Second order QCD corrections to SIDIS : Technicalities

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Probing Hadron Structure at the Electron-Ion Collider

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in collaboration with S. Goyal, S. Moch, V. Pathak, V. Ravindran

Electron Ion Collider

a machine to look inside the nucleus

EIC will take precision snapshots of the internal structures of the protons and neutrons, allowing us a better understanding of the strongest force in nature

A precise theoretical description of our current understanding (the Standard Model) is also necessary to find any agreement/disagreement with precise experimental data

The inclusive/semi-inclusive deep inelastic scattering (DIS/SIDIS) plays a crucial role. Parton model connects the partonic cross-section to the hadronic one through PDFs & FFs. We compute the partonic cross-section using the framework of perturbative QCD order by order in α_s.

Higher order corrections are essential to

- achieve sufficiently/comparably precise theoretical estimates
- reduce the uncertainties arising from the factorization scales

detailed talk by V. Ravindran

An example from the LHC : NNLO QCD for the SM Higgs

The NNLO QCD corrections played very important role in confirming the SM Higgs.

16.00 pb LO

+ 20.84 pb NLO (EFT)

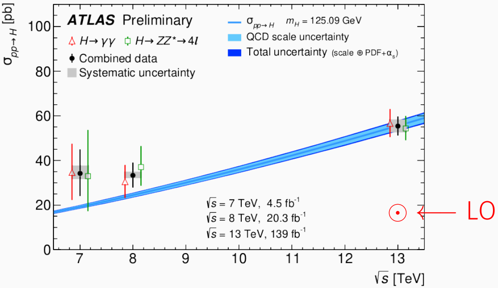
- 2.05 pb (m_t exact NLO)

+ 9.56 pb NNLO (EFT)

+ 0.34 pb NNLO ($1/m_t$)

+ 2.40 pb EW

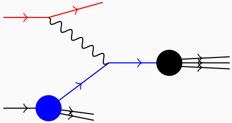
+ 1.49 pb N^3LO (EFT)



If we had considered LO only, we would have never found the SM Higgs.

For EIC also, higher order corrections will play important role!

Semi-Inclusive DIS



SIDIS

$$l + H \to l + H' + X$$

Phase-space:
Final state hadron is tagged!
Extra constrain on the phase-space

$$dPS|_{\text{SIDIS}} = dPS|_{\text{DIS}} \times \delta\left(z' - \frac{p_a \cdot p_b}{p_a \cdot q}\right)$$

DIS

$$l + H \to l + X$$

Phase-space:
All final states are fully integrated!

The hadronic part is characterized by two structure functions F_1 & F_2.

Parton model & perturbative expansion

$$F_i = x^{i-1} \sum_{a,b} \int_x^1 \frac{dx_1}{x_1} f_a(x_1, \mu_F^2) \int_z^1 \frac{dz_1}{z_1} D_b(z_1, \mu_F^2) \times \mathcal{F}_{i,ab}\left(\frac{x}{x_1}, \frac{z}{z_1}, Q^2, \mu_F^2\right)$$

$\Downarrow$

the finite coefficient functions which can be computed perturbatively

In QCD, we have a series expansion of the partonic cross sections in strong coupling constant α_s:

$$\begin{aligned}\mathcal{F}_{ab}(z) &= \mathcal{F}_{ab}^{(0)} \sum_{m=0}^{\infty} \alpha_s^m \, \mathcal{F}_{ab}^{(m)}(z) \\ &= \mathcal{F}_{ab}^{(0)} \left[1 + \alpha_s \mathcal{F}_{ab}^{(1)}(z) + \alpha_s^2 \mathcal{F}_{ab}^{(2)}(z) + \alpha_s^3 \mathcal{F}_{ab}^{(3)}(z) + \cdots\right]\end{aligned}$$

