Statistical Properties of the Navier-Stokes-Voigt Model

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Scales and Turbulence

Figure: Eye of the hurricane Elena

Incompressible Navier-Stokes Equations

The velocity vector field $\mathbf{u} = (u_1, u_2, u_3)$ of the fluid satisfies the incompressible Navier-Stokes equations in the domain Ω:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0,$$

The scalar p is the kinematic pressure, and the parameter $\nu > 0$ is the kinematic viscosity.

Boundary Conditions

- $\mathbf{u} = 0$ on $\partial\Omega$.
- Periodic.
- Flows in the whole space. ($\mathbb{R}^2$ or $\mathbb{R}^3$)

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Averages

One possible meaning of $\langle \Phi(\mathbf{u}) \rangle$ is the mathematical expectation of the functional Φ with respect to a measure in function space. For a set of smooth solutions of the NSE, one could consider invariant measures for such set, which are supported on the global attractor.

One could also consider measures supported on Navier-Stokes solution paths, which should be stable with respect to small random perturbations.

One could also consider stationary statistical solution measures, a notion defined by Foias, which generalizes the concept of invariant measure in the case that smooth solutions may not be availabe. This is a measure satisfying a Liouville-type equation.

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Functional formulation (no-slip boundary condition)

Let

$$\mathcal{V} = \{\varphi \in C_0^\infty(\Omega);\ \nabla \cdot \varphi = 0\}.$$

The two fundamental functional spaces in this work are

$$H = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3,$$

and

$$V = \text{closure of } \mathcal{V} \text{ in } H^1(\Omega)^3.$$

We denote by P_{LH} the (Leray-Helmhotz) orthogonal projector in $L^2(\Omega)^3$ onto the subspace H.

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The Stokes operator is defined by

$$A\mathbf{u} = -P_{\mathrm{LH}}\Delta\mathbf{u}.$$

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$$B(\mathbf{u},\mathbf{v}) = P_{\mathrm{LH}}((\mathbf{u}\cdot\nabla)\mathbf{v})$$

is a bilinear term associated with the inertial term.
The Navier-Stokes equations functional formulation in H is

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) = \mathbf{f} - \nu A\mathbf{u} - B(\mathbf{u},\mathbf{u}). \tag{1}$$

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Stationary Reynolds Equations

- Assuming statistical equilibrium, O. Reynolds considered the decomposition of turbulent flow into mean and fluctuation and derived the equations that describe their interaction.
- $$\nu A\langle \mathbf{u} \rangle + B(\langle \mathbf{u} \rangle, \langle \mathbf{u} \rangle) = \mathbf{f}_P - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle.$$
  where $\mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$.
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Projected energy balance

- $$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{u}_{\kappa',\kappa''}\right\|_2^2 + \nu\left\|\nabla\mathbf{u}_{\kappa',\kappa''}\right\|_2^2 = [e_{\kappa'} - e_{\kappa''}] + (\mathbf{f}, \mathbf{u}_{\kappa',\kappa''}),$$

- where

$$e_\kappa(\mathbf{u}) = e_\kappa^{\rightarrow}(\mathbf{u}) - e_\kappa^{\leftarrow}(\mathbf{u})$$

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$$e_\kappa^{\rightarrow}(\mathbf{u}) = -(B(\mathbf{u}_{\kappa_1,\kappa}, \mathbf{u}_{\kappa_1,\kappa}), \mathbf{u}_{\kappa,\infty})$$

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Kolmogorov picture of turbulence

Consider solutions of the 3D Navier-Stokes equations describing a homogeneous turbulent flow in a domain Ω with characteristic length L. Let $k_0 = L^{-1}$.
There exist two scales – Kolmogorov's scale k_ν and viscous scale k'_ν, such that in the statistical equilibrium the dynamics of $u_k(t)$ exhibits three different regimes

- The **inertial range**: $k_0 < |k| < k_\nu$. The dynamics is governed by the inviscid Euler ($\nu = 0$) equations.
- The **dissipation range**: $k_\nu < |k| < k'_\nu$. The energy from the inertial modes is absorbed and dissipated.
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K41 - The locality hypothesis

The central hypothesis of the Kolmogorov's theory of homogeneous turbulence states that in the inertial range, there is no interchange of energy between the *shell* $k' < |k| < k''$ and the *shell* $k_1 < |k| < k_2$ if the shells $[k', k'']$ and $[k_1, k_2]$ are separated by at least an order of magnitude.
One usually considers $2k'' < k_1$.

