E: The Primes

$$A_N f(x) = \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) f(x-n)$$

$$\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

<u>Thm</u> (Wierdl)

$$\| \sup_N |A_N f| \|_{\ell^p} \lesssim \|f\|_{\ell^p} \qquad 1 < p < +\infty$$

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<u>Thm</u> (Han, Krause, La, Yang)

A_N satisfies ℓ^p-improving, $1<p<2$.

Also additional results w/

Giannitsi, Mousavi, Rahimi

$$\hat{A}_N(\theta) = \frac{1}{N} \sum_{n=1}^{N} \Lambda(n)\, e(n\theta)$$

$$\theta = \frac{a}{q} \longrightarrow \hat{A}_N\left(\frac{a}{q}\right) = \frac{1}{N} \sum_{n=1}^{N} \Lambda(n)\, e\left(n\frac{a}{q}\right)$$

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primes

not equidistributed

in progressions

Dirichlet Theorem

For $(a,q)=1$

$$\frac{\phi(q)}{N}\,\#\{p\le N:\ p\equiv a \bmod q\}\to 1$$

$$\hat{A}_N(\theta) \overset{'}{=} \frac{1}{\phi(q)} \sum_{(r,q)=1} \frac{\phi(q)}{N} \sum_{m=1}^{N/q} \Lambda(mq+r)\, e(ra/q)$$

$$\simeq \frac{1}{\phi(q)} \sum_{(r,q)=1} e(ra/q)$$

Ramanujan sum

$$= \frac{\mu(q)}{\phi(q)}$$

μ = Mobius $f^{\underline{n}}$

special value of Ram. sum

Start Here

<u>Goal:</u> A major/minor decomposition for

$$\widehat{A_N}(\theta) = \frac{1}{N}\sum_{n \le N} \Lambda(n)\, e(n\theta)$$

for $\theta = a/q$, q much smaller than N

$$\hat{A}\left(\frac{a}{q}\right) \cong \frac{\mu(q)}{\phi(q)}$$

$\phi(q) \approx q^{1-\varepsilon}$

$\phi(q)$ = totient fn $= \# \{ 1 \le a < q : (a,q) = 1 \}$

$$\mu(q) = \begin{cases} 0 & p^2 \mid q \text{ for prime } p \\ (-1)^k & q = p_1 p_2 \cdots p_k \end{cases}$$

Next Step: θ close to rational.

Need a big upgrade on Dirichlet Theorem

Key Additional Ingredient (Major arcs)

Siegel – Walfitz (Primes in AP's)

"Dirichlet Thm, w/ error terms"

Siegel - Walfisz

$(a,q)=1$

$$\frac{\phi(q)}{N}\sum_{\substack{n\le N\\ n\equiv a \bmod q}} \Lambda(n) = 1 + O\left(\exp\left(-\sqrt{\log N}\right)\right)$$

This depends upon deep properties of L functions

Siegel-Walfitz allows a straight forward approach to the 'close to rational' case.

Key Additional Ingredient
(minor arcs)

Vaughan's Identity:

An intricate inequality for

$$\sum_{n \le N} f(n) \Lambda(n)$$

Main Approximation

$\varepsilon > 0$

For $1 \leq Q \leq \exp(c\sqrt{\log N})$

$$A_N = A_N^{\text{minor}} + A_N^{\text{major}}$$

Main Approximation $\quad \varepsilon > 0$

For $1 \leq Q \leq \exp(c\sqrt{\log N})$

$$A_N = A_N^{\text{minor}} + A_N^{\text{major}}$$

$$\| \widehat{A_N^{\text{minor}}} \|_\infty \lesssim \frac{1}{Q^{1-\varepsilon}}$$

Main Approximation

$\varepsilon > 0$

For $1 \leq Q \leq \exp(c\sqrt{\log N})$

$$A_N = Hi + Lo$$

$$\|\widehat{Hi}\|_\infty \lesssim \frac{1}{Q^{1-\varepsilon}}$$

Dirichlet kernel

$$\widehat{Lo}(\theta) = \sum_{q \in S_Q} \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \widehat{\alpha}_N\left(\theta - \frac{a}{q}\right) \underbrace{\widehat{\varphi}_Q\left(\theta - \frac{a}{q}\right)}_{\text{cutoff}}$$

This result (except for role of Q) goes back to Vinogradov

$$\boxed{S_Q = \{q : p \mid q \Rightarrow p < Q\}}$$

Vinogradov Approach

avg count for no. of representations of n as $p_1 + p_2 + p_3$

is

$$A_n * A_n * A_n (n)$$

$$= \int_{\mathbb{T}} \widehat{A_n}(\theta)\, e(-n\theta)\, d\theta$$

We will take the approach

$$A_n * A_n * A_n(n) \approx Lo * Lo * Lo(n)$$

Important caculation for low part, A

$$\widehat{B_q}(\theta) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \widehat{\alpha_N}(\xi - a/q)\, \widehat{\psi_Q}(\xi - a/q)$$

Lemma For square free q

$$B_q(x) = \underbrace{\alpha_N * \psi_Q(x)}_{\text{analytic part}} \; \underbrace{\frac{\mu_q\, C_q(-x)}{\phi(q)}}_{\text{arithmetic part}}$$

Proof

$$B_q(x) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \int \hat{\alpha}_N(\xi - a/q)\, \hat{\psi}_Q(\xi - a/q)\, e((\xi - a/q)x)\, e(ax/q)\, dx$$

