

E: The

Primes

$$A_N f(x) = \frac{1}{N} \sum_{n=1}^N \Lambda(n) f(x-n)$$

$$\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

Thm (Wiener)

$$\| \sup_N |A_N f| \|_{\ell^p} \leq \|f\|_{\ell^p} \quad |2p < \infty$$

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Thm (Han, Krause, ~~Lo~~, Yang)

A_N satisfies l^p -improving, $1 < p < 2$.

Also additional results w/

Giannitsi, Mousavi, Rahimi

$$\hat{A}_N(\theta) = \frac{1}{N} \sum_{n=1}^N \Lambda(n) e(n\theta)$$

$$\theta = \frac{a}{q} \rightarrow \hat{A}_N\left(\frac{a}{q}\right) = \frac{1}{N} \sum_{n=1}^N \Lambda(n) e\left(n \frac{a}{q}\right)$$

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$$\approx \frac{1}{q} \sum_{r=1}^q \frac{q}{N} \sum_{m=1}^{N/q} \Lambda(mq+r) e\left((mq+r) \frac{a}{q}\right)$$

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$$\approx \underbrace{\frac{1}{q} \sum_{r=1}^q}_{\text{primes}} \frac{q}{N} \sum_{m=1}^{N/q} \underbrace{\Lambda(mq+r) e\left((mq+r) \frac{a}{q}\right)}_{= e\left(\frac{ar}{q}\right)}$$

primes
not equi distributed
in progressions

Dirichlet Theorem

For $(a, q) = 1$

$$\frac{\phi(q)}{N} \# \{ p \leq N : p \equiv a \pmod{q} \} \rightarrow 1$$

$$\hat{A}_N(\theta) = \frac{1}{\phi(q)} \sum_{(r, q) = 1} \frac{\phi(q)}{N} \sum_{m=1}^{N/q} \Lambda(mq+r) e(ra/q)$$

$$\approx \frac{1}{\phi(q)}$$

$$\sum_{(r, q) = 1} e(ra/q)$$

Ramanujan sum

$$= \frac{\mu(q)}{\phi(q)}$$

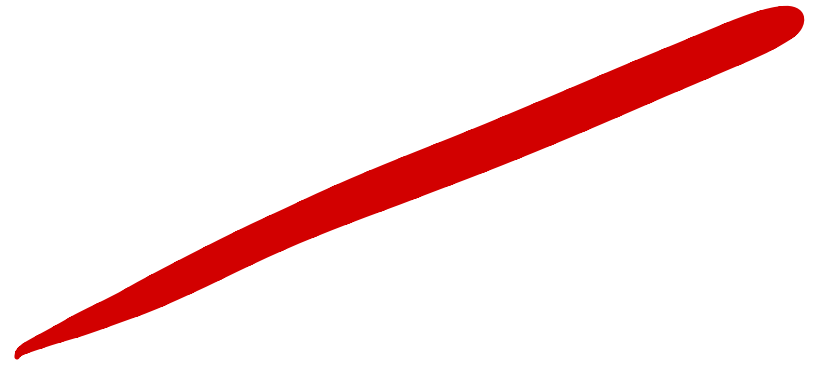
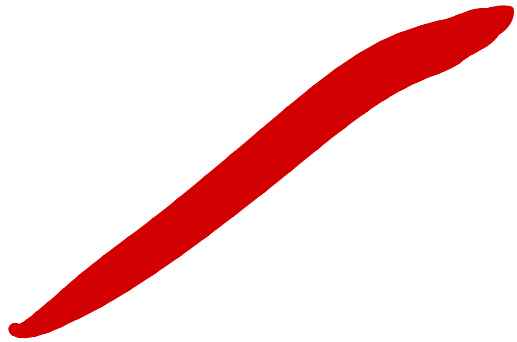
$\mu =$ Mobius f^{-1}

special value of

Ram. sum

Start

here



Goal: A major/minor
decomposition for

$$\hat{A}_N(\theta) = \frac{1}{N} \sum_{n \in N} \Lambda(n) e(n\theta)$$

for $\theta = a/q$, q much smaller than N

$$\hat{A}\left(\frac{a}{q}\right) \approx \frac{\mu(q)}{\phi(q)}$$

$$\phi(q) \approx q^{1-\varepsilon}$$

$$\phi(q) = \text{totient } f_q^a = \#\{1 \leq a < q : (a, q) = 1\}$$

$$\mu(q) = \begin{cases} 0 & p^2 \mid q \text{ for prime } p \\ (-1)^k & q = p_1 p_2 \dots p_k \end{cases}$$

Next Step θ close to
rational.