K41 - Universality

- **Anomalous dissipation conjecture**:

$$\epsilon := \lim_{\nu \to 0} \lim_{T \to \infty} \frac{\nu}{T} \int_0^T \left\| \nabla \mathbf{u}^{(\nu)}(t) \right\|_2^2 dt > 0,$$

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K41 - Scaling

Assuming statistical equilibrium, let us denote by $U_\kappa := \langle |\mathbf{u}_k|^2 \rangle^{1/2}$ and t_κ the characteristic velocity and time scales and length scale κ^{-1}, respectively.

- By the universality hypothesis the possible expressions for the characteristic time-scale at length scale κ^{-1} are:

$$t_\kappa \sim \frac{1}{kU_\kappa}, \quad \text{and} \quad t_\kappa = \frac{U_\kappa^2}{\epsilon}$$

Equating both expressions yields

$$U_\kappa \sim \epsilon^{1/3} \kappa^{-1/3}$$

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K41 - Scaling (Smallest Scales of motion)

- **Kolmogorov dissipative scale**: Similarly, equating

$$t_\kappa \sim \frac{1}{\epsilon^{1/3}\kappa^{2/3}} \quad \text{and viscous} \quad t_\kappa^\nu \sim \frac{1}{\nu\kappa^2}$$

yields

$$\kappa_\nu = \left(\frac{\epsilon}{\nu^3}\right)^{1/4}$$

The Kolmogorov length scale is

$$\eta := \kappa_\nu^{-1} = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

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Power law and Intermittency

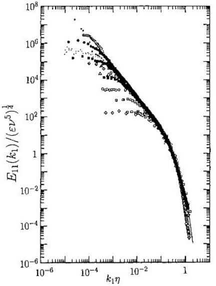

Fig. 4. The turbulence spectrum measured in many different flows. As the Reynolds number increases, the width of the spectrum increases, as the large scales are separated further from the small scales. At high Reynolds number, Kolmogorov (10) predicts a slope of −5/3, which is seen in these measurements. [Reproduced with permission from ref. 11 (Copyright 1994, Cambridge University Press).]

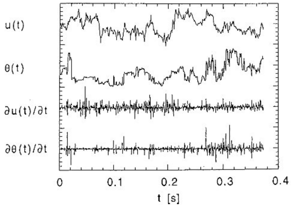

Fig. 5. Time series of the longitudinal velocity fluctuations, $u(t)$, and their derivative, $\partial u(t)/\partial t$. Also shown is the time series of scalar fluctuations, $\theta(t)$, and their derivative, $\partial\theta(t)/\partial t$, in the same flow. Notice the higher intermittency in the scalar (bottom trace). $R_\lambda = 582$. Measurements by L. Mydlarski and Z.W.

Some observed properties in a large class of viscoelastic flows

- Increasing the elasticity of polymers, the energy flux in the turbulent cascade is partially supressed and transferred to the elastic degrees of freedom. This supression remains partial even for large values of elasticity.
- ..the elastic dissipation removes only a finite fraction of the flux. Moreover the effect of polymers on turbulence is local in scales, i.e. large enough scales are essentially not affected by the presence of polymers.

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Navier-Stokes-Voigt equations

$$\begin{cases} \partial_t(\mathbf{u}^{(\alpha)} - \alpha^2 \Delta \mathbf{u}^{(\alpha)}) - \nu \Delta \mathbf{u}^{(\alpha)} + \mathbf{u}^{(\alpha)} \cdot \nabla \mathbf{u}^{(\alpha)} + \nabla p^{(\alpha)} = \mathbf{f}, \quad \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{u}^{(\alpha)} = 0, \quad \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}, t) = 0 \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{2}$$

in the domain Ω, where α is a length scale parameter such that α^2/ν is the relaxation time of the fluid. The scalar p is the kinematic pressure, and $\nu > 0$ is the kinematic viscosity.

