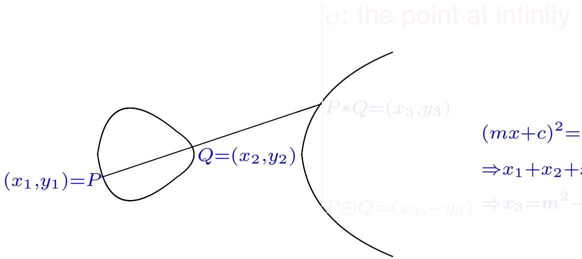

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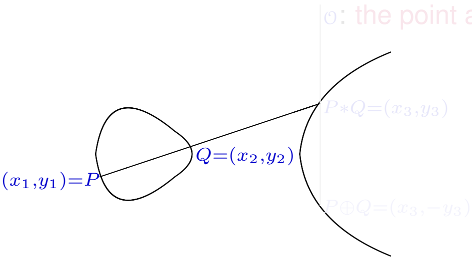

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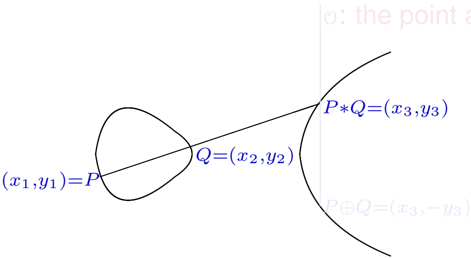

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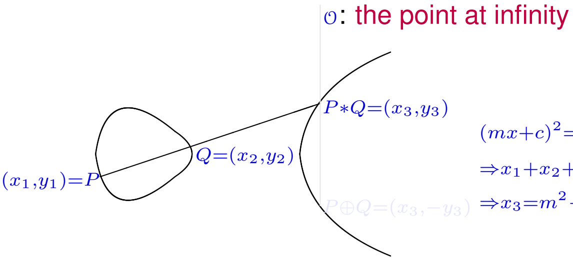

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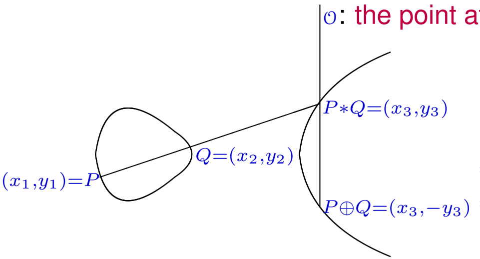

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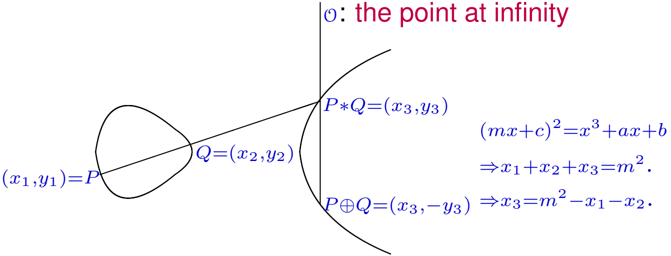

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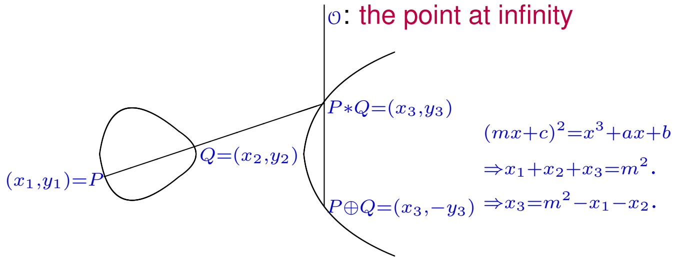

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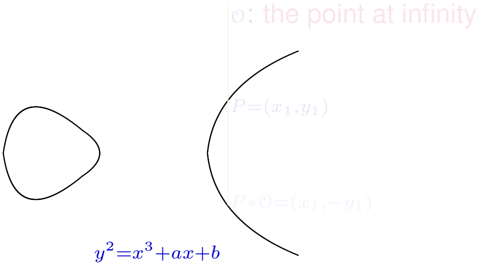