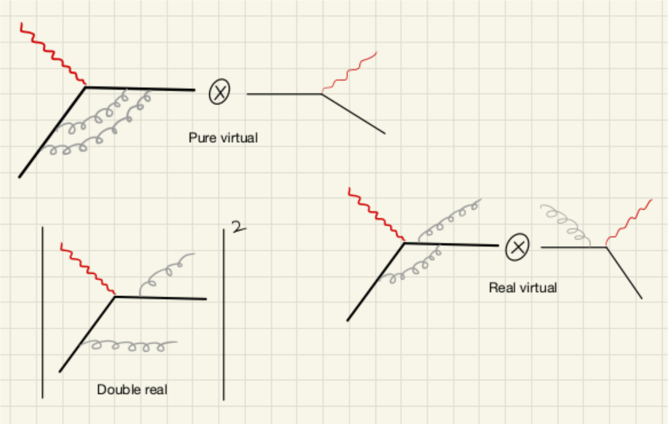
$\nearrow$

We are interested in the second order correction

Goal of this talk

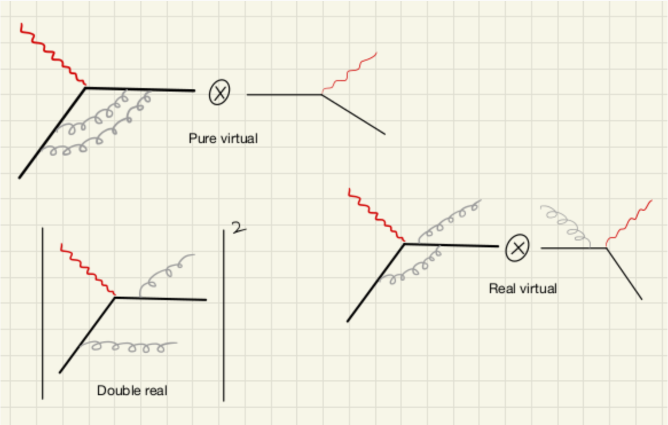
- Motivation, kinematics and the basics have been discussed on Monday
  talk by V. Ravindran
- In this talk, we discuss the details of the computational technology

Schematic diagrams for NNLO contributions to SIDIS



Each individual contribution is divergent : $\frac{1}{\epsilon}$ in dimensional regularization

Schematic diagrams for NNLO contributions to SIDIS



Sum of all degenerate processes: KLN theorem & mass factorization

Computational procedure

$d = 4 - 2\epsilon$

- Diagrammatic approach -> `QGRAF` to generate Feynman diagrams
- In-house `FORM` routines for algebraic manipulation : *Lorentz, Dirac and Color algebra*
- Reverse unitarity : phase-space integrals to loop integrals

$$\delta(k^2 - m^2) \sim \frac{1}{2\pi i}\left(\frac{1}{k^2 - m^2 - i0} - \frac{1}{k^2 - m^2 + i0}\right)$$

- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

- Identity relations among scalar integrals : IBPs, LIs & SRs
- Algebraic linear system of equations relating the integrals

$\Downarrow$

Master integrals (MIs)

- Computation of MIs : Method of differential equation (generic & canonical)
- UV renormalization and mass factorization
- Numerical evaluation using suitable PDFs and FFs

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Loop computation procedure: Integration-by-parts identities

Integration-by-parts identities

[Tkachov, Chetyrkin]

Generalization of Gauss's theorem in d dimension

Within dimensional regularization, all integrals in d dimension are well-defined and convergent $\Rightarrow$ integrand must be zero at boundary

$$\int \prod_{i=1}^{l} \mathcal{D}^d l_i \frac{d}{dl_j^\mu} \left(\frac{v^\mu}{D_1^{n_1} \dots D_m^{n_m}} \right) = 0 \quad \Big|_{v \equiv l, p}$$

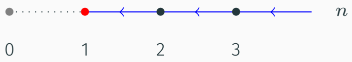
A very simple example 1:

$$\mathcal{I}(n) = \int \frac{d^d l}{(2\pi)^{d/2}} \frac{1}{(l^2 - m^2)^n}$$

The identity for $v \equiv l$ gives a recursion relation for $\mathcal{I}(n+1) \Rightarrow \mathcal{I}(n)$

$$\mathcal{I}(n+1) = \frac{(d-2n)}{2nm^2} \mathcal{I}(n)$$

The relation can be represented as



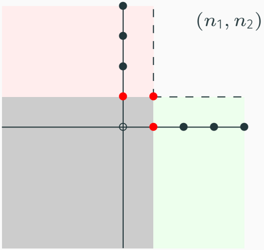
Integration-by-parts identities

[Tkachov, Chetyrkin]

Another simple example 2:

$$\mathcal{I}(n_1, n_2) = \int \frac{d^d l}{(2\pi)^{d/2}} \frac{1}{(l^2 - m_1^2)^{n_1}((l-q)^2 - m_2^2)^{n_2}}$$

The relations now depend on n_1 and n_2 and whether they are `positive` or `non-positive`.



- The relations are like translations from one point to another.
- The first goal is to choose the red points (the MIs).
- The second goal is to find an efficient path (IBP reduction rules) with minimal translation.

Integration-by-parts identities

[Tkachov, Chetyrkin]

Integral families with 7 propagators.

$$\mathcal{I}(n_1, n_2, \ldots, n_7)$$

For NNLO, it's not difficult! Thousands of Feynman integrals can be written in terms of only a few! However for three-loop onward, it becomes extremely challenging.