Some facts about the NSV equations

- Global well-posedness
  (A. Oskolkov, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **38** (1973), 98–136.)
- Finite dimension Global atrractor, determining modes.
  (V. Kalantarov, E.S. Titi, *Chinese Annals of Math.*, Ser. B, **30(6)** (2009), 697–714.)
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Reynolds equations

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NSV - Inviscid conserved quantity

The inviscid conserved quantity (the α-energy) is

$$S_2^\alpha = \frac{1}{2}\|\mathbf{u}\|_2^2 + \frac{\alpha^2}{2}\|\nabla \mathbf{u}\|_2^2 .$$

We denote by ϵ_α, the total energy dissipation rate for the NSV,

$$\epsilon_\alpha = \nu\langle \|\nabla \mathbf{u}\|_2^2\rangle.$$

Forward Cascade

Theorem

For any invariant measure of the NSV model, μ^α, and for every $\kappa' \leq \kappa''$ such that $\mathbf{f}_{\kappa',\kappa''} = 0$, we have

$$\nu \langle \left\| \nabla \mathbf{u}_{\kappa',\kappa''} \right\|^2 \rangle_\alpha = (\langle e_{\kappa'}(\mathbf{u}) \rangle_\alpha - \langle e_{\kappa''}(\mathbf{u}) \rangle_\alpha). \tag{3}$$

Moreover, for all $\kappa > \bar{\kappa}$, we have

$$\nu \langle \| \nabla \mathbf{u}_{\kappa,\infty} \|_2^2 \rangle_\alpha = \langle e_\kappa(\mathbf{u}) \rangle_\alpha. \tag{4}$$

NSV - Scaling

We define the following characteristic velocities at scale k:

$$\mathbf{U}_k^{(0)} = \langle |\mathbf{u}_k|^2 \rangle^{1/2},$$

and

$$\mathbf{U}_k^{(\alpha)} = (1 + \alpha^2 k^2) \langle |\mathbf{u}_k|^2 \rangle^{1/2}.$$

We denote the characteristic kinetic energy at scale k by

$$S_2(k) = \frac{1}{2} (\mathbf{U}_k^{(0)})^2 = \langle |\mathbf{u}_k|^2 \rangle, \tag{5}$$

We denote the characteristic α-energy at scale k by

$$S_2^\alpha(k) = \frac{1}{2} \mathbf{U}_k^{(0)} \mathbf{U}_k^{(\alpha)}, \tag{6}$$

With this notation, we can write the α-energy as

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$$\mathbf{U}_k^{(0)} = \langle |\mathbf{u}_k|^2 \rangle^{1/2},$$

and

$$\mathbf{U}_k^{(\alpha)} = (1 + \alpha^2 k^2) \langle |\mathbf{u}_k|^2 \rangle^{1/2}.$$

We denote the characteristic kinetic energy at scale k by

$$S_2(k) = \frac{1}{2} (\mathbf{U}_k^{(0)})^2 = \langle |\mathbf{u}_k|^2 \rangle, \tag{5}$$

We denote the characteristic α-energy at scale k by

$$S_2^{\alpha}(k) = \frac{1}{2} \mathbf{U}_k^{(0)} \mathbf{U}_k^{(\alpha)}, \tag{6}$$

With this notation, we can write the α-energy as

$$S_2^{\alpha} = \sum_k S_2^{\alpha}(k) = \frac{1}{2} \sum_k \mathbf{U}_k^{(0)} \mathbf{U}_k^{(\alpha)}.$$

NSV - Scaling

- Assuming the universality hypothesis the possible expressions for the characteristic time-scale at length scale κ^{-1} are:

$$t_k^\alpha = \frac{S_2^\alpha(k)}{\epsilon_\alpha},$$

and

$$t_k^\alpha = \frac{1}{\kappa U_\kappa}.$$

$$U_\kappa = \mathbf{U}_k^{(\alpha)} \quad \text{or} \quad U_\kappa = \mathbf{U}_k^{(0)} \quad ?$$

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This yields

- $$S_2^\alpha(k) \sim \epsilon_\alpha^{2/3} k^{-2/3} (1+\alpha^2 k^2)^{1/3}.$$
- $$U_k^{(0)} \sim \epsilon_\alpha^{1/3} k^{-1/3} (1+\alpha^2 k^2)^{-1/3}.$$
- and $$t_k^\alpha = \frac{1}{k U_k^{(0)}} = \frac{(1+\alpha^2 k^2)^{1/3}}{\epsilon_\alpha^{1/3} k^{2/3}}.$$

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NSV - Smallest scales of motion

Equating $t_k^{\text{dissip}} \sim \frac{1}{\nu\kappa^2}$ and t_k^{α}, we have that the smallest scale of motion is

- For $\alpha \lesssim \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$

$$\eta^{NSV} \sim \eta.$$

- For $\alpha \gtrsim \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$

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The incompressible Navier-Stokes equations