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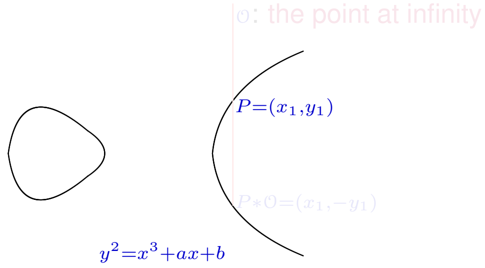

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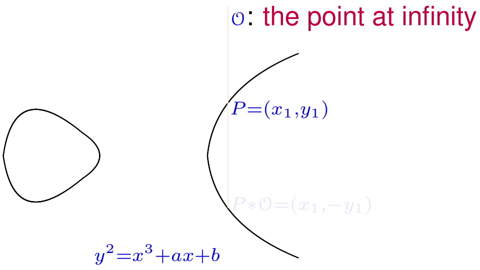

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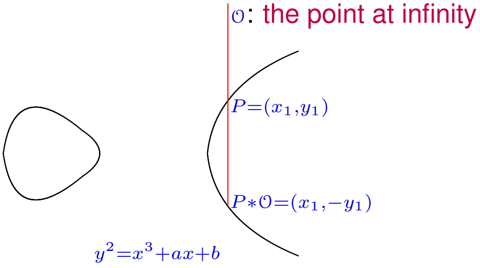

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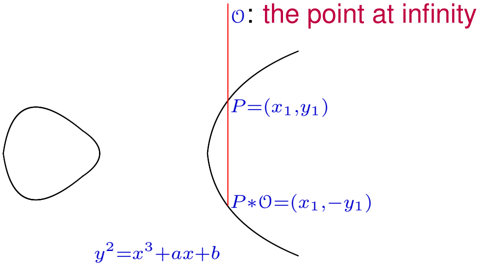

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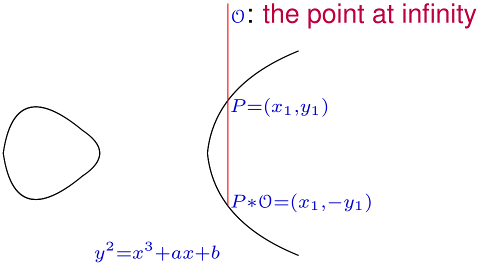

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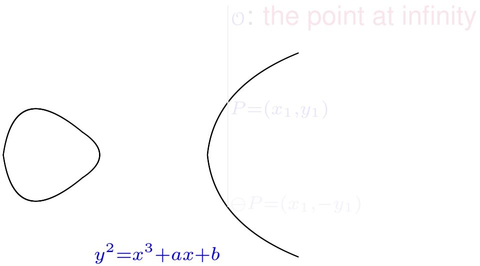
The Inverse of a Point



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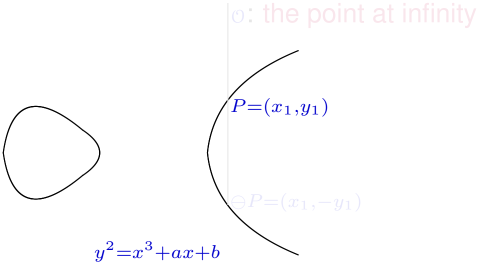

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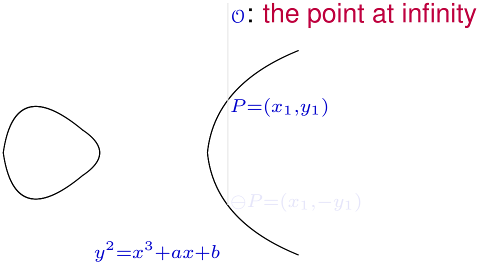

- The tangent at $\mathcal{O}$ hits the curve again at $\mathcal{O}$. This can be seen by substituting $u = \frac{X}{Y}, \quad v = \frac{Z}{Y}$ in $Y^2Z = X^3 + aXZ^2 + bZ^3$, which yields $v = u^3 + auv^2 + bv^3$. The tangent $v = 0$ at $\mathcal{O}(u = 0, v = 0)$ gives $u^3 = 0$.