Need a big upgrade on
Dirichlet Theorem

Key Additional Ingredient (Major arcs)

Siegel - Walfitz (Primes in APs)

"Dirichlet T_m , w/ error terms"

Siegel - Walfisz

$(a, q) = 1$

$$\frac{\phi(q)}{N} \sum_{\substack{n \leq N \\ n \equiv a \pmod{q}}} \Lambda(n) = 1 + O\left(\exp\left(-\sqrt{\log N}\right)\right)$$

This depends upon deep
properties of L functions

Siegel-Walfitz allows a
straight forward approach to
the 'close to rational' case.

Key Additional Ingredient
(minor axis)

Vaughan's Identity:

An intricate inequality for

$$\sum_{n \leq N} f(n) \Lambda(n)$$

Main Approximation

$\epsilon > 0$

For $1 \leq Q \leq \exp(\sqrt{\log N})$

$$A_N = A_N^{\text{minor}} \cup A_N^{\text{major}}$$

Main Approximation

$\varepsilon > 0$

For $1 \leq Q \leq \exp(c\sqrt{\log N})$

$$A_N = A_N^{\text{minor}} \oplus A_N^{\text{major}}$$

$$\| \widehat{A_N^{\text{minor}}} \|_{\infty} \leq \frac{1}{Q^{1-\varepsilon}}$$

Main Approximation

$\varepsilon > 0$

For $1 \leq Q \leq \exp(\sqrt{\log N})$

$$A_N = H_i + L_0$$

$$\| \widehat{H}_i \|_{\infty} \leq \frac{1}{Q} \varepsilon$$

$$\widehat{L_0}(\theta) =$$

$$\sum_{q \in S_Q}$$

$$\sum_{(q, p) \in \dots}$$

$$\frac{\mu(q)}{\phi(q)}$$

$$\widehat{\chi}_N(\theta - \frac{q}{p})$$

$$\widehat{\Psi}_Q(\theta - \frac{q}{p})$$

cutoff

Dirichlet kernel

This result (except for role of Q)

goes back to Vinogradov

$$S_Q = \frac{1}{2}q : p/q \Rightarrow \{ p < Q \}$$



Vinogradov Approach

arg count for no. of representations of n as $p_1 + p_2 + p_3$

is

$$A_n \times A_n \times A_n(n)$$

$$= \int_{\mathbb{T}} \widehat{A_n}(\theta) e(-n\theta) d\theta$$

We will take the approach

$$A_n \otimes A_n \otimes A_n(n) \approx \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}(n)$$

Important calculation for low part, A

$$\widehat{B}_q(\theta) = \sum_{(q, q_1)=1} \frac{\mu(q_1)}{\phi(q_1)} \widehat{\alpha}_N(\frac{\cdot}{q_1}) \widehat{\psi}_Q(\frac{\cdot}{q_1})$$

Lemma

For square free q

$$B_q(x) = \underbrace{\alpha_N * \psi_Q(x)}_{\text{analytic part}} \underbrace{\frac{\mu_q C_q(-x)}{\phi(q)}}_{\text{arithmetic part}}$$

analytic part

arithmetic part

Proof

$$B_q(x) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \int \hat{\alpha}_N(\xi - a/q) \hat{\psi}_Q(\xi - a/q) e\left(\left(\frac{\xi - a/q}{Q}\right) x\right) e\left(\frac{ax}{q}\right) dx$$

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$$= \hat{\alpha}_N * \hat{\psi}_q(x) \frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q) e(ax/q)$$

Proof

$$B_q(x) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \int \hat{\alpha}_N(\xi - a/q) \hat{\psi}_{2^k}(\xi - a/q) e\left(\left(\xi - \frac{a}{q}\right)x\right) e\left(\frac{ax}{q}\right) dx$$

$$= \underbrace{\hat{\alpha}_N * \hat{\psi}_{2^k}(x)}_{\text{nice avg}} \underbrace{\frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q) e(ax/q)}_{\text{evaluate this}}$$

nice avg

evaluate this

$$\frac{1}{\phi(q)} \sum_{(a, q)=1} \mu(q) e(ax/q)$$

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Ram. Sum

$$= \frac{1}{\phi(q)} \sum_{(a, q)=1} \left\{ \sum_{(r, q)=1} e(ra/q) \right\} e(ax/q)$$

$$\frac{1}{\phi(q)} \sum_{(r,q)=1} \mu(q) e(ax/q)$$

$$= \frac{1}{\phi(q)} \sum_{(r,q)=1} \left\{ \sum_{(r,q)=1} e(ra/q) \right\} e(ax/q)$$

reverse sum

$$= \frac{1}{\phi(q)} \sum_r \sum_a e((r+x)a/q) = \frac{1}{\phi(q)} \sum_{(r,q)=1} C_q^{(r+x)}$$