In the Fourier space variables the Navier-Stokes takes the form

$$\frac{du_k(t)}{dt} = i \sum_{m+l=k} \left(u_m(t) \cdot l u_l(t)\right) - \nu |k|^2 u_k(t) - ikp_k(t) + f_k, \tag{7a}$$

$$k \cdot u_k = 0, \tag{7b}$$

The Sabra shell model of turbulence

The Sabra shell model of turbulence describes the dynamics of a complex “Fourier” component of a scalar velocity field u_n.

$$\frac{du_n}{dt} = i(k_{n+1}u_{n+2}u^*_{n+1} - \epsilon k_n u_{n+1}u^*_{n-1} - (\epsilon-1)k_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n,$$

for $n = 1, 2, 3, \ldots$, with the boundary conditions $u_{-1} = u_0 = 0$. The associated one dimensional wave number are denoted by k_n, where the discrete index n is referred as the “shell index”. The wave numbers k_n are taken to be

$$k_n = k_0 2^n,$$

Quadratic invariants

We choose $0 < \epsilon < 1$, so that in the inviscid ($\nu = 0$) and unforced ($f_n = 0, \forall n$) case, the model has two quadratic invariants.

- The quantity representing the kinetic energy of the system

$$\mathbb{E} = \sum_{n=1}^{\infty} |u_n|^2.$$

- The sign-indefinite second quadratic invariant

$$\mathbb{W} = \sum_{n=1}^{\infty} \left(\frac{1}{\epsilon - 1} \right)^n |u_n|^2,$$

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The spectrum of the shell model in 3D-regime

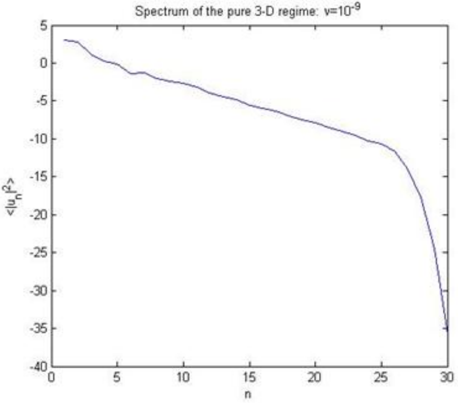
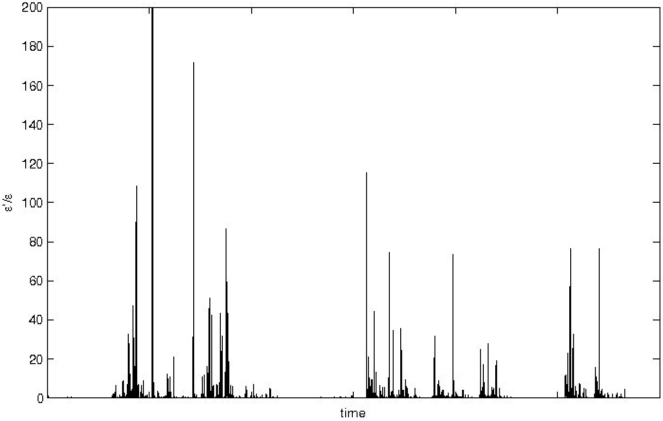


Figure: $U_k := \langle |\mathbf{u}_k|^2 \rangle^{1/2} \sim k^{-0.36}$

Intermittency - $\nu = 10^{-9}$



Preliminaries – the sequence spaces

- Define a space $H = \ell_2$ to be a space of square summable infinite sequences over $\mathbb{C}$, equipped with an inner product and norm

$$(u, v) = \sum_{n=1}^{\infty} u_n v_n^*, \quad |u|^2 = (u, u).$$

This space corresponds to the sequences with the finite kinetic energy.

- Denote a sequence analog of the Sobolev spaces

$$V_d = \{u = (u_1, u_2, u_3, \dots) \ : \ \sum_{n=1}^{\infty} k_n^{2d} |u_n|^2 < \infty\},$$

for some $d \in \mathbb{R}$. This space corresponds to sequences with the finite d-th "derivative".