Scalar integrals form a 'vector space' $\Rightarrow$ IBP reduction is a 'projection' to basis vectors

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Several technical advances have been made in recent years by improving system-solving strategy, either due to novel algorithms or to the development of software.

- `LiteRed` : Symbolical recursion relation
- `FIRE, Reduze, Kira` : Laporta algorithm; Solves for specific integer values

Loop computation procedure: Solving remaining integrals

The method of differential equations

A Feynman integral is a function of spacetime dimension d and kinematic invariants x, z.

$$J_i = \mathcal{N} \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 l_2^2 ((l_1 - l_2)^2 - m^2)(l_1 - q)^2 (l_2 - q)^2} \equiv f(d,\, x,\, z)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.\ x,\ z$ and solve it.

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$$\begin{aligned} \frac{d}{dz} J_i &= \text{some combinations of integrals} \\ &\Downarrow \text{ IBP identities/reduction} \\ &= \sum_{j=1}^{n} c_{ij} J_j \end{aligned}$$

c_{ij}'s are rational function of $d,\ x$ and z.

J_i is a basis 'vector' $\Rightarrow \frac{d}{dz}$ is a 'rotation' $\Rightarrow$ IBP reduction is a 'projection' to basis vectors

The method of differential equations

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The idea is to obtain a differential eqn. for the integral $w.r.t.\ x,\ z$ and solve it.

$$d_z \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_z \mathbb{J} = \mathbb{A}(d, z) \mathbb{J}$$

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The bullets ($\bullet$) indicate a non-zero rational function of $d,\ x$ and z.

To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form. Then, it can be solved using bottom-up approach.

The method of differential equations

- However, even a small 2×2 sub-system is difficult to solve in d-dimension.
- The solution is to expand the sub-system in ϵ and solve order-by-order in ϵ.

$$\frac{d}{dz} J_n(z,\epsilon) = \mathcal{C}_{nm}(z,\epsilon) J_n(z,\epsilon) + \mathcal{R}_n(z,\epsilon)$$

Taylor expansion in ϵ

$$J_n(z,\epsilon) = \sum_{k=-2}^{\infty} J_n^{(k)}(z)\epsilon^k \,, \mathcal{C}_n(z,\epsilon) = \sum_{k=0}^{\infty} \mathcal{C}_n^{(k)}(z)\epsilon^k \,, \mathcal{R}_n(z,\epsilon) = \sum_{k=-2}^{\infty} \mathcal{R}_n^{(k)}(z)\epsilon^k$$

The leading pole is fixed for a topology (process).

$$\frac{d}{dz} J_n^{(k)}(z) = \mathcal{C}_{nm}^{(0)}(z) J_n^{(k)}(z) + \sum_{p=1}^{k+2} \mathcal{C}_{nm}^{(p)}(z) J_n^{(k-p)}(z) + \mathcal{R}_n^{(k)}(z)$$

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The leading pole is fixed for a topology (process). The homogeneous part is same for all k!

$$\frac{d}{dz} J_n^{(k)}(z) = \mathcal{C}_{nm}^{(0)}(z) J_n^{(k)}(z) + \sum_{p=1}^{k+2} \mathcal{C}_{nm}^{(p)}(z) J_n^{(k-p)}(z) + \mathcal{R}_n^{(k)}(z)$$

Algorithm : to solve a system of linear first order diff. eqns.

- First step is to reduce the sub-system to a higher order eqn in a single unknown

$$d_z \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} a \frac{d^2 J_1}{dz^2} + b \frac{dJ_1}{dz} + cJ_1 + d = 0 \\ J_2 = a' \frac{dJ_1}{dz} + b' J_1 + c' \end{array}$$

- Start with the leading pole (ϵ^{-2}) - find the homogeneous solutions $(h_i(z))$ and best uncoupling procedure - solve for the nonhomogeneous part using the method of variation of constant
- Structure of homogeneous part is same at each order in ϵ-expansion. Hence the homogeneous solutions and uncoupling procedure are unique for any order
- Now at each order in ϵ, find the nonhomogeneous part $(r(y))$ keeping the uncoupling structure fixed and solve using variation of constant

$$g(z) = \sum_{i=1}^{m} h_i(z) \int dy \, \frac{r(y) W_i(y)}{W(y)}$$

The results are obtained in terms of iterated integrals (HPLs/GPLs).

Iterated integrals

From Feynman integrals to iterated integrals : What do we gain?

Direct numerical integration of Feynman integrals is tedious, unstable and challenging to obtain precise results.

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Direct numerical integration of Feynman integrals is tedious, unstable and challenging to obtain precise results.