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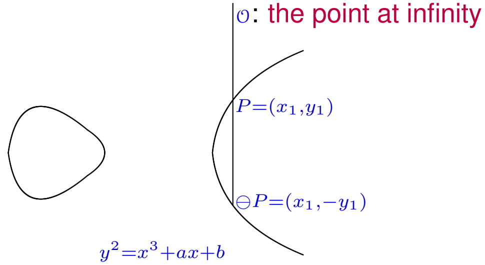

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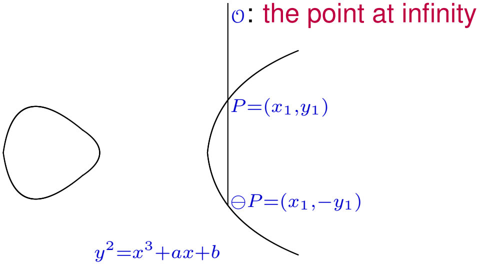

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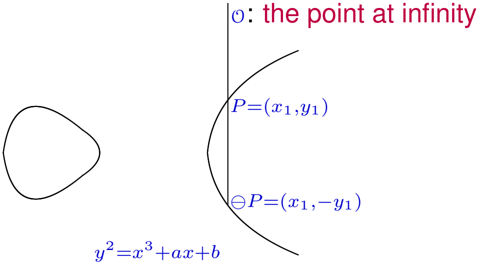

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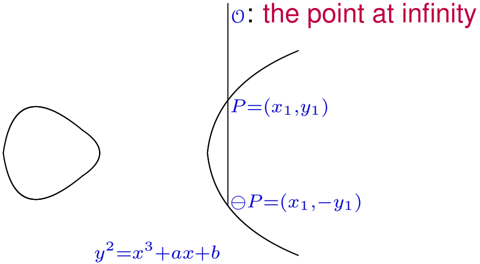

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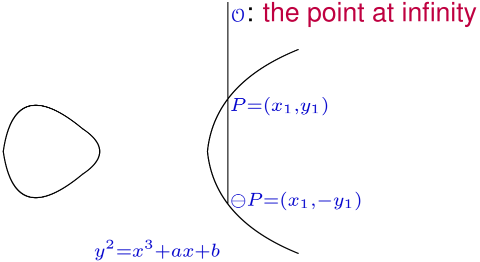
The Inverse of a Point

Introduction

The Point at Infinity

Group Structure



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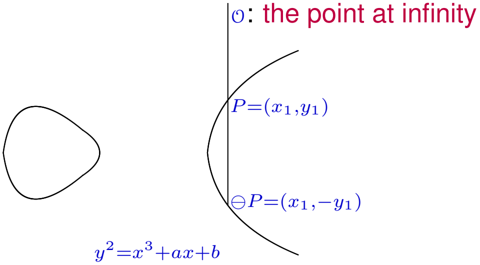

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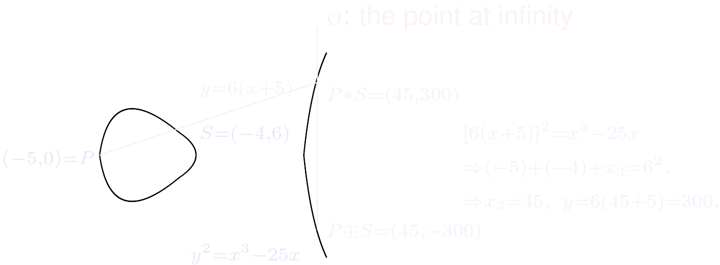
Addition on E: an Example



- The coefficient of x^2 will be the negative of the sum of the three roots. Thus $36 = (-5) + (-4) + x$, or $x(P * S) = 45$, and $y(P * S) = 6(45 + 5) = 300$.
- Note that the vertical line through $P * S = (45, 300)$ (and $\mathcal{O}$) hits the curve at the point $(45, -300)$.
- By our definition, $P \oplus S = (45, -300)$.