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Solutions of the viscous model ($\nu > 0$)

- The viscous Sabra shell model has a unique weak and strong global solutions in all the parameters regime for any $u^0, f \in H$, namely, forcing and initial conditions of the finite kinetic energy. (P. Constantin, B. Levant and E.S. Titi, *Physica D*, **219(2)** (2006), 120–141.)
- Moreover, if the forcing $f = (f_1, f_2, f_3, \dots)$ is applied to the finite number of modes (i.e., $f_n = 0$ for n greater than some N), then the solutions of the viscous shell model has an exponentially decaying spectrum

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- The inviscid Sabra shell model has a weak global solution with the finite kinetic energy

$$u(t) \in L^\infty([0, T], H),$$

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The α-energy, which is conserved in the unforced and inviscid case is

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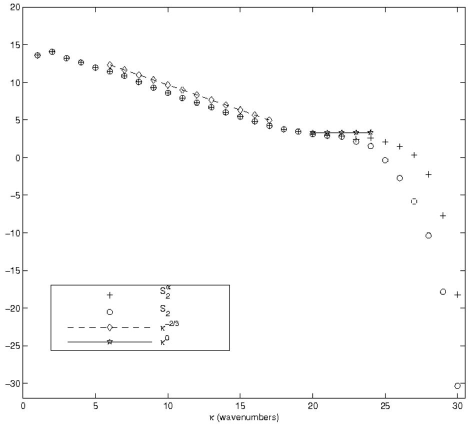
$\nu = 10^{-9}$, $\alpha = 10^{-6}$



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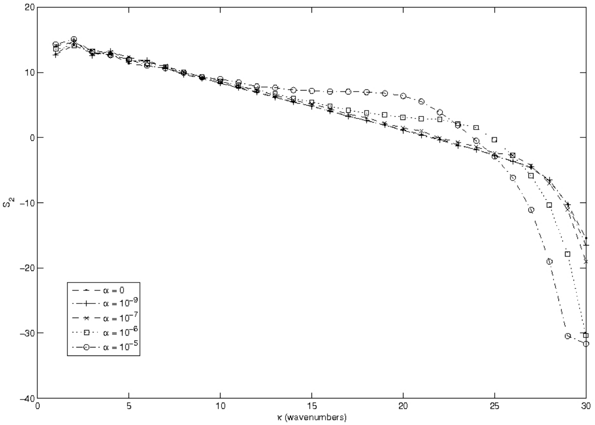

Navier-Stokes-Voigt model suppresses intermittency

- In the previous figures we have seen that for large values of the parameter α, that are larger than the Kolmogorov dissipation scale the Navier-Stoke-Voigh model slows the cascade of energy in the inertial range.
- As a consequence of the above fact we also show, in the next figures, that the Navier-Stokes-Voigt model suppresses the intermittency.

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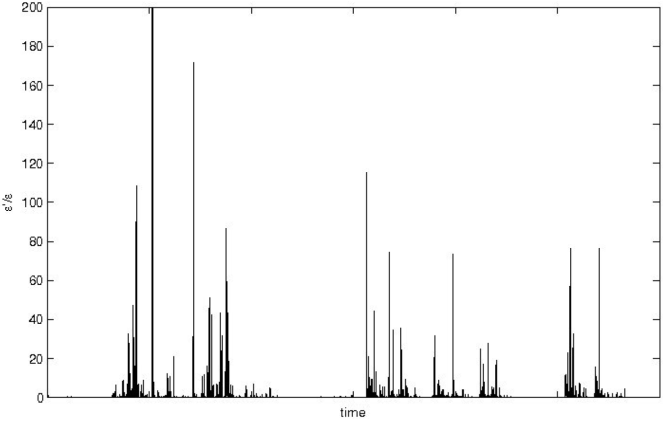
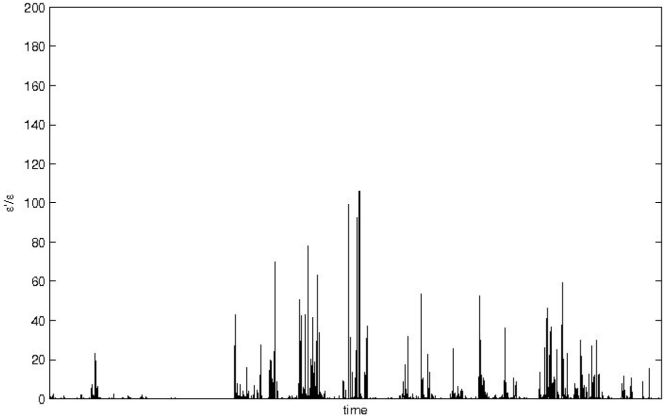
Intermittency - $\nu = 10^{-9}$, $\alpha = 0$