Iterated integrals are one-dimensional. They can be computed with great precision in a short amount of time. Besides, they have the following properties:

(a) **Shuffle algebra** : Allows to obtain a basis for a set of iterated integrals. Reduction to such a basis is extremely effective to reduce the computation time by few times.

(b) **Scaling invariance** : Allows to convert the limit of these integrals from kinematical variables (z) to constants (1). This makes the integration really precise.

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Till date, most of the MIs were solved in terms of GPLs

$\Downarrow$

the iterative kernel is a simple polynomial $\int_0^x \frac{dt}{a+t} \int_0^t \cdots$

What happens when we have (multiple) square-roots?!

Rationalization

Rationalizable

- Find a suitable transformation

Let's consider $\sqrt{4m^2 - s}$.
We can use Landau transformation $s = -m^2 \frac{(1-x)^2}{x}$ for this.

Non-rationalizable

- A single transformation can not rationalize all square-roots simultaneously.

$\Downarrow$

Square-roots will be present in the iterated integrals.

1) We can accept 'the fact' and evaluate them with appropriate analytic continuation.

or

2) Instead of using a single transformation rule to rationalize them, we write the system (each MI) as sum of functions of dependent variables and treat them separately. As a result, each sub-system has alphabet with 'good' letters with different argument.

Phase-space computation procedure: Reverse unitarity

Reverse unitarity

- The IBP identities and method of differential equations are state-of-the-art tools. They only depend on the 'form' of the object (Feynman integrals) and its variables.
- The phase-space integrals are challenging, specially the angular integration.

$$\mathcal{N}\int d^d l_1 d^d l_2 d^d p_b \frac{1}{(p_a-l_1)^2\cdots}\delta(l_1^2)\delta(l_2^2)\delta(p_b^2)\delta^d(p_a+q-l_1-l_2-p_b)\delta\left(z'-\frac{p_a.p_b}{p_a.q}\right)$$

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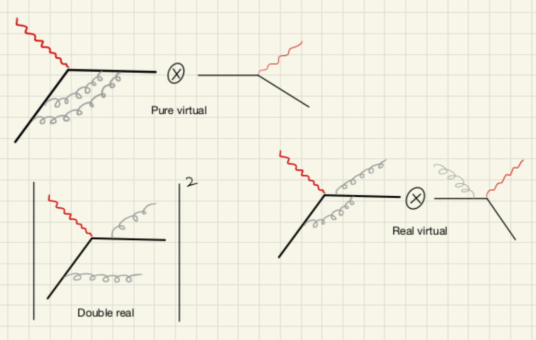
- The idea is to write the phase-space integrals in the loop-integral format and apply the methods (IBP & DE).
- Reverse unitarity

$$\delta(k^2-m^2)\sim\frac{1}{2\pi i}\left(\frac{1}{k^2-m^2-i0}-\frac{1}{k^2-m^2+i0}\right)$$

- We can consider only the first term, as the differential equation is independent of the sign of $i0$. Of course, we need a boundary conditions to solve differential equations and that is where the actual physics information (phase-space integrals) goes in!

To obtain the finite partonic cross-section

Compute & combine everything and mass factorization

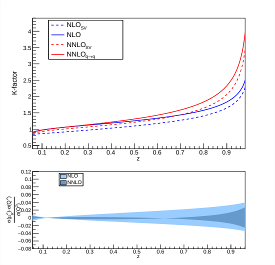


- We compute relevant Feynman diagrams & corresponding Feynman integrals analytically.
- We combine them appropriately, perform mass factorization and obtain the finite partonic cross-section.

Results!

Results

We have computed the non-singlet contributions to the quark initiated process with quark fragmenting to hadrons. The finite partonic cross-section has been convoluted with PDFs and FFs to obtain the hadronic cross-section through a FORTRAN code.



Checks

- The master integrals were computed using different methods!
- Mass factorization (universal) removes all remaining infrared singularities!
- Successful checks with available results in the threshold limit!
- The constraint (z') can be integrated in our analytic result. We found perfect agreement with the fully inclusive result.

Concluding remarks!

- EIC will unravel the mysteries of strong force.
- Theoretical precision studies are extremely necessary to fully exploit the EIC data.
- Our current (well-tested) theoretical understanding (the SM) is constrained by its perturbative nature and hence, higher order perturbative corrections are necessary to achieve precise theoretical predictions.

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- In this talk, we have presented the computational details to obtain the first results on NNLO QCD corrections to SIDIS.
- The technicalities are impressive and generic.

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- Aside the phenomenological impact of the result, it also sets a milestone for the computational technique.

Thank you for your attention!