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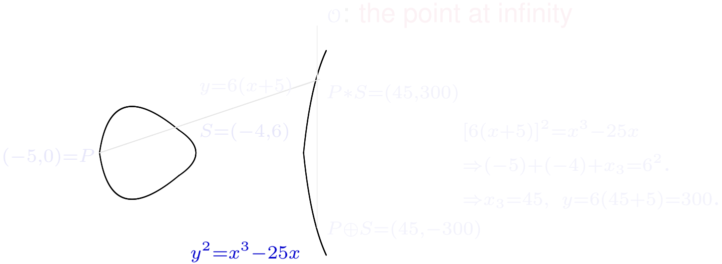

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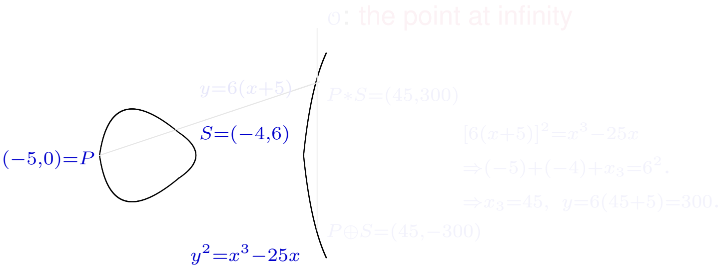

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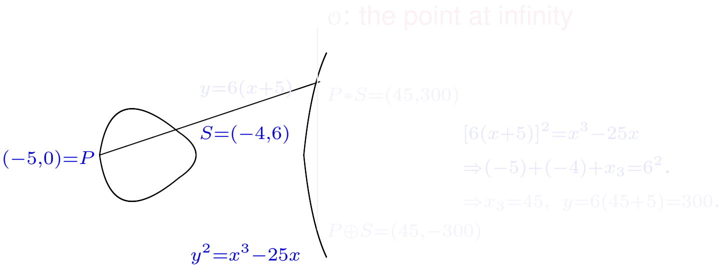

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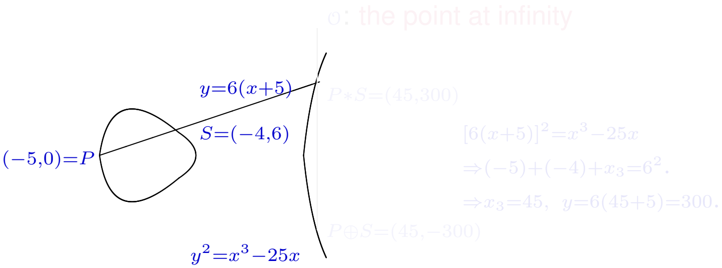

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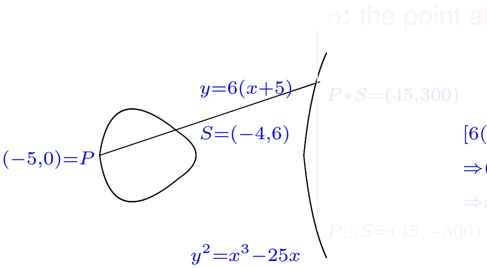

$[6(x+5)]^2 = x^3 - 25x$
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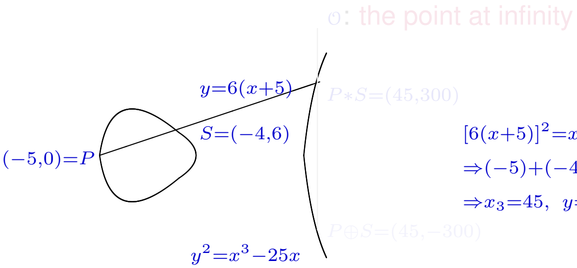

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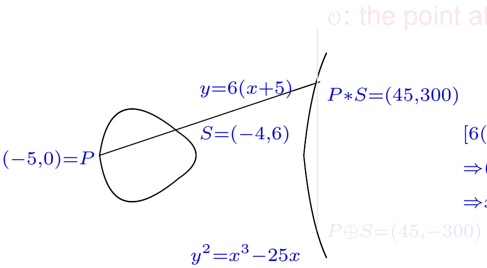

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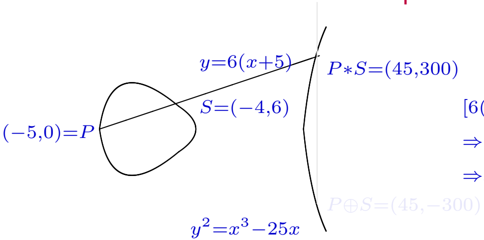