Intermittency - $\nu = 10^{-9}$, $\alpha = 10^{-7}$



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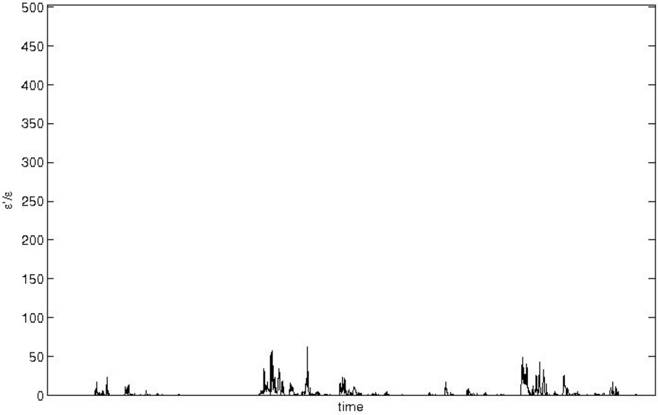

The limit $\alpha \to 0$

- Do the statistical properties of the viscoelastic turbulent flows, described by the NSV equations, converge to the statistical properties of the Navier-Stokes equations?
- Dramatic drag reduction in viscoelastic solutions, even for very small concentration of polymers, indicates that it is not a straightforward questions in some classes of realistic rheological models.
- Thus, the question is: Do invariant measures of the NSV model converge, as $\alpha \to 0$, to a measure associated to the Navier-Stokes equations with physical relevant properties?

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Invariant measures

For finite dimensional dynamical systems

$$\frac{du^{(\nu)}}{dt} = F(u^{(\nu)}), \quad u^{(\nu)}(t) \in \mathbb{R}^N,$$

invariant measures μ obey

$$\int_{\mathbb{R}^N} \nabla_v \Psi(v) F(v) d\mu^{(\nu)}(v) = 0.$$

for any test function Ψ. In infinite dimensions we need to restrict the test functions to a limited class of admissible functions.

Stationary Statistical Solutions

A stationary statistical solution is a Borel probability measure μ^ν on H such that:

(1) $\displaystyle\int_H \|u\|_V^2\, d\mu^\nu(u) < \infty$;

(2) $\displaystyle\int_H \langle u \cdot \nabla u - \mathbf{f}, \Psi'(u)\rangle + \nu\langle \nabla_x u, \nabla_x \Psi'(u)\rangle d\mu^\nu(u) = 0$

for any test functional $\Psi \in \mathcal{T}$, and

(3) $\displaystyle\int_{E_1 \le \|u\|_H \le E_2} \left\{\nu\, \|\nabla_x u\|_2^2 - (\mathbf{f}, u)\right\} d\mu^\nu(u) \le 0,\ E_1, E_2 > 0.$

In practice, in order to be relevant to an experiment, the meaning of $\langle \Phi(u) \rangle$ has to be a specific empirical average (long time average, or long time and space average).

Stationary Statistical Solutions of the Navier-Stokes equations and long time averages.

We construct stationary statistical solutions Navier-Stokes equations by the Krylov-Bogoliubov procedure of taking long time averages.

$$\operatorname{LIM}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(u^{(\nu)}(t))dt = \int_{L^2(\mathbb{R}^2)} \Phi(u) d\mu^{\nu}(u)$$

Zero elasticity limit

Definition

A strong stationary statistical solution of the 3D Navier-Stokes equations is a Borel probability measure μ on V such that

(1) $\displaystyle\int_V \|\mathbf{u}\|_{D(A)}^2 \, d\mu(\mathbf{u}) < \infty;$

(2) $\displaystyle\int_V (\Psi'(\mathbf{u}), \mathbf{f} - \nu A\mathbf{u} - B(\mathbf{u}, \mathbf{u})) d\mu(\mathbf{u}) = 0,$
for any test functional $\Psi \in \mathcal{T}_1^\alpha$,

(3) $\displaystyle\int_{E_1 \le \|\mathbf{u}\|_2^2 \le E_2} \left\{ \nu \|\nabla \mathbf{u}\|_2^2 - (\mathbf{f}, \mathbf{u}) \right\} d\mu(\mathbf{u}) \le 0, \quad E_1, E_2 > 0.$