Ram Sum
↙

$$\frac{1}{\phi(q)} \sum_{(r,q)=1} \mu(q) e(ax/q)$$

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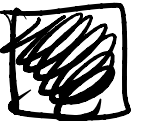
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Ram Sum
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$$= \frac{1}{\phi(q)} \mu(q) C_q(-x)$$

Cohen's Identity





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Some relevant facts:



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$$\bullet C_p(a) = \begin{cases} -1 & a \neq 0 \\ \phi(p) = p-1 & a = 0 \end{cases}$$

$$\bullet \mu(q) = 0 \text{ if } p^2 | q \text{ } p \text{ prime}$$



$C_q(\cdot)$ is only typically small.

Some relevant facts:

$$* C_p(a) = \begin{cases} -1 & a \neq 0 \\ \phi(p) = p-1 & a = 0 \end{cases}$$

• $\mu(q) = 0$ if $p^2 \mid q$ p prime

• $C_*(*)$ & $\mu(\cdot)$ are multiplicative.

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$$Q \in \mathbb{N}, \quad S_{Q, \mathbb{N}} = \{q : q \text{ sq free} \\ p | q \Rightarrow p < Q\}$$

$$\log \prod_{p < Q} p = \sum_{p < Q} \log p \approx Q$$

$$\text{i.e. } \max_{q \in S_Q} \ln e^Q$$

Lemma Let $Q < \exp(\sqrt{\log N})$, $\varepsilon > 0$

$$|L_{O_{Q,N}}(x)| \leq \log N \alpha_N * \psi_0(x)$$

Recall :

$$\widehat{B}_q(\theta) = \sum_{(q, q')=1} \frac{\mu(q')}{\phi(q')} \widehat{\alpha}_N(\frac{\theta}{q'}) \widehat{\Psi}_Q(\frac{\theta}{q'})$$

Lemma

For square free q

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analytic part

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Lemma

Let $S_Q = \{q: \mu(q) \neq 0, q \text{ in } Q\text{-smoothly}\}$

Then

$$\left| \sum_{q \in S_Q} \frac{\mu(q) C_q(x)}{q} \right| \leq \log Q$$

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Proof

Divide

S_Q

$$Q_1 = \{p: p < Q, p \nmid x\}$$

$$Q_2 = \{p < Q: p \mid x\}$$

Then for $g \in S_Q$, x fixed

$$g = g_1 g_2, \quad g_1 \in S_{Q_1}, \quad g_2 \in S_{Q_2}$$

Then for $q \in SQ$, x fixed

$$q = q_1 q_2, \quad q_1 \in SQ_1, \quad q_2 \in SQ_2$$

Then

$$\frac{\mu(q) C_q(-x)}{\phi(q)} = \frac{\mu(q_1) C_{q_1}(-x)}{\phi(q_1)} \frac{\mu(q_2) C_{q_2}(-x)}{\phi(q_2)}$$

↑
multiplicative

Then for $q \in SQ$, x fixed

$$q = q_1 q_2, \quad q_1 \in SQ_1, \quad q_2 \in SQ_2$$

$$\text{Then } \frac{\mu(q) C_q(-x)}{\phi(q)} = \frac{\mu(q_1) C_{q_1}(-x)}{\phi(q_1)} \frac{\mu(q_2) C_{q_2}(-x)}{\phi(q_2)} = 1$$

(Note: In the original image, $C_{q_1}(-x)$ is circled in red with the label $= \mu(q_1)$ above it, and the entire right-hand side fraction is circled in red.)

$$\frac{\mu(q) C_q(x)}{\phi(q)} = \frac{\mu(q_2)}{\phi(q_1)}$$

$$\sum_{q \in Q_1} \frac{1}{\phi(q)} = \prod_{\substack{p|x \\ p < Q}} \left(1 + \frac{1}{\phi(p)}\right) - 1$$

$$= \prod_{\substack{p|x \\ p < Q}} \frac{1}{p-1} - 1 \ll \log Q$$

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$$= \prod_{\substack{p|x \\ p \leq Q}} \frac{1}{p-1} - 1 \ll \log Q$$

$$S_2 = \sum_{q \in S_{Q_2}} \mu(q) \quad , \quad d = |Q_2|$$

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$$= \sum_{i=1}^d \binom{d}{i} (-1)^i$$

$$= -1 + \sum_{i=0}^d \binom{d}{i} (-1)^i = -1 + (-1+1)^d$$

$$= -1$$

(*)

Improving

(*)

Sparse (P, P)

$$1 < P \leq 2$$