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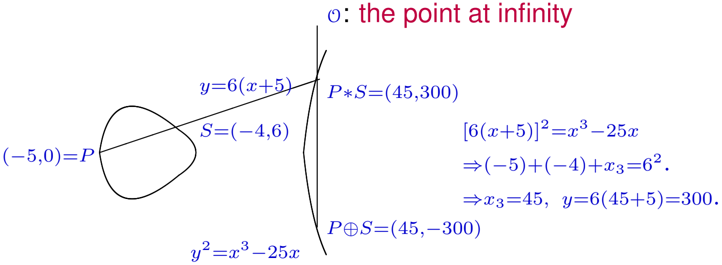

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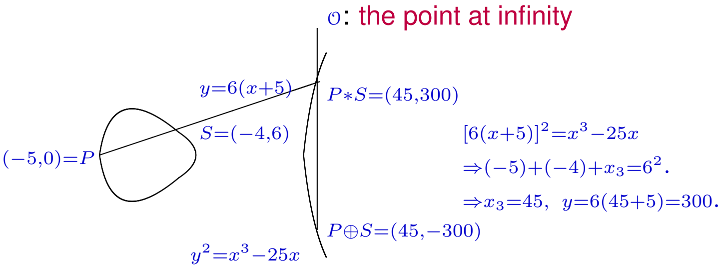

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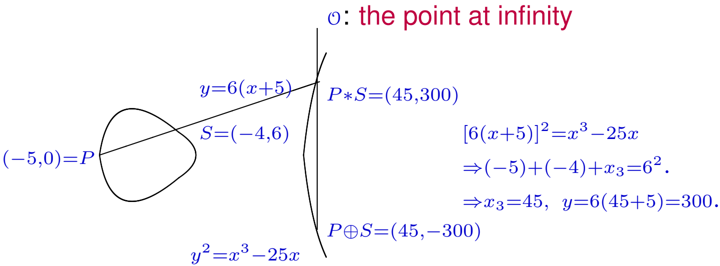

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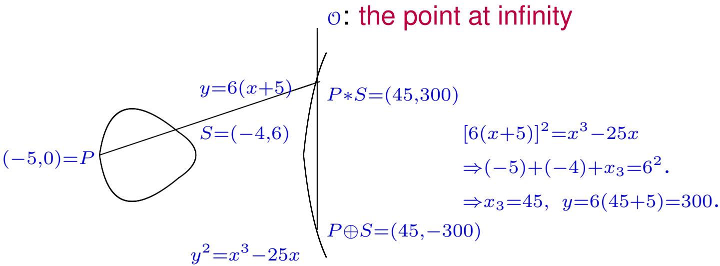

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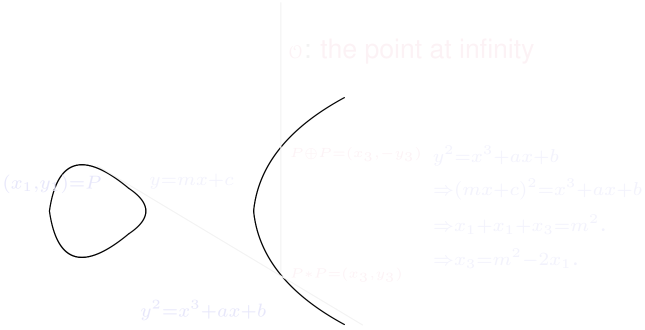

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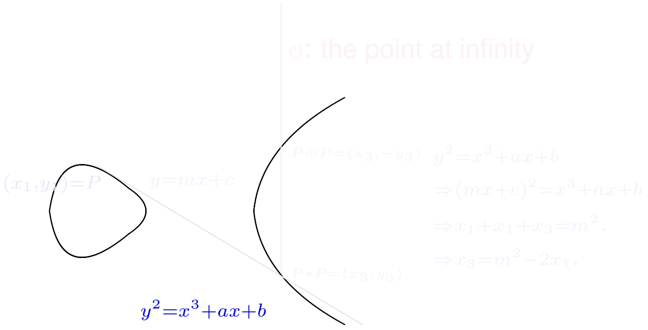