Zero elasticity limit

Definition

The class of test functions $\mathcal{T}_1^\alpha$ is the set of functions $\Psi : V \to \mathbb{R}$ of the form

$$\Psi(\mathbf{u}) := \Psi_l(u) = \psi\left((\mathbf{u}, \mathbf{w}_1), \ldots, (\mathbf{u}, \mathbf{w}_m)\right), \tag{9}$$

or

$$\Psi(\mathbf{u}) := \Psi_m^\beta(\mathbf{u}) = \psi\left((\beta_\chi(\mathbf{u}), \mathbf{w}_1), \ldots, \beta_\chi(\mathbf{u}), \mathbf{w}_m)\right), \tag{10}$$

where

$$\beta_\chi(\mathbf{u}) = \chi(|\mathbf{u}_\epsilon|^2)\mathbf{u}_\epsilon,$$

and the subscript ϵ means mollification. The function ψ is a C^1 scalar valued function defined on $\mathbb{R}^m$, $m \in \mathbb{N}$; $\mathbf{w}_1, \ldots, \mathbf{w}_m$ belong to $C_0^3(\Omega)$. The function χ belongs to the class $C^\infty(\mathbb{R})$.

Zero elasticity limit

Theorem (F. Ramos and E.S. Titi, Dis. Cont. Dyn. Sys., 28(1) (2010), 375–403.)

Given a sequence of invariant measures of the 3D Navier-Stokes-Voigt model, $\{\mu^{\alpha_n}\}$, with $\alpha_n \to 0$, as $n \to \infty$, there exists a subsequence, denoted also by $\{\mu^{\alpha_n}\}$, and a Borel probability measure μ on H^1, such that

$$\lim_{n\to\infty} \int_{H^1} \Phi(\mathbf{u}) d\mu^{\alpha_n}(\mathbf{u}) = \int_{H^1} \Phi(\mathbf{u}) d\mu(\mathbf{u}), \tag{11}$$

for all weakly continuous bounded real-valued functions Φ. Furthermore, the weak limit measure μ is a strong stationary statistical solution of the 3D Navier-Stokes equations.

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Now, because the function $\mathbf{u} \mapsto |\mathbf{u}|^2$ is weakly continuous and bounded for $\mathbf{u} \in \operatorname{supp} \mu^\alpha$, for $\alpha \in [0, \alpha_0]$, we have

Theorem

Consider a sequence of invariant measures, $\{\mu^{\alpha_n}\}$, of the 3D Navier-Stokes-Voigt model, with $\alpha_n \to 0$, as $n \to \infty$, converging weakly to a stationary statistical solution μ of the Navier-Stokes equations. Then,

$$\lim_{n\to\infty} \int_{H^1} |\mathbf{u}|^2 \, d\mu^{\alpha_n}(\mathbf{u}) = \int_{H^1} |\mathbf{u}|^2 \, d\mu(\mathbf{u}). \tag{12}$$

Theorem

Consider a sequence of invariant measures, $\{\mu^{\alpha_n}\}$, of the Navier-Stokes-Voigt model, with $\alpha_n \to 0$, as $n \to \infty$, converging weakly to a stationary statistical solution μ of the Navier-Stokes equations. Then, for every pair of wavenumbers $\kappa' \leq \kappa''$ such that $\mathbf{f}_{\kappa',\kappa''} = 0$, the convergence of the net rate of energy transfer between κ' and κ'':

$$(\langle e_{\kappa'}(\mathbf{u})\rangle^{\alpha} - \langle e_{\kappa''}(\mathbf{u})\rangle^{\alpha}) \to (\langle e_{\kappa'}(\mathbf{u})\rangle - \langle e_{\kappa''}(\mathbf{u})\rangle) \qquad (13)$$

holds.

Blowup Criterion for the Navier-Stokes and Euler equations

Theorem: Larios-Titi

Let $\mathbf{u}^{(\alpha)}$ be a solution of the Navier-Stokes-Voigt$-\alpha$ (correspondingly Euler-Voigt$-\alpha$) model in the interval $[0, T]$, with initial data $\mathbf{u}_0$. Suppose that

$$\limsup_{\alpha \to 0} \sup_{t \in [0,T]} \alpha^2 \|\nabla \mathbf{u}^{(\alpha)}\|_2^2 > 0$$

Then the strong solution of the Navier-Stokes (correspondingly Euler) equations with the same initial data $\mathbf{u}_0 \in H^s$, for $s \geq 3$, must blow up at the interval $[0, T]$.

Thank You!!