__Proof__

$$B_q(x) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \int \hat{\alpha}_N(\xi - a/q)\, \hat{\psi}_Q(\xi - a/q)\, e((\xi - a/q)x)\, e(ax/q)\, dx$$

$$= \alpha_N * \psi_Q(x)\, \frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q)\, e(ax/q)$$

Proof

$$B_q(x) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \int \hat{\alpha}_N(\xi - a/q)\, \hat{\psi}_{2^s}(\xi - a/q)\, e((\xi - a/q)x)\, e(ax/q)\, dx$$

$$= \underbrace{\alpha_N * \psi_{2^s}(x)}_{\text{nice avg}} \underbrace{\frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q)\, e(ax/q)}_{\text{evaluate this}}$$

$$\frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q)\, e(ax/q)$$

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Ram. Sum

$$= \frac{1}{\phi(q)} \sum_{(a,q)=1} \left\{ \sum_{(r,q)=1} e(ra/q) \right\} e(ax/q)$$

$$\frac{1}{\phi(q)} \sum_{(a,q)=1} \mu(q)\, e(ax/q)$$

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reverse sum

Ram Sum

$$= \frac{1}{\phi(q)} \sum_{r} \sum_{a} e((r+x)a/q) = \frac{1}{\phi(q)} \sum_{(r,q)=1} c_q(r+x)$$

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Ram Sum

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Cohen's Identity

$$= \frac{1}{\phi(q)} \mu(q)\, c_q(-x)$$

$\blacksquare$

$C_q(\cdot)$ is only typically small.

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- $$C_p(a) = \begin{cases} -1 & a \neq 0 \\ \phi(p) = p-1 & a = 0 \end{cases}$$
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- $C_\cdot(*)$ & $\mu(\cdot)$ are multiplicative.

Move to redefine
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$Q \in \mathbb{N}$, $\quad S_{Q,N} = \{ q : q$ sq free $p \mid q \Rightarrow p < Q \}$

$$\log \prod_{p<Q} p = \sum_{p<Q} \log p \simeq Q$$

i.e. max $q \in S_Q$ is e^Q

Lemma Let $Q < \exp(\sqrt{\log N})$, $\varepsilon > 0$

$$|Lo_{Q,N}(x)| \lesssim \log N \ \alpha_N * \psi_0(x)$$

Recall:

$$\widehat{B_q}(\theta) = \sum_{(a,q)=1} \frac{\mu(q)}{\phi(q)} \widehat{\alpha_N}(\xi - a/q)\, \widehat{\psi_Q}(\xi - a/q)$$

Lemma For square free q

$$B_q(x) = \underbrace{\alpha_N * \psi_Q(x)}_{\text{analytic part}} \cdot \underbrace{\frac{\mu(q)\, c_q(-x)}{\phi(q)}}_{\text{arithmetic part}}$$

Lemma Let $S_Q = \{ q : \mu(q) \neq 0, \ q \text{ is } Q\text{-smooth} \}$

Then

$$\left| \sum_{q \in S_Q} \frac{\mu(q) \, c_q(-x)}{q} \right| \lesssim \log Q$$

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Proof Dive S_Q . .

$$Q_1 = \{ p : p < Q,\ p \nmid x \}$$

$$Q_2 = \{ p < Q : p \mid x \}$$

Then for $q \in S_Q$, x fixed

$$q = q_1 q_2, \quad q_1 \in S_{Q_1}, \quad q_2 \in S_{Q_2}$$

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$q = q_1 q_2$, $q_1 \in S_{Q_1}$, $q_2 \in S_{Q_2}$

Then
$$\frac{\mu(q)\, c_q(-x)}{\phi(q)} = \frac{\mu(q_1)\, c_{q_1}(-x)}{\phi(q_1)} \cdot \frac{\mu(q_2)\, c_{q_2}(-x)}{\phi(q_2)}$$

multiplicative

Then for $q \in S_Q$, x fixed

$$q = q_1 q_2, \quad q_1 \in S_{Q_1}, \quad q_2 \in S_{Q_2}$$

Then

$$\frac{\mu(q)\, C_q(-x)}{\phi(q)} = \frac{\mu(q_1)\, \underbrace{C_{q_1}(-x)}_{=\mu(q_1)}}{\phi(q_1)} \cdot \frac{\mu(q_2)\, C_{q_2}(-x)}{\phi(q_2)}$$

$$= 1$$

$$\frac{\mu(q)\, c_q(-x)}{\phi(q)} = \frac{\mu(q_2)}{\phi(q_1)}$$

$$\sum_{q \in Q_1} \frac{1}{\phi(q)} = \prod_{\substack{p \mid x \\ p < Q}} \left(1 + \frac{1}{\phi(p)}\right) - 1$$

$$= \prod_{\substack{p \mid x \\ p < Q}} \frac{1}{p-1} - 1 \ll \log Q$$

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$$= \prod_{\substack{p \mid \leq x \\ p < Q}} \frac{1}{p-1} - 1 \ll \log Q$$

$$S_2 = \sum_{q \in S_{Q_2}} \mu(q) \quad , \quad d = |Q_2|$$

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$$= \sum_{i=1}^{d} \binom{d}{i} (-1)^i$$

$$= -1 + \sum_{i=0}^{d} \binom{d}{i} (-1)^i = -1 + (-1+1)^d$$

$$= -1$$

- Improving
- Sparse (p,p) $\quad 1 < p \leq 2$