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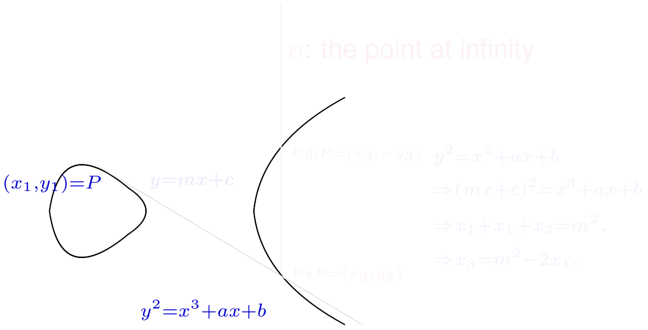

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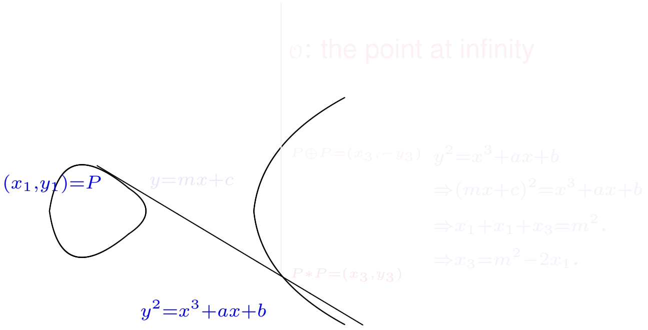

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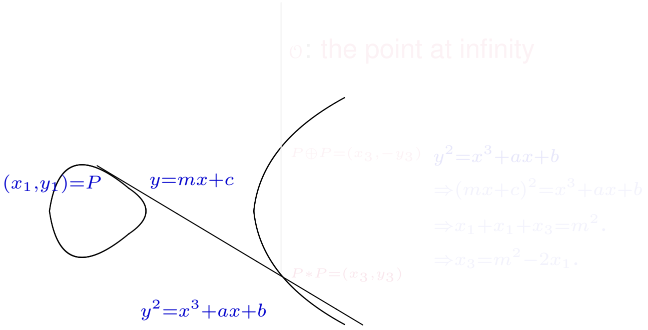

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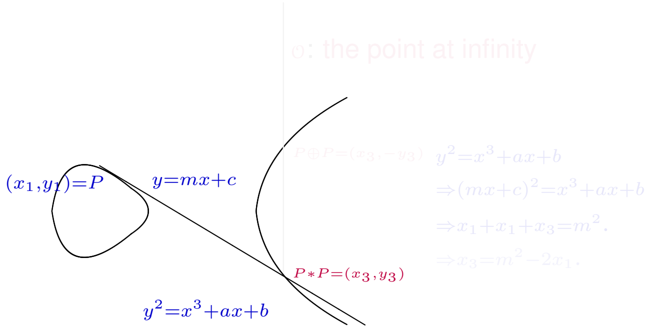

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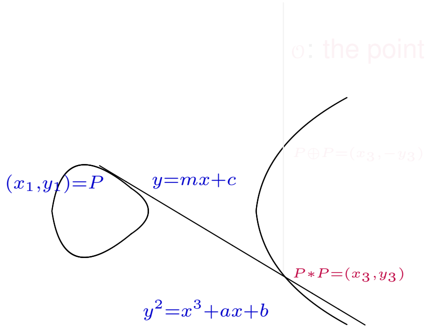

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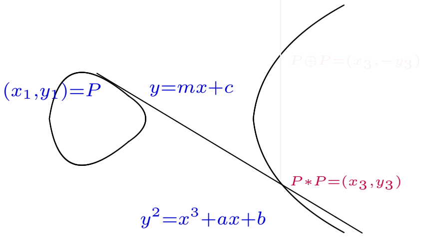

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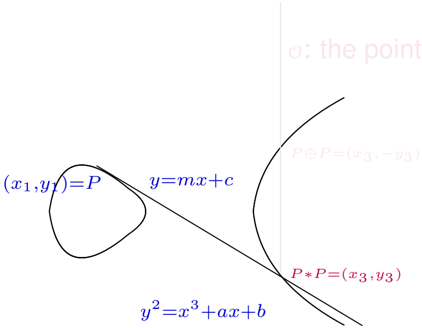

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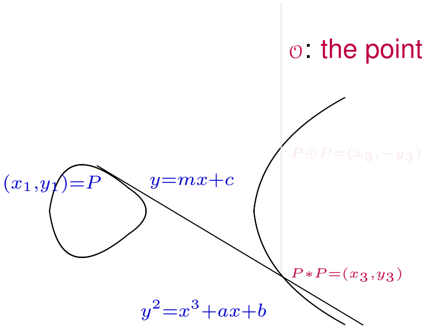

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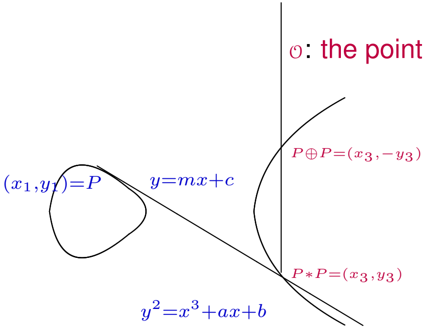

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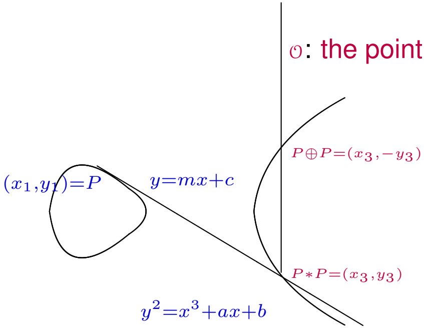

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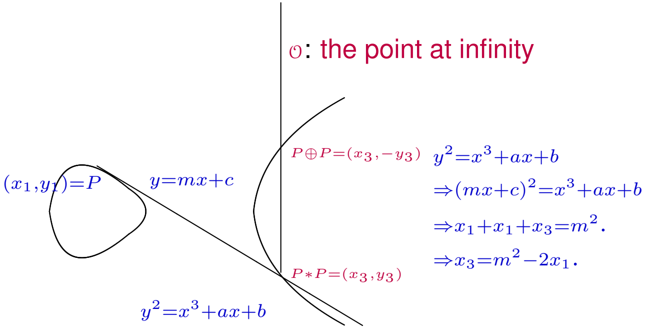

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Doubling a Point: an example

- Consider the elliptic curve $E: \quad y^2 = x^3 + x + 1$ over the finite field $\mathbb{F}_{23} = \{0,\ 1,\ \ldots, 22\}$. Note that $4.1^3 + 27.1^3 \neq 0$ in $\mathbb{F}_{23}$.
- $P = (3, 10)$ is a point on E as $3^3 + 3 + 1 = 31 = 10^2$ in $\mathbb{F}_{23}$.
- To compute $P * P = (x_3, y_3)$, the slope of the tangent at P is

$$m = \frac{3x^2+1}{2y} = \frac{3.3^2+1}{2.10} = \frac{7}{5} = \frac{7\times 9}{5\times 9} = \frac{63}{-1} = 6 \in \mathbb{F}_{23}.$$

- The tangent at $P = (3, 10)$ is
$$y - 10 = 6(x-3), \text{ i.e., } y = 6x - 8.$$

$$(6x_3 - 8)^2 = x_3^3 + x_3 + 1$$
$$\implies x_3 = 36 - (3+3) = 30 = 7 \in \mathbb{F}_{23},$$
$$y_3 = 6 \times 7 - 8 = 11 \in \mathbb{F}_{23},$$
$$\implies 2P = (7, -11) = (7, 12).$$

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Doubling a Point: an example

- Consider the elliptic curve $E: \quad y^2 = x^3 + x + 1$ over the finite field $\mathbb{F}_{23} = \{0,\ 1,\ \ldots, 22\}$. Note that $4.1^3 + 27.1^3 \neq 0$ in $\mathbb{F}_{23}$.
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$$m = \frac{3x^2 + 1}{2y} = \frac{3.3^2 + 1}{2.10} = \frac{7}{5} = \frac{7 \times 9}{5 \times 9} = \frac{63}{-1} = 6 \in \mathbb{F}_{23}.$$

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Multiplication by Integers

- We can add P to itself and get another point $2P = P \oplus P$, and in general,

$$[n]P = nP = \underbrace{P \oplus P \oplus \ldots \oplus P}_{n\text{-times}}, \qquad n = 1, 2, 3, \ldots.$$

- For a negative integer n, we can define nP as

$$nP = \ominus[(-n)P],$$

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K-Rational Points on Elliptic Curve

- Let K be a field, and E be an elliptic curve defined over K.
- The points (x, y) on E where $x, y \in K$ are are called K-rational points on E and denoted by $E(K)$.
- If we start with two points on E with K-rational co-ordinates, their sum also has K-rational co-ordinates:

The straight line $y = mx + c$ joining $(x_1, y_1), (x_2, y_2) \in E(K)$ has slope $c, m = \frac{y_2 - y_1}{x_2 - x_1} \in K$. The third point of intersection (x_3, y_3) of this line with $y^2 = x^3 + ax + b$ is obtained as

$$\begin{aligned} (mx + c)^2 &= x^3 + ax + b \\ \implies x_1 + x_2 + x_3 &= m^2 \\ \implies x_3 &= m^2 - x_1 - x_2 \in K, \\ y_3 &= mx_3 + c \in K. \end{aligned}$$

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The Mordell-Weil Group

- Let $E(K) = \{(x, y) \mid x,\ y \in K,\ \ y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$. Then

$$P, Q \in E(K) \implies P \oplus Q \in E(K).$$

- The abelian group $E(K)$ is called the Mordell-Weil group of E/K.
- Mordell-Weil Theorem: For any finite extension K of $\mathbb{Q}$, $E(K)$ is a finitely generated abelian group, i.e.,

$$E(K) \cong \mathbb{Z}^{r_E(K)} \oplus E(K)_{tors}.$$

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The Torsion Subgroup

- **Lutz-Nagell Theorem**: Let $E : y^2 = x^3 + ax + b$ be an elliptic curve with $a,\ b \in \mathbb{Z}$. If $(x, y) \in E(\mathbb{Q})$ is a non-zero torsion point, then $x, y \in \mathbb{Z}$, and $y = 0$ or $y^2 \mid (4a^3 + 27b^2)$.

- **Mazur's Theorem**: Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E_{tors}(\mathbb{Q})$ is isomorphic to one of the following 15 groups:

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, &\qquad \text{with } 1 \le N \le 10 \text{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}, &\qquad \text{with } 1 \le N \le 4. \end{aligned}$$

  Further, each of these subgroup occurs as $E_{tors}(\mathbb{Q})$ for infinitely many elliptic curves $E/\mathbb{Q}$. (*Conjectured by Levi 1906*)

- **Merel**: For every integer $d \geq 1$, there is a constant $N(d)$ such that for all number fields $[K : \mathbb{Q}] \leq d$ and all elliptic curves E/K,

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Points of Infinite Order

- Bhargava and Shankar showed that at least 83% of elliptic curves have rank either 0 or 1.
- We have example of $E(\mathbb{Q})$ with the highest possible rank 28.
- Conjecture: There exist elliptic curves $E/\mathbb{Q}$ of arbitrarily large rank.
- Mordell's Conjecture (for plane curves): Let C be a non-singular curve over $\mathbb{Q}$ of degree strictly greater than 3. Then the set $C(\mathbb{Q})$ of rational point on C is finite. Faltings proved the conjecture in the 1980's.
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- Mordell's Conjecture (for plane curves): Let C be a non-singular curve over $\mathbb{Q}$ of degree strictly greater than 3. Then the set $C(\mathbb{Q})$ of rational point on C is finite. Faltings proved the conjecture in the 1980's.
- There is no effective algorithm to compute the rank of $E(\mathbb{Q})$.
- The first part of the BSD Conjecture connects the rank of $E(\mathbb{Q})$ to an analytic object, called the Hasse-Weil L-function of the elliptic curve.

Points of Infinite Order

- Bhargava and Shankar showed that at least 83% of elliptic curves have rank either 0 or 1.
- We have example of $E(\mathbb{Q})$ with the highest possible rank 28.
- Conjecture: There exist elliptic curves $E/\mathbb{Q}$ of arbitrarily large rank.
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THANK YOU